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AUTOMORPHISMS OF INDUCTIVE LIMIT C^*-ALGEBRAS

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Introduction.

After the remarkable results obtained in [2], [7] and especially in [8], the study of C^*-algebra inductive limits of finite direct sums of matrix algebras over commutative C^*-algebras (suggested by E. G. Effros in [5]), and also of their automorphism groups, become obviously more attractive and important.

In this note we prove some results concerning mainly the automorphisms of a certain class of C^*-algebras which we call *almost constant* (see Definition 1). These are some C^*-algebra inductive limits of matrix algebras over commutative C^*-algebras, including many Goodearl algebras [10] of real rank zero [3] and, in particular, all the Bunce-Deddens algebras [4].

The main purpose of this paper is to show that a similar type of results with those given in [13] can be obtained for a large class of algebras. (Compare the very recent work in a similar direction in [9]).

Let $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ be an almost constant C^*-algebra and consider the UHF algebra $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}}) \subset A$. If moreover $K_0(A)$ is *weakly torsion free* (see Definiton 3) and A has cancellation it is shown that any endomorphism of A is approximately inner with respect to the trace seminorm (see Theorem 1 for a much more complete and general result). Necessary and sufficient conditions for an automorphism of B to be extended to an automorphism of A are given, provided that the (unique) trace of A is faithful (see Theorem 2). A key fact in proving these results is that B is dense in A with respect to the trace seminorm (see Proposition 2b)). Also it is shown that the centralizer of $\{\Phi \in \mathrm{Aut}(B): \Phi = \tilde{\Phi}_{|B}$ for some $\tilde{\Phi} \in \mathrm{Aut}(A)\}$ in $\mathrm{Aut}(B)$ is trivial and if moreover $A \cap A' = \mathbf{C} \cdot 1_A$ then the centralizer of $\{\Phi \in \mathrm{Aut}(A): \Phi(B) = B\}$ in $\mathrm{Aut}(A)$ is also trivial (see Propositions 3 and 4 for much more general situations.)

We shall present now some notations used in this paper. We shall work only with *unital C^*-algebras*. For a compact space X and a C^*-algebra A we shall

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consider the embedding $A \subset C(X, A)$, where each element in A is seen as a constant map on X and also the embedding $C(X) \ni f \to f \otimes 1_A \in C(X) \otimes A = C(X, A)$. We denote by $U(A)$ the unitary group of the C^*-algebra A. By a homomorphism of C^*-algebras we shall mean a unital $*$-homomorphism and by an automorphism of a C^*-algebra, a $*$-automorphism. We denote by $\mathrm{Hom}(A, B)$ the homomorphisms $A \to B$ and by $\mathrm{Aut}(A)$ the automorphisms of A. By M_n we mean the $n \times n$ complex matrices. $K_0(A)$ will denote the K_0-group of the C^*-algebra A and by a trace on A we shall mean a tracial state on A. Let A be a C^*-algebra and let τ be a trace on A. We shall denote by $\|\cdot\|_\tau$ the seminorm on A given by $\|a\|_\tau = \tau(a^*a)^{\frac{1}{2}}$, $a \in A$. When $(x_n)_{n \geqq 1}$ is a sequence in A and $\|x_n - x\|_\tau \to 0$ for some $x \in A$, we shall write $\tau - \lim x_n = x$.

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Results.

We begin with some definitions:

DEFINITION 1. We shall say that a C^*-algebra A is *almost constant* if there is an inductive system $(C(X_n, M_{p(n)}), \Phi_{n,m})$ such that $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ and:

1) each X_n is a compact space

2) for any $n < m$, the homomorphism $\Phi_{n,m}\colon C(X_n, M_{p(n)}) \to C(X_m, M_{p(m)})$ is given by:

$$\Phi_{n,m}(f) = \mathrm{diag}(f \circ \phi_{n,m}^{(1)}, f \circ \phi_{n,m}^{(2)}, \dots, f \circ \phi_{n,m}^{(p(m)/p(n))})$$

for any $f \in C(X_n, M_{p(n)})$, where $\phi_{n,m}^{(i)}\colon X_m \to X_n$ are some continuous maps

(3) for any $n \in N$ we have:

$$\lim_{n \leqq m \to \infty} \frac{\mathrm{card}\{i \mid \phi_{n,m}^{(i)} \text{ is constant}\} \cdot p(n)}{p(m)} = 1$$

EXAMPLES. Let $A = \varinjlim (C(X, M_{p(n)}), \Phi_n)$ be a Goodearl algebra [10] of real rank zero, where X is a compact separable space which is not totally disconnected. Then $A = \varinjlim (C(X, M_{p(n)}), \Phi_n)$ is almost constant (see [10, Theorem 9]). Note also that by Elliott's remarkable classification results from [7], any Bunce-Deddens algebra A [4] can be written as a Goodearl algebra $A = \varinjlim (C(S^1, M_{p(n)}), \Phi_n)$ of real rank zero, where the multiplicity of the identity map in each Φ_n is 1 (use also [10]). Now it is obvious that $A = \varinjlim (C(S^1, M_{p(n)}), \Phi_n)$ is an almost constant C^*-algebra.

DEFINITION 2. We shall say that (A, B) is an *inductive pair* if $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ is an almost constant C^*-algebra and $B \subset A$ is the UHF algebra $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}})$.

DEFINITION 3. Let A be a C^*-algebra. We shall say that $K_0(A)$ is *weakly torsion free* if $nx = ny = [1_A]$ in $K_0(A)$ for some $n \in N$ and $x, y \in K_0(A)_+$ implies $x = y$.

The following Theorem shows that for certain C^*-algebras, any two homomorphisms between them are approximately inner equivalent in a weak sense.

THEOREM 1. *Let (A, B) be an inductive pair and et C be a C^*-algebra with a unique trace τ and such that C has cancellation and $K_0(C)$ is weakly torsion free. If $\Phi, \Psi \in \mathrm{Hom}(A, C)$, then there exists a sequence $(u_n)_{n \geqq 1}$ in $U(C)$ such that:*

$$\Phi(a) = \tau - \lim u_n \Psi(a) u_n^*, \quad a \in A$$

and

$$\Phi(b) = \lim u_n \Psi(b) u_n^*, \quad b \in B.$$

To prove this Theorem we shall need the following two Propositions:

PROPOSITION 1. *Let A be a* UHF *algebra and B a C^*-algebra with cancellation and such that $K_0(B)$ is weakly torsion free. If $\Phi, \Psi \in \mathrm{Hom}(A, B)$, then there is a sequence $(u_n)_{n \geqq 1}$ in $U(B)$ such that:*

$$\Phi(a) = \lim u_n \Psi(a) u_n^*, \quad a \in A.$$

PROOF. Denote $A = \varinjlim (M_{p(n)}, \Phi_n)$, where each homomorphism $\Phi_n \colon M_{p(n)} \to M_{p(n+1)}$ is unital. Let $(e_{ij}^{(n)})_{1 \leqq i,j \leqq p(n)}$ be a system of matrix units for $A_n := M_{p(n)}$. Since:

$$p(n)[\Phi(e_{11}^{(n)})] = p(n)[\Psi(e_{11}^{(n)})] = [1_B]$$

in $K_0(B)$, by hypothesis it follows that:

$$\Phi(e_{11}^{(n)}) = v_n \Psi(e_{11}^{(n)}) v_n^*$$

for some v_n in $U(B)$, $n \geqq 1$. Define $u_n := \sum_{i=1}^{p(n)} \Phi(e_{i1}^{(n)}) v_n \Psi(e_{1i}^{(n)})$, $n \geqq 1$. It is easy to see that each u_n is a unitary in B and:

$$\Phi(x) = u_n \Psi(x) u_n^*, \quad x \in A_n.$$

It follows that:

$$\Phi(a) = \lim u_n \Psi(a) u_n^*, \quad a \in A.$$

REMARK 1. The statement and proof of the above Proposition is a slight variation of a result of Effros and J. Rosenberg ([6, Theorem 3.8]); the argument is apparently due to Elliott. This argument has been used many times since by various authors.

PROPOSITION 2. *Let (A, B) be an inductive pair. Then:*
a) *A has a unique trace τ.*
b) *B is dense in A with respect to the seminorm $\|\cdot\|_\tau$.*
c) *If A has trivial center then $B' \cap A = \mathbf{C} \cdot 1_A$.*

PROOF. a) and b). Assume that $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ and $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}}) (\subset A)$ where:

$$\Phi_{n,m}(f) = \operatorname{diag}(f \circ \phi_{n,m}^{(1)}, \ldots, f \circ \phi_{n,m}^{(p(m)/p(n))}), f \in C(X_n, M_{p(n)})$$

for some continuous maps $\phi_{n,m}^{(i)}: X_m \to X_n$, and:

$$\lim_{n \leqq m \to \infty} \frac{\operatorname{card}\{i \mid \phi_{n,m}^{(i)} \text{ is constant}\}\, p(n)}{p(m)} = 1$$

for any $n \in N$.

Denote $A_n := C(X_n, M_{p(n)}), n \geqq 1$. Fix $0 \neq a \in A_n$ for some n. Then, for any $k \in N$ there is $m(k) = m(k, n) > n$ such that:

$$\frac{\operatorname{card}\{i \mid \phi_{n,m(k)}^{(i)} \text{ is not constant}\}\, p(n)}{p(m(k))} \leqq 2^{-k} \|a\|^{-2}.$$

Let $I_k := \{i \mid \phi_{n,m(k)}^{(i)}$ is not constant$\}$. Define $b_k \in A_{m(k)}$ to be the element of $A_{m(k)}$ obtained replacing in $\Phi_{n,m(k)}(a)$ the block $a \circ \phi_{n,m(k)}^{(i)}$ with 0 only for $i \in I_k$. Denote by $\mu_k: A_k \to A$ the canonical homomorphisms. It is obvious that $(\mu_{m(k)}(b_k))_{k \geqq 1}$ is a sequence in B.

Now, let σ be an *arbitrary* trace of A. Denote $\sigma_{m(k)} = \sigma \circ \mu_{m(k)}$. We have:

$$\begin{aligned}
\|\mu_n(a) - \mu_{m(k)}(b_k)\|_\sigma^2 &= \|\Phi_{n,m(k)}(a) - b_k\|_{\sigma_{m(k)}}^2 \\
&\leqq \sum_{i \in I_k} \|a \circ \phi_{n,m(k)}^{(i)}\|^2 \cdot \frac{p(n)}{p(m(k))} \\
&\leqq (\operatorname{card} I_k) \cdot \|a\|^2 \frac{p(n)}{p(m(k))} \leqq \frac{p(m(k))}{p(n)} 2^{-k} \cdot \|a\|^{-2} \cdot \|a\|^2 \cdot \frac{p(n)}{p(m(k))} \\
&= 2^{-k}, \quad k \geqq 1.
\end{aligned}$$

Hence, we have:

$$\sigma(\mu_n(a)) = \lim_{k \to \infty} \sigma(\mu_{m(k)}(b_k)) = \lim_{k \to \infty} \lambda(\mu_{m(k)}(b_k))$$

where λ is the unique trace of the UHF algebra B. Since $\bigcup_{k \geqq 1} \mu_k(A_k)$ is dense in A in the C^*-algebra norm, it follows that A has a unique trace and B is dense in A with respect to the trace seminorm.

c) follows from the following more general fact: Let C be a C^*-algebra with trivial center and let σ be a trace of C. Consider a unital C^*-subalgebra D of C which is dense in C with respect to $\|\cdot\|_\sigma$. Then $D' \cap C$ is trivial.

The proof is similar with that of [13, Corollary 3] and will be not given.

PROOF OF THEOREM 1. By Proposition 1 it follows that there is a sequence $(u_n)_{n \geqq 1}$ in $U(C)$ such that

$$\Phi(b) = \lim u_n \Psi(b) u_n^*, \quad b \in B.$$

Since A has a unique trace denoted σ (see Proposition 2a)), we have:

$$\|\Phi(a)\|_\tau = \|a\|_\sigma = \|\Psi(a)\|_\tau, \quad a \in A.$$

Define $\Psi_n: A \to C$ by $\Psi_n(a) := u_n \Psi(a) u_n^*$, $a \in A (n \geqq 1)$. Then $\Psi_n, \Phi: (A, \|\cdot\|_\sigma) \to (C, \|\cdot\|_\tau)$ are linear and continuous. Since $\|\Psi_n(b) - \Phi(b)\|_\tau \to 0$, $b \in B$ and $\|\Psi_n(a)\|_\tau = \|a\|_\sigma$, $a \in A$, by Proposition 2b) we deduce that $\|\Psi_n(a) - \Phi(a)\|_\tau \to 0$, $a \in A$.

REMARK 2. Let (A, B) be an inductive pair and suppose that A has cancellation and $K_0(A)$ is weakly torsion free. Denote by τ the trace of A. Then, if Φ is an endomorphism of A, the above Theorem implies that there is a sequence $(u_n)_{n \geqq 1}$ in $U(A)$ such that:

$$\Phi(a) = \tau - \lim u_n a u_n^*, \quad a \in A$$

and

$$\Phi(b) = \lim u_n b u_n^*, \quad b \in B.$$

Note that Φ isn't approximately inner in general; indeed, e.g. by [13, Proposition 3] there are automorphisms of Bunce-Deddens algebras which don't induce the identity of the K_1-group (see also [1, problem 10.11.5 (b)] and [12]]).

Now we are interested to find necessary and sufficient conditions under which an automorphism of the "canonical" UHF subalgebra of an almost constant C^*-algebra can be extended to an automorphism of the whole C^*-algebra.

NOTATION. Let A be a C^*-algebra with a unique trace τ, which is faithful. We shall denote by $L^2(A)$ the completion of A with respect to the norm $\|\cdot\|_\tau$. The induced norm on $L^2(A)$ will be also denoted by $\|\cdot\|_\tau$. If $(x_n)_{n \geqq 1}$ is a sequence in $(L^2(A), \|\cdot\|_\tau)$ we shall denote by $\tau - \lim x_n \in L^2(A)$ the corresponding limit.

THEOREM 2. *Let (A, B) be an inductive pair, where $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ as in Definition 1, $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}})$ and suppose that the trace τ of A is* *faithful. Consider* $\Phi \in \mathrm{Aut}(B)$ *and let* $(u_n)_{n \geqq 1} \subset U(B)$ *such that* $\Phi(x) = \lim u_n x u_n^*$, $x \in B$. *Denote by* $\mu_n\colon C(X_n, \mathrm{M}_{p(n)}) \to A$ *the canonical homomorphisms. Then the following two conditions are equivalent:*

a) Φ *extends to an automorphism of* A

b) $\tau - \lim u_n f u_n^*$ *and* $\tau - \lim u_n^* f u_n$ *exist in* A *for any* $f \in \bigcup_{m=1}^{\infty} \mu_m(C(X_m))$.

Moreover, when Φ *extends, it has a unique extension* $\tilde{\Phi} \in \mathrm{Aut}(A)$, *where*

$$\tilde{\Phi}(x) = \tau - \lim u_n x u_n^*$$

and

$$\tilde{\Phi}^{-1}(x) = \tau - \lim u_n^* x u_n$$

for any $x \in A$.

The following rigidity result will be needed to prove the above Theorem:

LEMMA 1. *Let* (A, B) *be an inductive pair and* C *a* C^**-algebra with a unique trace. If* $\Phi, \Psi \in \mathrm{Hom}(A, C)$ *and* $\Phi_{|B} = \Psi_{|B}$ *then* $\Phi = \Psi$.

PROOF. In fact the folowing more general result is true:

Let M be a C^*-algebra with a unique trace τ and N a unital C^*-subalgebra of M such that N is dense in M with respect to $\|\cdot\|_\tau$. If P is a C^*-algebra with a unique trace, $\Phi, \Psi \in \mathrm{Hom}(M, P)$ and $\Phi_{|N} = \Psi_{|N}$ then $\Phi = \Psi$.

The proof of this fact is immediate and similar with that of [13, Lemma 2] and therefore will not be given.

PROOF OF THEOREM 2. This proof is inspired by that of [13, Theorem 2].

First of all observe that the unicity of the extension (when it exists), follows from the above Lemma.

a) $\Rightarrow$ b). Consider $\tilde{\Phi} \in \mathrm{Aut}(A)$ such that $\tilde{\Phi}_{|B} = \Phi$. By the proof of Theorem 1 and the above observation, we obtain:

$$\tilde{\Phi}(x) = \tau - \lim u_n x u_n^*, \quad x \in A.$$

It follows that $\tau - \lim u_n f u_n^* = \tilde{\Phi}(f) \in A$ for any $f \in \bigcup_{m=1}^{\infty} \mu_m(C(X_m))$. Working with Φ^{-1} we obtain the other relations.

b) $\Rightarrow$ a). Recall that by $L^2(A)$ (resp. $L^2(B)$) we mean the completion of A (resp. B) with respect to the trace norm $\|\cdot\|_\tau$ (resp. $\|\cdot\|_\sigma$), where $\sigma = \tau_{|B}$ is the trace of B. Observe that by Proposition 2b) we have $L^2(A) = L^2(B)$.

If B is seen in its GNS representation in $\mathscr{B}(L^2(B))$ associated with σ, we have:

$$\Phi(x) = U x U^*, \quad x \in B.$$

Here $U \in U(\mathscr{B}(L^2(B)))$ and $U(b) := \Phi(b), \quad b \in B$.

Since by a previous observation we have also $U \in U(\mathscr{B}(L^2(A)))$, one can define $\tilde{\Phi} \in \mathrm{Hom}(A, \mathscr{B}(L^2(A)))$ by:

$$\tilde{\Phi}(x) = UxU^*, \quad x \in A,$$

where A is seen in its GNS representation in $\mathscr{B}(L^2(A))$ associated with τ. It is clear that $\tilde{\Phi}_{|B} = \Phi$.

Consider an arbitrary element f in $\bigcup_{m=1}^{\infty} \mu_m(C(X_m))$. By a version of Kaplansky's Density Theorem (use a slight modification of the proof of [11, Lemma 3.11]) there is a sequence $(b_k)_{k \geqq 1}$ in B such that $\|b_k - f\|_\tau \to 0$ and $\|b_k\| \leqq \|f\|$. Since $x_n \xrightarrow{\|\cdot\|_\tau} 0$ in A means $x_n \xrightarrow{\text{so}} 0$ in $\mathscr{B}(L^2(A))$ when $\{\|x_n\|\}$ is bounded, we have:

$$\tilde{\Phi}(f) = UfU^* = \text{so-lim}\, Ub_kU^* = \text{so-lim}\, \Phi(b_k)$$

$$= \text{so-lim}_k\, (\text{so-lim}_n\, u_n b_k u_n^*)$$

It is not difficult to see that $\tau - \lim u_n x u_n^*$ exists in $L^2(A)$ for any $x \in A$ (indeed, the limit exists for all $x \in B$, by Proposition 2b) B is dense in A with respect to $\|\cdot\|_\tau$ and $\|u_n x u_n^*\|_\tau = \|x\|_\tau$ for any $x \in A$ and $n \in N$). Hence:

$$\|\tau - \lim_n u_n b_k u_n^* - \tau - \lim_n u_n f u_n^*\|_\tau = \lim_n \|u_n(b_k - f)u_n^*\|_\tau = \|b_k - f\|_\tau$$

which implies that in $L^2(A)$ one has:

$$\tau - \lim_k (\tau - \lim_n u_n b_k u_n^*) = \tau - \lim_n u_n f u_n^*.$$

Since $\tau - \lim u_n f u_n^* \in A$ by hypothesis and $\|x_n\|_\tau \to 0$ in A means $x_n \xrightarrow{\text{so}} 0$ in $\mathscr{B}(L^2(A))$ if $\{\|x_n\|\}$ is bounded, we can write:

$$\tilde{\Phi}(f) = \text{so-lim}_k\, (\text{so-lim}_n\, u_n b_k u_n^*)$$

$$= \text{so-lim}_n\, u_n f u_n^* \in A$$

But A is the C^*-algebra generated by B and $\bigcup_{m=1}^{\infty} \mu_m(C(X_m))$ and we already knew that $\tilde{\Phi}$ belongs to $\mathrm{Hom}(A, \mathscr{B}(L^2(A))$ and $\tilde{\Phi}(B) = B$. It follows that $\tilde{\Phi}(A) \subset A$ and as in the proof of a) $\Rightarrow$ b) one obtains:

$$\tilde{\Phi}(x) = \tau - \lim_n u_n x u_n^*, \quad x \in A.$$

Repeating the above argument for Φ^{-1}, where $\Phi^{-1}(x) = \lim u_n^* x u_n$, $x \in B$, finally we get $\tilde{\Phi}^{-1} \in \mathrm{Aut}(A)$.

The following two Propositions give, in particular, additional informations about $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$, where (A, B) is an inductive pair. Their proofs are similar with those of [13, Proposition 4 and Proposition 5] and therefore will be not given.

PROPOSITION 3. *Let A be a C^*-algebra and let B be a* UHF *algebra which is a unital C^*-subalgebra of A. Then the centralizer of $\{\Phi \in \mathrm{Aut}(B): \Phi = \tilde{\Phi}_{|B}$ for some $\tilde{\Phi} \in \mathrm{Aut}(A)\}$ in Aut(B) is trivial.*

PROPOSITION 4. *Let A and B be as in the above Proposition. Suppose moreover that:*

a) *the center of A is trivial.*

b) *A has a unique trace τ.*

c) *B is dense in A with respect to the trace seminorm $\|\cdot\|_\tau$.*

Then the centralizer of $\{\Phi \in \mathrm{Aut}(A): \Phi(B) = B\}$ in $\mathrm{Aut}(A)$ is trivial.

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