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WHEN DOES bvca(Σ, X) CONTAIN A COPY OF l_∞?

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Abstract.

Let X be a Banach space and let Σ be a σ-algebra of subsets of a set Ω. Denoting by bvca(E, X) the Banach space of all X-valued countably additive measures of bounded variation defined on Σ endowed with the variation norm, we show that if X has the Radon-Nikodym property with respect to each element of ca$^+(\Sigma)$, bvca(Σ, X) has acopy of l_∞ if and only if X does.

Our notation is standard (see [1] and [2]). In what follows X will be a Banach space, Ω a set and Σ a σ-algebra of subsets of Ω. We denote by ca(Σ, X) the Banach space of all X-valued countable additive measures equipped with the semivariation norm. By ca(Σ) we denote the Banach space of all scalar countably additive measures defined on Σ provided with the variation norm, whose positive members we denote by ca$^+(\Sigma)$. If $\mu \in$ ca$^+(\Sigma)$ and $1 \leqq p < \infty$. $L_p(\mu, X)$ will stand for the Banach space of all (classes of) X-valued Bochner p-integrable functions defined on Ω equipped with their usual norms. By bvca(Σ, X) we denote the Banach space of all X-valued countably additive measures F of bounded variation defined on Σ with the variation norm $|F| = |F|(\Omega)$.

If $\mu \in$ ca$^+(\Sigma)$ we denote by bvca(Σ, μ, X) the linear subspace of bvca(Σ, X) formed by all those measures that are μ-continuous. The linear operator $T\colon L_1(\mu, X) \to$ bvca(Σ, μ, X) defined by $(Tf)(E) = \int_E f\,d\mu$ (integral of Bochner) for all $E \in \Sigma$, is an isometry onto if and only if X has the Radon-Nikodym property with respect to μ.

The main aim of this note is to demonstrate the theorem below. Our proof is based upon the proofs of Lemma 4 and Theorem 2 of [4].

THEOREM. *If X has the Radon-Nikodym property with respect to each $\mu \in$ ca$^+(\Sigma)$, then the space* bvca(Σ, X) *contains an isomorphic copy of l_∞ if and only if X does.*

This result can be aligned with those given in [3] and [4], where conditions are

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imposed on the spaces $\mathrm{cca}(\Sigma, X)$ and $\mathrm{ca}(\Sigma, X)$ in order to ensure that they contain copies of l_∞ and c_0. We shall need the following lemmas.

LEMMA 1 (Rosenthal, [7]). *Let T be a bounded linear operator from l_∞ into X. If X does not contain a copy of l_∞, T is weakly compact.*

LEMMA 2 (Mendoza, [6]). *If $1 \leqq p < \infty$, then $L_p(\mu, X)$ contains an isomorphic copy of l_∞ if and only if X does.*

PROOF OF THE THEOREM. The *if* part of the theorem is trivial. So we concentrate on the converse. Suppose that $\mathrm{bvca}(\Sigma, X)$ contains a copy of l_∞. Let J denote a canonical isomorphism from l_∞ into $\mathrm{bvca}(\Sigma, X)$ and let (e_n) be the unit vector basis of c_0. We are going to define some useful linear operators (these definitions have been taken from [4, Lemma 4]).

If X does not contain any copy of l_∞, then for each E in Σ the operator $J_E\colon l_\infty \to X$, defined by $J_E(\xi) = (J\xi)(E)\ \forall \xi \in l_\infty$, is weakly compact by Lemma 1, and therefore the series $\sum_n J_E e_n$ is unconditionally convergent. Hence the operator $T\colon l_\infty \to \mathrm{ba}(\Sigma, X)$ defined by $T\xi(E) = \sum_n \xi_n J_E e_n$ for each $\xi \in l_\infty$ and each $E \in \Sigma$ is well-defined. It is bounded too, since if $\{E_i, 1 \leqq i \leqq n\}$ is a partition of Ω by elements of Σ and we write $\xi^n := (\xi_1, \dots, \xi_n, 0, 0, \dots, 0, \dots)$, then

$$\sum_{i=1}^{n} \|T\xi(E_i)\| = \sum_{i=1}^{n} \|\textstyle\sum_j \xi_j J_{E_i} e_j\| = \sum_{i=1}^{n} \lim_k \|J_{E_i}\xi^k\| = \lim_k \sum_{i=1}^{n} \|J\xi^k(E_i)\|$$

$$\leqq \sup_k \sum_{i=1}^{n} \|J\xi^k(E_i)\| \leqq \sup_k |J\xi^k| \leqq \sup_k \|J\|\,\|\xi^k\|_\infty \leqq \|J\|\,\|\xi\|_\infty$$

Thus $T\xi$ is of bounded variation for each $\xi \in l_\infty$, and $\|T\| \leqq \|J\|$. So setting $v := \sum_n 2^{-n} |Je_n|$, clearly $Je_n \in \mathrm{bvca}(\Sigma, v, X)$ for each $n \in \mathbb{N}$, and hence $J\xi^n \ll v$ for each $\xi \in l_\infty$ and each $n \in \mathbb{N}$. Now, since $\lim_n J\xi^n(E) (= T\xi(E) \in X)$ exists $\forall \xi \in l_\infty$ and $\forall E \in \Sigma$, the Vitali-Hahn-Saks theorem guarantees that $T\xi \ll v$ for each $\xi \in l_\infty$. Therefore $T(l_\infty) \subseteq \mathrm{bvca}(\Sigma, v, X)$.

As $Te_n(E) = J_E e_n = Je_n(E)$ for each $E \in \Sigma$ and each $n \in \mathbb{N}$, then $Te_n = Je_n$ for each $n \in \mathbb{N}$ and hence $\inf_n \|Te_n\| > 0$. Now a well-known theorem of Rosenthal ([17]) assures that there is some $\mathbb{M} \subseteq \mathbb{N}$ with $\mathrm{card}\, \mathbb{M} = \aleph_0$ such that the restriction of T to $l_\infty(\mathbb{M})$ is an isomorphism. So the space $\mathrm{bvca}(\Sigma, v, X)$ has a copy of l_∞. But $\mathrm{bvca}(\Sigma, v, X)$ is isometric to $L_1(v, X)$, since X has the Radon-Nikodym property with respect to the positive measure v. Hence Lemma 2 applies to get the contradiction.

COROLLARY 1. *If Σ is such that each $\mu \in \mathrm{ca}^+(\Sigma)$ is purely atomic, then* $\mathrm{bvca}(\Sigma, X)$ *has a copy of l_∞ if and only if X does.*

REMARK. If X has the Radon-Nikodym property with respect to each μ in $\mathrm{ca}^+(\Sigma)$ and bvca(Σ, X) has a copy L of c_0, then there clearly exists a $\mu \in \mathrm{ca}^+(\Sigma)$ so that $L \subseteq \mathrm{bvca}(\Sigma, \mu, X)$. According to a well-known theorem of Kwapien, [5], X must contain a copy of c_0.

COROLLARY 2. *If X is a reflexive Banach space, then* bvca(Σ, X) *does not contain any copy of* c_0.

PROOF. Since X is reflexive, X has the Radon-Nikodym property and hence does not contain any copy of c_0. Therefore bvca(Σ, X) cannot have a copy of c_0.

PROPOSITION. *Suppose that the σ-algebra Σ is countably generated and that* bvca(Σ, μ, Y) *is separable whenever Y is a closed separable subspace of X and* $\mu \in \mathrm{ca}^+(\Sigma)$. *Then* bvca$(\Sigma, X)$ *contains a copy of l_∞ if and only if X does.*

PROOF. Let J be an isomorphism from l_∞ into bvca(Σ, X) and assume X has not any copy of l_∞. Then define for each $E \in \Sigma$ the weakly compact linear operator J_E as in the Theorem and denote by Z the closed linear hull of $\{J_E e_n, n \in \mathrm{N}, E \in \mathscr{A}\}$, where $\mathscr{A}$ denotes a sequence of elements of Σ containing Ω which generates Σ. Obviously, Z is a separable Banach space.

Next we shall see that $J_E e_n \in Z \; \forall E \in \Sigma$. In fact, given a family $\mathscr{B}$ of elements of Σ, denote by $\mathscr{B}^*$ the family of all countable unions of sets of $\mathscr{B}$ and all the complementary sets of sets of $\mathscr{B}$. Let ω be the first ordinal of uncountable cardinal. Set $\Sigma_0 = \mathscr{A}$ and for each ordinal α with $0 < \alpha < \omega$ define $\Sigma_\alpha = \{\cup \{\Sigma_\beta, \beta < \alpha\}\}^*$. Note that $\Sigma_\beta \subseteq \Sigma_\alpha \; \forall \beta \leqq \alpha$ and $\Sigma = \cup \{\Sigma_\alpha, \alpha < \omega\}$. We shall proceed by transfinite induction.

We know that $J_E e_n \in Z$ for each $E \in \Sigma_0$. Suppose that, if $0 < \alpha < \omega$, $J_E e_n \in Z$ for each $E \in \Sigma_\beta$ with $\beta < \alpha$. As $\Sigma_\alpha = \{\cup \{\Sigma_\beta, \beta < \alpha\}\}^*$, choosing $E = \cup \{E_k, k \in \mathrm{N}\}$ with $E_k \in \Sigma_{\beta_k}$ and $\beta_k < \alpha$ for each k, then one has that $J_E e_n = Je_n(E) = \lim_k Je_n\left(\bigcup_{j=1}^{k} E_j\right) \in Z$. On the other hand, if $E \in \Sigma_\beta$ with $\beta < \alpha$, then $J_{\Omega \setminus E} e_n = Je_n(\Omega \setminus E) = Je_n(\Omega) - Je_n(E) \in Z$.

Using the same notation of the Theorem, since $T\xi(E) = \sum_n \xi_n J_E e_n$ for each $\xi \in l_\infty$ and $E \in \Sigma$, and since all $J_E e_n \in Z$ as we have just seen, we have $T\xi(E) \in Z$ for each $\xi \in l_\infty$ and each $E \in \Sigma$. Besides, reasoning as in the last part of the proof of the Theorem, there exists a $\mu \in \mathrm{ca}^+(\Sigma)$ such that $T\xi << \mu$ for each $\xi \in l_\infty$. This shows that T is a bounded linear operator of l_∞ into bvca(Σ, μ, Z). Thus Rosenthal's theorem implies that bvca(Σ, μ, Z) has a copy of l_∞. But we suppose that bvca(Σ, μ, Z) is separable, a contradiction.

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