MATH. SCAND. 74 (1994), 275–292

AMENABILITY IN GROUP ALGEBRAS AND BANACH ALGEBRAS

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0. Introduction.

In this paper, we will examine some aspects of the relationship between amenability in groups and amenability in Banach algebras. While leaving specific characterizations of these properties for the next section, we have the following results from [13].

0.1. Proposition. *A locally compact group G is amenable if and only if $L^1(G)$ is amenable.*

0.2. Proposition. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras and $v\colon \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism with range dense in $\mathfrak{B}$. If $\mathfrak{B}$ is amenable then $\mathfrak{B}$ is amenable.*

0.2.1. Corollary. *If X is a locally compact Hausdorff topological space, then $C_0(X)$ is amenable.*

0.2.2. Corollary. *If H is a Hilbert space then $\mathscr{K}(H)$ is amenable.*

Each of these Corollaries is proven by constructing an amenable locally compact group G and a dense-ranged continuous homomorphism $L^1(G) \to \mathfrak{A}$. It is natural to ask whether *any* amenable Banach algebra $\mathfrak{A}$ can be shown to be amenable by a similar construction. It will be shown in Sections 2 and 3 that this is not the case, and for certain classes of Banach algebras we will develop some necessary and sufficient conditions for there to exist such a homomorphism.

The research presented in this paper was undertaken for the degree of Doctor of Philosophy at the Australian National University, and I would like to thank my supervisors Dr R. J. Loy and Dr G. A. Willis for their encouragement and suggestions. Many thanks also to Professor B. E. Johson for his proof of Proposition 2.3, Professors U. Haagerup and P. C. Curtis for the suggestion

Received April 13, 1994.

leading to the results in Section 3, and to Dr N. Grønbæk for the suggestion that lead to the results in Section 4.

1. Notation and Preliminary Results.

Throughout, $G, G_1, \ldots$ will denote locally compact groups, each represented multiplicatively with unit denoted e. If H is a closed normal subgroup of G, T_H will denote the epimorphism $L^1(G) \to L^1(G/H)$ as described by Reiter in [20, 3.3.2–3.5.3]. When G is Abelian, its dual group will be denoted Γ, and this will be identified with $\Phi_{L^1(G)}$, the maximal ideal space of the group algebra $L^1(G)$. If $\mathfrak{A}$ is a commutative semisimple Banach algebra, the *hull* of a set $X \subseteq \mathfrak{A}$ is $Z(X) = \{\varphi \in \Phi_{\mathfrak{A}}: \varphi(X) = 0\}$ and the *kernel* of a set $S \subseteq \Phi_{\mathfrak{A}}$ is $\mathscr{I}(S) = \{a \in \mathfrak{A}: \varphi(a) = 0, (\varphi \in S)\}$.

Let $C_b(G)$ be the space of continuous bounded functions on G, then $M \in C_b(G)^*$ is called a *mean* when $\inf f(G) \leqq M(f) \leqq \sup f(G)$ for each $f \in C_b(G)$ with rng $f \subseteq \mathsf{R}$. A group G is *amenable* if there exists a mean M on $C_b(G)$ that is left-invariant, that is, $M(f) = M({}_xf)$, $(f \in C_b(G), x \in G)$. Locally compact groups that are Abelian or compact are amenable. There are many equivalent characterizations to the above, including the existence of left and/or right-invariant means on other function spaces on G, such as $L^\infty(G)$, $UC(G)$, $UC_r(G), \ldots$ and a variety of structural conditions on G, referred to as Følner conditions. See [10] or [17] for more on invariant means and Følner conditions.

Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ (defined as in [3, Chapter 6]) be endowed with its canonical structure as a Banach $\mathfrak{A}$-bimodule, given by $(a \otimes b) \cdot c = a \otimes (bc)$ and $a \cdot (b \otimes c) = (ab) \otimes c$. Let π be the mapping $\mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$ given by extending $a \otimes b \mapsto ab$ by linearity and continuity. An *approximate diagonal* for $\mathfrak{A}$ is a bounded net $\{d_n\}_{n \in \Delta} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that for each $a \in A$, $\pi(d_n)a \to a$, and $d_n \cdot a - a \cdot d_n \to 0$. If an approximate diagonal exists, then we say that $\mathfrak{A}$ is *amenable*. Again, there are other characterizations equivalent to this, such as the condition that every derivation from $\mathfrak{A}$ into a dual Banach $\mathfrak{A}$-bimodule is inner (see [3, Theorem 43.9]), or that $\mathfrak{A}$ has bounded approximate identity and $\ker \pi$, when considered as an ideal of the algebra $\mathfrak{A} \hat{\otimes} \mathfrak{A}^{\mathrm{op}}$, has a bounded approximate identity (see [8]).

We say a Banach algebra $\mathfrak{A}$ has *property* **(G)** if there exists an amenable locally compact group G and a continuous homomorphism ν: $L^1(G) \to \mathfrak{A}$ with range dense in $\mathfrak{A}$. Then as noted in the introduction, a Banach algebra with property **(G)** is amenable.

We present some basic results on property **(G)** which will aid us in later sections, when we will characterize property **(G)** for certain Banach algebras $\mathfrak{A}$.

1.1. PROPOSITION. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras with property* **(G)**, *then $\mathfrak{A} \oplus \mathfrak{B}$ and $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ have property* **(G)**.

PROOF. By hypothesis, there exist amenable locally compact groups G_1 and G_2 and continuous homomorphisms $\nu_1: L^1(G_1) \to \mathfrak{A}$ and $\nu_2: L^1(G_2) \to \mathfrak{B}$ with $\overline{\operatorname{rng} \nu_1} = \mathfrak{A}$ and $\overline{\operatorname{rng} \nu_2} = \mathfrak{B}$. Then the continuous homomorphisms $\nu_1 \oplus \nu_2$: $L^1(G_1) \oplus L^1(G_2) \to \mathfrak{A} \oplus \mathfrak{B}$ and $\nu_1 \otimes \nu_2: L^1(G_1) \hat{\otimes} L^1(G_2) \to \mathfrak{A} \hat{\otimes} \mathfrak{B}$ have dense range, so it suffices to show that $L^1(G_1) \oplus L^1(G_2)$ and $L^1(G_1) \hat{\otimes} L^1(G_2)$ have property (G). For this, note that the groups $G_1 \times G_2$ and $G_1 \times G_2 \times \mathbf{Z}_2$ are amenable with

$$L^1(G_1 \times G_2) \cong L^1(G_1) \hat{\otimes} L^1(G_2),$$
$$L^1(G_1 \times G_2 \times \mathbf{Z}_2) \cong L^1(G_1 \times G_2) \hat{\otimes} \mathbf{C}^2$$
$$\cong L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2),$$

and $T_{G_2} \oplus T_{G_1}: L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2) \to L^1(G_1) \oplus L^1(G_2)$ is an epimorphism.

1.2. PROPOSITION. *Suppose G is a locally compact group and ν: $L^1(G) \to \mathfrak{A}$ is a continuous homomorphism into a commutative Banach algebra, then there is a locally compact Abelian group G' and a continuous homomorphism ν': $L^1(G') \to$ with* $\operatorname{rng} \nu = \operatorname{rng} \nu'$.

PROOF. Let C be the closure of the commutator subgroup of G, then by [12, Theorem 23.8, G/C is Abelian. We show firstly that the kernel of T_C: $L^1(G) \to L^1(G/C)$ is $\mathscr{J}$, the commutator ideal of $L^1(G)$. The inclusion $\mathscr{J} \subseteq \ker T_C$ follows from the observation that T_C is a continuous homomorphism into a commutative Banach algebra.

Conversely, by [20, 3.6.4], we have $\ker T_C = \overline{\operatorname{span}}\{f - {}_xf: x \in C, f \in L^1(G)\}$. Put $H = \{x \in G: f - {}_xf \in \mathscr{J}, (f \in L^1(G))\}$, a closed subgroup of G, and let $\{e_n\}_{n \in \Delta}$ be a bounded approximate identity for $L^1(G)$. For each $x, y \in G$ and each $f \in L^1(G)$,

$$\|{}_{yx}f - {}_ye_n * {}_xe_n * f\| \leqq \|{}_y({}_xf - e_n * {}_xf)\| + \|{}_ye_n * {}_x(f - e_n * f)\|$$
$$\leqq \|{}_xf - e_n * {}_xf\| + \|e_n\| \, \|f - e_n * f\|$$
$$\to 0,$$

so that $({}_ye_n * {}_xe_n - {}_xe_n * {}_ye_n) * f \to {}_{yx}f - {}_{xy}f$. Furthermore, $({}_ye_n * {}_xe_n - {}_xe_n * {}_ye_n) * f \in \mathscr{J}$, which is closed and translation-invariant, so that ${}_{x^{-1}y^{-1}xy}f - f \in \mathscr{J}$ and $x^{-1}y^{-1}xy \in H$. But C is the closed subgroup generated by $\{x^{-1}y^{-1}xy: x, y \in G\}$, so $C \subseteq H$. Hence $\ker T_C \subseteq \mathscr{J}$.

Now, since $\mathfrak{A}$ is commutative, $\ker T_C \subseteq \ker \nu$, and so $\nu \circ T_C^{-1}: L^1(G/C) \to \mathfrak{A}$ defines a continuous algebra homomorphism, as required.

2. Subalgebras of Commutative Group Algebras.

In this section we will examine closed subalgebras of commutative group algebras, which we call *group subalgebras*, and for certain classes of these, develop necessary and sufficient conditions for property (**G**).

By Proposition 1.2, a subalgebra $\mathfrak{A}$ of $L^1(G)$ has property (**G**) if and only if there is a locally compact Abelian group G' and a continuous homomorphism $\nu\colon L^1(G') \to L^1(G)$ with $\mathfrak{A} = \overline{\operatorname{rng} \nu}$. Thus we can use the Theorem of Cohen, [6, Theorem 1], which characterizes homomorphisms between commutative group algebras. For this we define the terms *coset ring*, *affine*, *piecewise affine* and *proper*. The *coset ring* of a locally compact Abelian group Γ, denoted $\mathscr{R}(\Gamma)$, is the Boolean ring generated by the open cosets in Γ. If $E \subseteq \Gamma$, a map $\psi\colon E \to \Gamma'$ is *affine* if for any $\gamma_1, \gamma_2, \gamma_3 \in E$, $\psi(\gamma_1\gamma_2^{-1}\gamma_3) = \psi(\gamma_1)\psi(\gamma_2)^{-1}\psi(\gamma_3)$. If $E \subseteq \Gamma$, a map $\psi\colon S \to \Gamma'$ is *piecewise affine* if there exist disjoint $S_1, \ldots, S_n \in \mathscr{R}(\Gamma)$ such that $S = S_1 \cup \ldots \cup S_n$ and for each $1 \leqq k \leqq n, \alpha|_{S_k}$ has a continuous affine extension $\alpha_k\colon E_k \to \Gamma'$. (Here it is understood that E_k is a coset containing S_k.) If X and Y are locally compact topological spaces then a map $\psi\colon X \to Y$ is *proper* if for any compact $C \subseteq Y$, $\psi^{-1}(C)$ is compact.

Then with $\nu\colon L^1(G') \to L^1(G)$ as above, it follows from [6, Theorem 1] that $Y = \{\gamma \in \Gamma\colon \nu^*(\gamma) \neq 0\} \in \mathscr{R}(\Gamma)$ and $\alpha = \nu^*|_Y$ is a proper piecewise affine map. Furthermore, any such α uniquely determines a homomorphism ν by the relations $\widehat{\nu(f)}(\gamma) = \hat{f} \circ \alpha(\gamma)$ if $\gamma \in Y$ and $\widehat{\nu(f)}(\gamma) = 0$ if $\gamma \notin Y$.

In the paper [14], it was shown that for such a homomorphism, $Y = \Gamma \backslash Z(\operatorname{rng} \nu)$ and $\operatorname{rng} \nu = \kappa(\alpha)$, where

$$\kappa(\alpha) = \{f \in L^1(G)\colon \hat{f} = 0 \text{ off } Y \text{ and } \hat{f}(\gamma_1) = \hat{f}(\gamma_2) \text{ whenever } \alpha(\gamma_1) = \alpha(\gamma_2)\},$$

which is closed. Thus we can classify the group subalgebras with property (**G**) as follows.

2.1. Proposition. *If G is a locally compact Abelian group and $\mathfrak{A}$ is a closed subalgebra of $L^1(G)$, then $\mathfrak{A}$ has property* (**G**) *if and only if*

(i) *$Y = \Gamma \backslash Z(\mathfrak{A}) \in \mathscr{R}(\Gamma)$, and*

(ii) *there is a locally compact Abelian group Γ' and a proper piecewise affine map $\alpha\colon Y \to \Gamma'$ with $\mathfrak{A} = \kappa(\alpha)$.*

We now consider specific classes of group subalgebras and develop necessary and sufficient conditions for amenability and property (**G**). The simplest such class consists of the closed ideals of commutative group algebras. For this, define the *discrete coset ring* of a locally compact Abelian group Γ to be $\mathscr{R}(\Gamma_d)$, the coset ring of Γ with its discrete topology. We denote this $\mathscr{R}_d(\Gamma)$, and it is the Boolean ring generated by *all* cosets in Γ.

We will, in fact, mainly be interested in sets in

$$\mathscr{R}_c(\Gamma) = \{X \in \mathscr{R}_d(\Gamma): X \text{ is a closed subset of } \Gamma\},$$

as these are the only sets in $\mathscr{R}_d(\Gamma)$ that can be hulls of ideals. The fact that an ideal has hull in $\mathscr{R}_c(\Gamma)$ if and only if it has bounded approximate identity is vital in the next theorem and subsequent results.

2.2. THEOREM. *Let $\mathscr{I}$ is a closed ideal of $L^1(G)$, and put $E = Z(\mathscr{I})$. Then $\mathscr{I}$ is amenable if and only if $E \in \mathscr{R}_c(\Gamma)$, whereas $\mathscr{I}$ has property* (**G**) *if and only if $E \in \mathscr{R}(\Gamma)$. In either case, $\mathscr{I} = \mathscr{I}(E)$.*

PROOF. The first part of this is [16, Theorem 1]. For the second, we have by Proposition 2.1 that if $\mathscr{I}$ is an ideal with property (**G**), then $Y = \Gamma\backslash E \in \mathscr{R}(\Gamma)$, and so $E \in \mathscr{R}(\Gamma)$. Conversely, if $E = Z(\mathscr{I}) \in \mathscr{R}(\Gamma)$, then E is clopen, and consequently of synthesis, so that $\mathscr{I} = \mathscr{I}(E)$. Moreover, if we define $\alpha: Y \to \Gamma$ to be the inclusion mapping, then α is a proper piecewise affine map with $\kappa(\alpha) = \mathscr{I}(E)$, so by the above discussion, $\mathscr{I}$ has property (**G**).

REMARK. In the above proof, the epimorphism $\nu: L^1(G) \to \mathscr{I}$ determined by α has $\nu(f) = \chi_{\Gamma\backslash E} \cdot \hat{f}$. This is clearly a multiplicative projection. Then by [5, Theorem 1], there is an idempotent measure $\mu \in M(G)$ with $\hat{\mu} = \chi_{\Gamma\backslash E}$, so that ν is given by $f \mapsto f * \mu$. This is a demonstration of the fact that $M(G)$ is the multiplier algebra of $L^1(G)$.

We now turn to another construction of closed subalgebras of $L^1(G)$ that are amenable and yet lack property (**G**). Suppose $\mathfrak{A}$ is a commutative Banach algebra and H is a group of automorphisms of $\mathfrak{A}$. Put

$$\mathfrak{A}_H = \{a \in \mathfrak{A}: h(a) = a, (h \in H)\}$$

certainly $\mathfrak{A}_H$ is a closed subalgebra of $\mathfrak{A}$. We then have the following result, whose proof in this generality was kindly suggested by Professor B.E. Johnson.

2.3. PROPOSITION. *If $\mathfrak{A}$ is a commutative amenable Banach algebra and H is a finite group of automorphisms of $\mathfrak{A}$, then $\mathfrak{A}_H$ is an amenable Banach algebra.*

PROOF. Let $\{d_n\}_{n\in\Delta} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ be an approximate diagonal for $\mathfrak{A}$, and let H have identity ι and cardinality N. Put $K = \max_{h\in H} \|h\|$.

The group $H \times H$ can be made into a group of automorphisms on $\mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H$ via $(h_1, h_2)(a_1 \otimes a_2) = h_1(a_1) \otimes h_2(a_2)$ and then $\mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H = (\mathfrak{A} \hat{\otimes} \mathfrak{A})_{(H\times H)}$. For each $n \in \Delta$, put $d'_n = \frac{1}{N}\sum_{h\in H}(h, h)(d_n)$. Then $\{d'_n\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{A}$ with $(h, h)(d'_n) = d'_n$, for each $h \in H$; let $M = \sup_{n\in\Delta} \|d'_n\|$. Now put

$$d''_n = e \otimes e - \prod_{h\in H} (e \otimes e - (h, \iota)(d'_n)),$$

where this product is in the algebra $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, and the term $e \otimes e$ plays a purely formal rôle as a multiplicative identity. It is clear that $\{d_n''\}_{n \in \Delta}$ is a bounded net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$. Moreover, if $(h_1, h_2) \in H \times H$, then

$$\begin{aligned}(h_1, h_2)(d_n'') &= e \otimes e - \prod_{h \in H} (e \otimes e - (h_1 h, h_2)(d_n')) \\ &= e \otimes e - \prod_{h \in H} (e \otimes e - (h_1 h h_2^{-1}, \imath)(d_n')) = d_n'',\end{aligned}$$

so that $d_n'' \in \mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H$. Also, if $a \in \mathfrak{A}_H$ then

$$\begin{aligned}\|a - a\pi(d_n'')\| &= \left\| a \prod_{h \in H} \pi(e \otimes e - (h, \imath)(d_n')) \right\| \\ &\leqq \|a - a\pi(d_n')\| \prod_{h \in H \setminus \{\imath\}} \|e \otimes e - (h, \imath)(d_n')\| \\ &\leqq \|a - a\pi(d_n')\| (1 + KM)^{N-1} \\ &\to 0,\end{aligned}$$

so that $\{\pi(d_n'')\}_{n \in \Delta}$ is an approximate left identity for $\mathfrak{A}_H$. Finally, we have

$$d_n'' = \sum_{\varnothing \neq S \subseteq H} (-1)^{|S|} \prod_{h \in S} (h, \imath)(d_n'),$$

so if we let $S \mapsto h_S \in S$ be a choice function, then for each $a \in \mathfrak{A}_H$,

$$\begin{aligned}\|d_n'' \cdot a - a \cdot d_n''\| &\leqq \sum_{\varnothing \neq S \subseteq H} \|(h_S, \imath)d_n' \cdot a - a \cdot (h_S, \imath)d_n'\| \prod_{h \in S \setminus \{h_S\}} \|(h, \imath)\| \, \|d_n'\| \\ &\leqq \sum_{\varnothing \neq S \subseteq H} \|(h_S, \imath)\| \, \|d_n' \cdot a - h_S^{-1}(a) \cdot d_n'\| \, (KM)^{|S|-1} \\ &\leqq (2^N - 1)K \, \|d_n' \cdot a - a \cdot d_n'\| \, (KM)^{N-1} \\ &\to 0.\end{aligned}$$

Hence $\{d_n''\}_{n \in \Delta}$ is an approximate diagonal for $\mathfrak{A}_H$, and so $\mathfrak{A}_H$ is amenable.

So we see that if G is a locally compact Abelian group and H is a finite group of automorphisms of $L^1(G)$, then $L^1_H(G) = L^1(G)_H$ is amenable. To determine when $L^1_H(G)$ has property (**G**), note that, by [6, Theorem 1], the automorphisms of $L^1(G)$ are characterized by the piecewise affine homeomorphisms of Γ. Hence we can consider H as a finite group of piecewise affine homeomorphisms $\Gamma \to \Gamma$. Then

$$\begin{aligned}L^1_H(G) &= \{f \in L^1(G): \hat{f} \circ h = \hat{f} \quad (h \in H)\} \\ &= \{f \in L^1(G): \hat{f} \text{ is constant on each orbit } H(\gamma)\}.\end{aligned}$$

So, applying Proposition 2.1, we see that $L^1_H(G)$ has property (**G**) if and only if there is a proper piecewise map α from $Y = \Gamma\backslash Z(L_H^1(G))$ into some other locally compact Abelian group such that $L_H^1(G) = \kappa(\alpha)$. However,

$$\kappa(\alpha) = \{f \in L^1(G): \hat{f}(\Gamma\backslash Y) = 0 \text{ and } \hat{f} \text{ is constant on each set } \alpha^{-1}\{\alpha(\gamma)\}\},$$

so it would seem that the partition of Γ into orbits $H(\gamma)$ is identical to the partition of Γ into sets on which α is constant. The following lemma delivers precisely this result.

2.4. LEMMA. *Let v: $L^1(G') \to L^1(G)$ be a homomorphism between commutative group algebras with $Y \in \mathscr{R}(\Gamma)$ and α: $Y \to \Gamma'$ as above, and let H be a finite group of piecewise affine homeomorphisms of Γ. If* rng $v = L_H^1(G)$, *then $Y = \Gamma$ and for γ_1, $\gamma_2 \in \Gamma$, $\alpha(\gamma_1) = \alpha(\gamma_2) \Leftrightarrow H(\gamma_1) = H(\gamma_2)$.*

PROOF. For each $\gamma \in \Gamma$, $H(\gamma)$ is finite, and since $L^1(G)^\wedge$ separates points of Γ, there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma)) = \{1\}$. Put $\tilde{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h$, then $\tilde{f} \in (L_H^1(G))^\wedge$ and $\hat{f}(\gamma) = 1$. Hence $\gamma \in \Gamma\backslash Z(L_H^1(G)) = Y$, so $Y = \Gamma$.

Now suppose $\gamma_1, \gamma_2 \in \Gamma$ are such that $H(\gamma_1) = H(\gamma_2)$. For each $f \in L^1(G')$, $v(f) \in L_H^1(G)$, so $v(f)(\gamma_1) = v(f)(\gamma_2)$. Thus $\hat{f}(\alpha(\gamma_1)) = \hat{f}(\alpha(\gamma_2))$, and since $A(\Gamma')$ separates points, $\alpha(\gamma_1) = \alpha(\gamma_2)$.

On the other hand, if $H(\gamma_1) \neq H(\gamma_2)$, then $H(\gamma_1)$ and $H(\gamma_2)$ are finite disjoint sets, so there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma_1)) = \{0\}$ and $\hat{f}(H(\gamma_2)) = \{1\}$. Then $\hat{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h \in (L_H^1(G))^\wedge = (\kappa(\alpha))^\wedge$ and $\hat{f}(\gamma_1) \neq \hat{f}(\gamma_1)$, so $\alpha(\gamma_1) \neq \alpha(\gamma_2)$.

We now use the above to characterize property (G) in algebras $L_H^1(G)$ in the case where H is a finite group of automorphisms of Γ. This is a natural situation to consider, as we then have a finite group of automorphisms on G, given by H^*, the group of adjoints of elements of H. Then $L_H^1(G) = \{f \in L^1(G): f \circ h^* = f, (h^* \in H^*)\}$, which is $L^1(G^{H^*})$, a convolution algebra on the *orbit hypergroup* $G^{H^*} = \{H^*(g): g \in G\}$. The amenability of hypergroups and hypergroup algebras is studied further in [23].

We will need a stronger characterization of the terms "coset ring" and "piecewise affine". For more details of this, see [14, Section 2]. Define $\mathscr{R}_0(\Gamma)$ to be the subset of $\mathscr{R}(\Gamma)$ of sets of the form $S = E_0\backslash(\bigcup_1^m E_k)$, where $E_0, \ldots, E_m$ are clopen cosets in Γ and each of $E_1, \ldots, E_m$ is a subcoset of infinite index in E_0. Then E_0 is the coset generated by S, which we denote $E_0 = E_0(S)$. Also, any member of $\mathscr{R}(\Gamma)$ can be represented as a finite disjoint union of elements of $\mathscr{R}_0(\Gamma)$, and so we can suppose that in the definition of *piecewise affine*, each S_k is in $\mathscr{R}_0(\Gamma)$ and each α_k has domain $E_0(S_k)$.

In the situation where $\kappa(\alpha) = L_H^1(G)$, we have seen that we have $Y = \Gamma$. The following lemmas allow us to obtain further special properties of such a piecewise affine map.

2.5. LEMMA. *Suppose G is an Abelian group and $E_1, \ldots, E_n$ are cosets in G such that $G = \bigcup_1^n E_k$, then for some $1 \leqq k \leqq n$, E_k is a subgroup of finite index in G.*

PROOF. Without loss, we have that for some $0 \leqq m \leqq n$, $E_1, \ldots, E_m$ are subcosets of finite index in G and $E_{m+1}, \ldots, E_n$ are subcosets of infinite index in G. For $1 \leqq k \leqq m$, let H_k be the subgroup $E_k E_k^{-1} \subseteq G$, then H_k is of finite index in G, and so $H = \bigcap_1^m H_k$ is of finite index in G. (If $m = 0$, put $H = G$.) For each $k > m$, $H \cap E_k$ is empty or a coset of infinite index in H, so by [22, Theorem 4.3.3], $\bigcup_{m+1}^n (H \cap E_k)$ is a proper subset of H. However, $H = \bigcup_1^n (H \cap E_k)$, so for some $k \leqq m$, $H \cap E_k \neq \emptyset$, so that $H_k \cap E_k \neq \emptyset$. Hence $H_k = E_k$, and we are done.

2.5.1. COROLLARY. *Suppose Γ_1, Γ_2 are locally compact Abelian groups and $\alpha: \Gamma_2 \to \Gamma_1$ is a piecewise affine map. Then there is a set $S \in \mathscr{R}_0(\Gamma_2)$ such that $E_0(S)$ is a subgroup of finite index in Γ_2 and $\alpha|_S$ has a continuous affine extension $\alpha_0: E_0(S) \to \Gamma_1$. Further, if α is proper, then so is α_0.*

PROOF. By the discussion above, that is, [14, Lemmas 2.1 & 3.1].

2.6. LEMMA. *Suppose $S \in \mathscr{R}_0(\Gamma)$ is such that $E_0(S)$ a subgroup of finite index in Γ, and H is a finite group of automorphisms of Γ. Then $\tilde{S} = \bigcap_{h \in H} h(S) \in \mathscr{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \bigcap_{h \in H} h(E_0(S))$ is a subgroup of finite index in Γ.*

PROOF. Suppose $S = E_0 \backslash (\bigcup_1^m E_k)$, as in the definition of $\mathscr{R}_0(\Gamma)$, and put $\tilde{E}_0 = \bigcap_{h \in H} h(E_0)$. Each of $\{h(E_0): h \in H\}$ is a subgroup of finite index in Γ, so $\tilde{E}_0$ is a subgroup of finite index in Γ. Also, $\tilde{S} = \tilde{E}_0 \backslash (\bigcup_{h \in H} \bigcup_1^m (h(E_k) \cap \tilde{E}_0))$, with each $h(E_k) \cap \tilde{E}_0$ being empty or of infinite index in $\tilde{E}_0$. Hence $\tilde{S} \in \mathscr{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \tilde{E}_0$.

2.6.1. COROLLARY. *With Γ_1, Γ_2 and $\alpha: \Gamma_2 \to \Gamma_1$ as in Corollary 2.5.1, if H is a finite group of automorphisms of Γ_2, then we can obtain S with the additional properties that $h(S) = S$ and $h(E_0(S)) = E_0(S)$, for each $h \in H$.*

In the following theorem, we will use the natural generalizations of [6, Theorem 1] and [14, Theorem A] to the situation where we have a homomorphism between two algebras, each a finite direct sum of commutative group algebras.

Suppose we have $\mathfrak{A} = L^1(G_1) \oplus \cdots \oplus L^1(G_n)$, where $G_1, \ldots, G_n$ are locally compact Abelian groups. We can naturally identify $\Phi_{\mathfrak{A}}$ with $\Gamma_1 \cup \cdots \cup \Gamma_n$, the disjoint union of the duals of $G_1, \ldots, G_n$. We can also define the coset ring of $\Gamma_1 \cup \cdots \cup \Gamma_n$, denoted $\mathscr{R}(\Gamma_1 \cup \cdots \cup \Gamma_n)$, to be

$$\{Y \subseteq \Gamma_1 \cup \cdots \cup \Gamma_n: Y \cap \Gamma_k \in \mathscr{R}(\Gamma_k) \quad (1 \leqq k \leqq n)\},$$

which happens to be the Boolean ring generated by all the open cosets of each of

$\Gamma_1, \ldots, \Gamma_n$. Similarly, for $G'_1, \ldots, G'_m$ locally compact Abelian groups, and $Y \in \mathscr{R}(\Gamma'_1 \cup \cdots \cup \Gamma'_m)$, we can define a map α: $Y \to \Gamma_1 \cup \cdots \cup \Gamma_n$ to be piecewise affine if we can partition Y into sets $\{Y_{jk}: 1 \leqq j \leqq m, 1 \leqq k \leqq n\}$ such that for each j, k, $Y_{jk} \in \mathscr{R}(\Gamma'_j)$, $\alpha(Y_{jk}) \subseteq \Gamma_k$, and $\alpha_{jk} = \alpha|_{Y_{jk}}$: $Y_{jk} \to \Gamma_k$ is piecewise affine.

With such notation, it is elementary to show that a homomorphism v from $\mathfrak{A} = L^1(G_1) \oplus \cdots \oplus L^1(G_n)$ into $\mathfrak{B} = L^1(G'_1) \oplus \cdots \oplus L^1(G'_m)$ is uniquely determined by the proper piecewise affine map $v^*|_Y$, where $Y = \Phi_{\mathfrak{B}} \backslash Z(\operatorname{rng} v)$. Moreover, the proof of [14, Theorem A] generalizes naturally to considering such homomorphisms, giving the conclusion $\operatorname{rng} v = \kappa(\alpha)$. This is merely an extension of the observation made in Section 4 of [14] regarding homomorphisms $L^1(G) \to L^1(G_1) \oplus \cdots \oplus L^1(G_n)$.

2.7. THEOREM. *Suppose H is a finite group of automorphisms of a locally compact Abelian group Γ. Then the following are equivalent:*

(i) *$L^1_H(G)$ has property* **(G)**,

(ii) *the subgroup $\Lambda = \{\gamma \in \Gamma: H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, and*

(iii) *$L^1_H(G)$ is isomorphic to a finite direct sum of group algebras.*

PROOF. Supposing (i), then by Proposition 2.1 and Lemma 2.4, there is a locally compact Abelian group G' and a proper piecewise affine map α: $\Gamma \to \Gamma'$ such that the level sets of α are precisely the orbits of the action of H on Γ. By Corollary 2.6.1, there exists $S \in \mathscr{R}_0(\Gamma)$ such that $E_0(S)$ is a subgroup of finite index in Γ, $\alpha|_S$ has a proper continuous affine extension α_0: $E_0(S) \to \Gamma'$, and for each $h \in H$, $h(S) = S$ and $h(E_0(S)) = E_0(S)$.

Now, $\alpha \circ h = \alpha$, for each $h \in H$, so $\Lambda_0 = \{\gamma \in E_0(S): \alpha_0 \circ h(\gamma) = \alpha_0(\gamma), (h \in H)\}$ is a subgroup of $E_0(S)$ with $S \subseteq \Lambda_0$. Since $E_0(S)$ is the coset generated by S, we have that $\Lambda_0 = E_0(S)$, and so $\alpha_0 \circ h = \alpha_0$, $(h \in H)$. Put $\Xi = \{\gamma \in E_0(S): \alpha_0(\gamma) = \alpha_0(e)\} = \alpha_0^{-1}\{\alpha_0(e)\}$, a subgroup of $E_0(S)$. For each $\gamma \in S$, $H(\gamma) \subseteq S$, so

$$\begin{aligned} \gamma' \in H(\gamma) &\Leftrightarrow \alpha(\gamma') = \alpha(\gamma) \\ &\Leftrightarrow \alpha_0(\gamma') = \alpha_0(\gamma) \text{ and } \gamma' \in S, \end{aligned}$$

so $H(\gamma) = \gamma\Xi \cap S$. Thus $\{\gamma \in E_0(S): H(\gamma) \subseteq \gamma\Xi\}$, a subgroup of $E_0(S)$, contains S. It follows that $H(\gamma) \subseteq \gamma\Xi$ for all $\gamma \in E_0(S)$. For each $h \in H$, let $\tilde{h}$: $E_0(S) \to \Xi$ be the homomorphism defined by $\tilde{h}(\gamma) = h(\gamma)\gamma^{-1}$, so that $\Lambda = \bigcap_{h \in H} \tilde{h}^{-1}\{e\}$. It remains to be proven that Ξ is finite, for then each $\tilde{h}^{-1}\{e\}$ is of finite index in $E_0(S)$, which is in turn of finite index in Γ.

By [14, Lemma 2.2], there exists $\gamma_1, \ldots, \gamma_N \in E_0(S)$ such that $E_0(S) = \bigcup_1^N \gamma_k S$, giving

$$\Xi = \Xi \cap E_0(S) = \bigcup_{1 \leqq k \leqq N} \gamma_k(\gamma_k^{-1}\Xi) \cap S) = \bigcup_{1 \leqq k \leqq N} \gamma_k H(\gamma_k^{-1}),$$

which is evidently finite.

Now assume (ii). For each coset $\gamma\Lambda$ of Λ, and each $h\in H$, $h(\gamma\Lambda)$ is the coset $h(\gamma)\Lambda$, so that H acts on Γ/Λ. Let $H(\gamma_1\Lambda),\ldots,H(\gamma_N\Lambda)$ be the orbits of this action, and for each $1\leqq k\leqq N$, let $h_{k,1},\ldots,h_{k,n_k}\in H$ be such that the cosets of Λ that make up $H(\gamma_k\Lambda)$ are $\{h_{k,j}(\gamma_k\Lambda)\colon 1\leqq j\leqq n_k\}$.

For each $1\leqq k\leqq N$, $H_k=\{h\in H\colon h(\gamma_k)\in\gamma_k\Lambda\}$ is a subgroup of H, and $\Lambda_k=\{h(\gamma_k)\gamma_k^{-1}\colon h\in H_k\}$ is a subgroup of Λ. Furthermore, H_k acts on $\gamma_k\Lambda$ by $h(\gamma_k\lambda)=(\gamma_k\lambda)\cdot(h(\gamma_k)\gamma_k^{-1})$, that is, by translations by elements of Λ_k. For $1\leqq j\leqq n_k$, define $\alpha_{kj}\colon h_{kj}(\gamma_k)\Lambda\to\Lambda/\Lambda_k$ by $\alpha_{kj}(h_{kj}(\gamma_k)\lambda)=\lambda\Lambda_k$. This is continuous and affine, and since $\alpha_{kj}^{-1}(\lambda\Lambda_k)=h_{kj}(\gamma_k)\lambda\Lambda_k$ is finite, α_{kj} is also proper.

Each coset of Λ in Γ is of the form $h_{kj}(\gamma_k)\Lambda$, for some unique k and j, so we can define a proper piecewise affine map $\alpha\colon\Gamma\to\Lambda/\Lambda_1\cup\cdots\cup\Lambda/\Lambda_N$ by "piecing together" all the α_{kj}. For each $\gamma\in\Gamma$, say $\gamma=h_{kj}(\gamma_k)\lambda$, we have $H(\gamma)=H(h_{kj}(\gamma_k\lambda))=H(\gamma_k\lambda)$. Also $\alpha_{kj}^{-1}(\lambda\Lambda_k)=h_{kj}(\gamma_k\lambda\Lambda_k)=h_{kj}(H_k(\gamma_k\lambda))$. Hence

$$\alpha^{-1}\{\alpha(\gamma)\}=\bigcup_{1\leqq j\leqq n_k}\alpha_{kj}^{-1}(\lambda\Lambda_k)=\bigcup_{1\leqq j\leqq n_k}h_{kj}(H_k(\gamma_k\lambda))=H(\gamma_k\lambda)=H(\gamma),$$

and as this holds for each $\gamma\in\Gamma$, $\kappa(\alpha)=L^1_H(G)$. Now, by the extension of Cohen's characterization of group algebra homomorphisms, as outlined above, α determines a homomorphism $\nu\colon A(\Lambda/\Lambda_1)\oplus\cdots\oplus A(\Lambda/\Lambda_N)\to A(\Gamma)$ with range $\kappa(\alpha)$. Also, $\ker\nu=\mathscr{I}(\operatorname{rng}(\alpha))$ and since α is surjective, we have that ν is a monomorphism. Hence $A_H(\Gamma)=\kappa(\alpha)\cong A(\Lambda/\Lambda_1)\oplus\cdots\oplus A(\Lambda/\Lambda_N)$.

The last implication (iii) $\Rightarrow$ (i) follows from Proposition 1.1.

So we see that the amenable algebras of the form $L^1_H(G)$ will usually fail to have property (**G**). For instance, if Γ is connected, then for $L^1_H(G)$ to have property (**G**), we must have $\Lambda=\Gamma$, and so $H=\{\iota\}$ and $L^1_H(G)=L^1(G)$.

If G is a locally compact Abelian group, we always have the automorphism η on G given by $x\mapsto x^{-1}$. (Although occasionally we have $\eta=\iota$, as we will see.) Then $H=\{\iota,\eta\}$ is a finite group of automorphisms of G and $L^1_H(G)=L^1_{\mathrm{sym}}(G)$, the subalgebra of symmetric (or even) functions in $L^1(G)$. We now apply the preceding theorem to this case

2.8. Theorem. *If G is a locally compact Abelian group, the following are equivalent:*

(i′) *$L^1_{\mathrm{sym}}(G)$ has property* (**G**),

(ii′) *$G\cong\sum_{\mathfrak{a}}\mathbf{Z}_2\times\prod_{\mathfrak{b}}\mathbf{Z}_2\times F$, for some cardinals $\mathfrak{a}$ and $\mathfrak{b}$ and some finite group F, and*

(iii′) *$L^1_{\mathrm{sym}}(G)$ is isomorphic to a group algebra.*

Proof. Suppose (i′), then we have by Theorem 2.7 that $\Lambda=\{\gamma\in\Gamma\colon H(\gamma)=\{\gamma\}\}$ is of finite index in Γ, say $|\Gamma/\Lambda|=N$. Then $\gamma\in\Gamma\Rightarrow\gamma^N\in\Lambda\Rightarrow\gamma^N=\gamma^{-N}\Rightarrow\gamma^{2N}=e$, so that Γ is of bounded order. Thus, by [12, Theorem A.25], there is an *algebraic* isomorphism $\psi\colon \Gamma_{2,d} \to \sum_{i\in \mathbf{I}} \mathbf{Z}_{n_i}$, where $\mathbf{I}$ is an index set and $\{n_i\colon i\in \mathbf{I}\}$ is a bounded set of integers greater than 2. Then $\{\gamma\in\Gamma\colon \gamma^2 = e\} = \Lambda$ is of finite index, so $F = \psi^{-1}(\sum_{n_i>2}\mathbf{Z}_{n_i})$ is a finite subgroup of Γ with $(\Gamma/F)_d \cong \sum_{n_i=2}\mathbf{Z}_2$.

Let Λ_0 be a compact open subgroup of Γ, which we can assume to contain F. If we now apply the argument of the above paragraph to $\hat{\Lambda}_0$, we obtain that $\Lambda_0 \cong F \times \prod_{\mathfrak{a}}\mathbf{Z}_2$, for some cardinal $\mathfrak{a}$. By continuing with an argument similar to that used in [12, 25.29], or by a straightforward application of Zorn's Lemma, we can obtain a complement to Λ_0, which will be isomorphic to $\sum_{\mathfrak{b}}\mathbf{Z}_2$ for some cardinal $\mathfrak{b}$, giving $G \cong F \times \sum_{\mathfrak{a}}\mathbf{Z}_2 \times \prod_{\mathfrak{b}}\mathbf{Z}_2$.

For the implication (ii′) ⇒ (iii′), we show that for $G = \sum_{\mathfrak{a}}\mathbf{Z}_2 \times \prod_{\mathfrak{b}}\mathbf{Z}_2 \times F$, $L^1_{\mathrm{sym}}(G)$ is isomorphic to a group algebra. Let $H = \sum_{\mathfrak{a}}\mathbf{Z}_2 \times \prod_{\mathfrak{b}}\mathbf{Z}_2$, so that $H^{(2)} = \{e\}$ and $G = H \times F$. Let $\Psi\colon L^1(G) \to L^1(H)\hat{\otimes} l^1(F)$ be the natural isomorphism. It is easily verified that $\Psi(L^1_{\mathrm{sym}}(G)) = L^1(H)\hat{\otimes} l^1_{\mathrm{sym}}(F)$. Now, $l^1_{\mathrm{sym}}(F)$ is a finite-dimensional commutative semisimple Banach algebra, so $l^1_{\mathrm{sym}}(F) \cong \mathbf{C}^m \cong l^1(\mathbf{Z}_m)$, where $m = \dim(l^1_{\mathrm{sym}}(F))$. Consequently $L^1_{\mathrm{sym}}(G) \cong L^1(H)\hat{\otimes} l^1(\mathbf{Z}_m) \cong L^1(H\times \mathbf{Z}_m)$.

The final implication (iii′) ⇒ (i′) is trivial.

In light of the conclusion (iii′) in Theorem 2.8, it is natural to ask whether we can reach the same conclusion in Theorem 2.7. We will give an example of an Abelian group G with a finite group of automorphisms H such that $l^1_H(G)$ has property (**G**), but is not isomorphic to a group algebra.

Let U and V be as constructed in [15, p. 616–7]. That is, U is a countably infinite torsion-free Abelian group and V is a non-isomorphic subgroup that is of index 2 in U. Let $\Upsilon = \hat{U}$, a connected compact Abelian group, then $\Xi = \mathrm{Ann}_\Upsilon(V)$ is a two-element group, say $\Xi = \{e, \xi\}$, and $\Upsilon/\Xi = \hat{V}$ is also compact and connected. Put $G = U \times \mathbf{Z}_2$, so that $\Gamma = \Upsilon \times \mathbf{Z}_2$, and define $\eta\in\mathrm{Aut}(\Gamma)$ by $\eta(v,0) = (v,0)$ and $\eta(v,1) = (v\xi, 1)$. Then $\eta^2 = \iota$, so $H = \{\iota, \eta\}$ is a finite group of automorphisms of G which clearly satisfies the criterion (ii) in Theorem 2.7. It then follows that $l^1_H(G)$ is isomorphic to a finite direct sum of groups algebras, and by applying the construction in the proof of Theorem 2.7, we obtain $l^1_H(G) \cong l^1(U) \oplus l^1(V)$, which has maximal ideal space $\Upsilon \cup \Upsilon/\Xi$.

Suppose $l^1(U) \oplus l^1(V)$ is isomorphic to a group algebra $L^1(G')$, so that there exists a piecewise affine homeomorphism $\alpha\colon \Upsilon \cup \Upsilon/\Xi \to \Gamma'$. Thus Γ' has two connected components, which are necessarily affinely homeomorphic. It follows that Υ and Υ/Ξ are topologically isomorphic, and so U and V are isomorphic. (Contradiction.)

Many thanks to Dr Laci Kovács for suggesting the group U used in this example.

3. A Non-commutative Amenable Banach Algebra Without Property (G).

In this section we examine some amenable Banach algebras which we show to lack property (G) by methods entirely different to those in Section 2. For the results presented in this section, I am indebted to a suggestion of U. Haagerup, and its communication through P.C. Curtis and George Willis.

3.1. LEMMA. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are unital Banach algebras and $\mathscr{I}$ is a closed left ideal of $\mathfrak{A}$ with a left approximate identity $\{e_n\}_{n\in\Delta}$, bounded by $M > 0$. If $\nu: \mathscr{I} \to \mathfrak{B}$ is a continuous homomorphism with* $\operatorname{rng}\nu \cap \mathfrak{B}^{-1} \neq \emptyset$, *then there is a unique homomorphism $\tilde{\nu}: \mathfrak{A} \to \mathfrak{B}$ extending ν. Moreover, $\tilde{\nu}(e) = e = \lim_{n\in\Delta} \nu(e_n)$, $\overline{\nu(\mathscr{I})} = \overline{\tilde{\nu}(\mathfrak{A})}$, and $\|\tilde{\nu}\| \leqq M\|\nu\|$.*

PROOF. Suppose $a \in \mathscr{I}$ is such that $\nu(a) \in \mathfrak{B}^{-1}$, then any homomorphism $\tilde{\nu}: \mathfrak{A} \to \mathfrak{B}$ extending ν must satisfy $\tilde{\nu}(x) = \nu(xa)[\nu(a)]^{-1}$, $(x \in \mathfrak{A})$. Define $\tilde{\nu}$ to be exactly this. Then $\tilde{\nu}$ is a continuous linear extension of ν with $\tilde{\nu}(e) = e$.

For each $n \in \Delta$, and each $x \in \mathfrak{A}$,

$$\nu(xe_n) - \tilde{\nu}(x) = \nu(x(e_n a - a))[\nu(a)]^{-1} \to 0$$

so $\tilde{\nu}(x) = \lim_{n\in\Delta} \nu(xe_n)$. Hence $\tilde{\nu}(\mathfrak{A}) \subseteq \overline{\nu(\mathscr{I})}$, $\|\tilde{\nu}\| \leqq M\|\nu\|$, and $e = \tilde{\nu}(e)$ $\lim_{n\in\Delta} \nu(e_n)$. Also, if $x, y \in \mathfrak{A}$, then $ya \in \mathscr{I}$, so $\nu(xya) = \lim_{n\in\Delta} \nu(xe_n ya)$. However, $\nu(xya) = \tilde{\nu}(xy)\nu(a)$ and $\lim_{n\in\Delta} \nu(xe_n ya) = [\lim_{n\in\Delta} \nu(xe_n)]\nu(ya) = \tilde{\nu}(x)\tilde{\nu}(y)\nu(a)$, and since $\nu(a) \in \mathfrak{A}^{-1}$, we have that $\tilde{\nu}(xy) = \tilde{\nu}(x)\tilde{\nu}(y)$, as desired.

3.2. PROPOSITION. *Suppose $\mathfrak{A}$ is a Banach algebra with unit e, G is a locally compact group, and $\nu: L^1(G) \to \mathfrak{A}$ is a continuous homomorphism. If* $\operatorname{rng}\nu \cap \mathfrak{A}^{-1} \neq \emptyset$, *then ν has a unique extension to a homomorphism $\tilde{\nu}: M(G) \to \mathfrak{A}$. Further, $\|\tilde{\nu}\| = \|\nu\|$ and $\overline{\nu(L^1(G))} = \overline{\tilde{\nu}(l^1(G))} = \overline{\tilde{\nu}(M(G))}$.*

PROOF. Let Δ be the set of compact neighbourhoods of $e \in G$, and order Δ by $\supseteq$. For each $U \in \Delta$, take $e_U \in C_{00}^+(G)$ with support within U and $\|e_U\| = 1$. Then $\{e_U\}_{U\in\Delta}$ is a bounded approximate identity for $L^1(G)$, a closed ideal of $M(G)$. Hence, by Lemma 3.1, ν has a unique extension to a homomorphism $\tilde{\nu}: M(G) \to \mathfrak{A}$, with $e = \tilde{\nu}(\delta_e) = \lim_{U\in\Delta} \nu(e_U)$, $\|\tilde{\nu}\| = \|\nu\|$, and $\overline{\nu(L^1(G))} = \overline{\tilde{\nu}(M(G))}$.

Since $\overline{\tilde{\nu}(l^1(G))} \subseteq \overline{\tilde{\nu}(M(G))}$, it remains to be proven that $\nu(L^1(G)) \subseteq \overline{\tilde{\nu}(l^1(G))}$. For this it suffices to prove that $\nu(C_{00}^+(G)) \subseteq \overline{\tilde{\nu}(l^1(G))}$.

For this we can use a portion of the proof of existence and uniqueness of Haar measure, as given in [12, 15.5–6]. (As is done in [24, Lemma 2.1].) This states that for $f \in C_{00}^+(G)$ and $\varepsilon > 0$, there exists $U \in \Delta$ such that if $g \in C_{00}^+(G)$ is zero off U with $\|g\| = 1$, then there exists $h \in l^1(G)$ with $\|h\| \leqq \|f\|$ and $\|f - h * g\| < \varepsilon$. Take $V \in \Delta$ with $V \subseteq U$, and $\|\nu(e_V) - e\| < \varepsilon$. Then $e_V \in C_{00}^+(G)$ is zero off U, so we can take $h \in l^1(G)$ with $\|f - h * e_V\| < \varepsilon$. Then

$$\begin{aligned}\|v(f) - \tilde{v}(h)\| &\leqq \|v(f - h * e_V)\| + \|\tilde{v}(h)(v(e_V) - e)\| \\ &\leqq \|v\| \, \|f - h * e_V\| + \|\tilde{v}\| \, \|h\| \, \|v(e_V) - e\| \\ &< \|v\| \, \varepsilon + \|v\| \, \|f\| \, \varepsilon.\end{aligned}$$

Hence $v(f) \in \overline{\tilde{v}(l^1(G))}$.

In the following, $\mathscr{Z}(\mathfrak{A})$ is the *centre* of $\mathfrak{A}$, that is, $\mathscr{Z}(\mathfrak{A}) = \{a \in \mathfrak{A}: ab = ba, (b \in \mathfrak{A})\}$.

3.3. THEOREM. *Suppose $\mathfrak{A}$ is a unital Banach algebra with property* (**G**). *Then*

$$\overline{\operatorname{span}}\{ab - ba: a, b \in \mathfrak{A}\} \cap \mathscr{Z}(\mathfrak{A}) = \{0\}.$$

PROOF. Let G be an amenable locally compact group and $v: L^1(G) \to \mathfrak{A}$ be a dense-ranged homomorphism. Then $\mathfrak{A}^{-1}$ is open, so $\operatorname{rng} v \cap \mathfrak{A}^{-1} \neq \varnothing$, and we can apply Lemma 3.2 to obtain an extension $\tilde{v}: M(G) \to \mathfrak{A}$ with $\|\tilde{v}\| = \|v\|$, $\mathfrak{A} = \overline{\tilde{v}(l^1(G))}$ and $\tilde{v}(\delta_e) = e$. Then

$$\begin{aligned}\overline{\operatorname{span}}\{ab - ba: a, b \in \mathfrak{A}\} &= \overline{\operatorname{span}}\{a\tilde{v}(f) - \tilde{v}(f)a: a \in \mathfrak{A}, f \in l^1(G)\} \\ &= \overline{\operatorname{span}}\{a\tilde{v}(\delta_x) - \tilde{v}(\delta_x)a: a \in \mathfrak{A}, ax \in G\} \\ &= \overline{\operatorname{span}}\{\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x) - a: a \in \mathfrak{A}, x \in G\}.\end{aligned}$$

Thus it suffices to show that for each $z \in \mathscr{Z}(\mathfrak{A})$, there is an element of $\mathfrak{A}^*$ that annihilates each $\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x) - a$, but not z.

Take $z \in \mathscr{Z}(\mathfrak{A})$. Let $\psi \in \mathfrak{A}^*$ be such that $\psi(z) \neq 0$ and $\|\psi\| < 1$. For each $a \in \mathfrak{A}$, define the function ψ_a on G by $\psi_a(x) = \psi(\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x))$, $(x \in G)$. Then $\sup_{x \in G} |\psi_a(x)| \leqq \|v\|^2 \, \|a\|$, so $\psi_a \in l^\infty(G)$. Define $\Psi: \mathfrak{A} \to l^\infty(G)$ by $\Psi(a) = \psi_a$. Then Ψ is linear with $\|\Psi\| \leqq \|v\|^2$. If $a \in \operatorname{rng} v$, say $a = v(f)$, then for each $x \in G$, $\psi_a(x) = \psi \circ v(\delta_{x^{-1}} * f * \delta_x)$, so $\psi_a \in C_b(G)$. Hence $\Psi(\operatorname{rng} v) \subseteq C_b(G)$, a closed subalgebra of $l^\infty(G)$, and since Ψ is continuous, $\Psi(\mathfrak{A}) \subseteq C_b(G)$.

Now, if $a \in \mathfrak{A}$ and $x, y \in G$, then

$${}_y(\Psi(a))(x) = \psi_a(yx) = \psi(\tilde{v}(\delta_{x^{-1}})\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y)\tilde{v}(\delta_x)) = \Psi(\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y))(x),$$

so that if M is a left-invariant mean on $C_b(G)$, then

$$M \circ \Psi(a) = M({}_y\Psi(a)) = M \circ \Psi(\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y)).$$

Hence $M \circ \Psi \in \mathfrak{A}^*$ annihilates each $\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y) - a$. But $\psi_z(x) = \psi(\tilde{v}(\delta_{x^{-1}}) z\tilde{v}(\delta_x)) = \psi(\tilde{v}(\delta_{x^{-1}})\tilde{v}(\delta_x)z) = \psi(z)$ is the constant function $\psi(z)$. Hence $M \circ \Psi(z) = \psi(z) \neq 0$.

Suppose H is a separable Hilbert space, n is an integer greater than 2, and $H_1, \ldots, H_n$ are orthogonal closed infinite-dimensional subspaces with $H_1 + \cdots +$

$H_n = H$. For each $1 \leqq k \leqq n$, let S_k be a linear isometry $H \to H_k$. Then $S_1, \ldots, S_n \in \mathfrak{B}(H)$ and $I = S_1^* S_1 = \cdots = S_n^* S_n = S_1 S_1^* + \cdots + S_n S_n^*$. Let $\mathcal{O}_n$ be the C^*-algebra generated by $S_1, \ldots, S_n$, which we call the *Cuntz algebra on n generators.* This algebra was introduced in [7], where it is shown not to depend on the actual isometries $S_1, \ldots, S_n$ chosen, but only on n. In [21], it is shown that the Cuntz algebras are amenable. However,

$$(S_1^* S_1 - S_1 S_1^*) + \cdots + (S_n^* S_n - S_n S_n^*) = (n-1)I \in \mathscr{Z}(\mathcal{O}_n),$$

so we see that $\mathcal{O}_n$ cannot have property (**G**).

This seems related to other properties of the Cuntz algebras related to amenability. In particularly, the Cuntz algebras are amenable, but not strongly amenable. (Strong amenability is a property of C^*-algebras defined in [13]. The Cuntz algebras were shown to not be strongly amenable in [21].)

Suppose $\mathfrak{A}$ is a C^*-subalgebra of $\mathfrak{B}(H)$, and $v: L^1(G) \to \mathfrak{A}$ is a homomorphism with $\operatorname{rng} v \cap \mathfrak{A}^{-1} \neq \emptyset$. By Proposition 3.2, we have a homomorphism $l^1(G) \to \mathfrak{A}$, which gives a continuous representation $\pi: G \to \mathfrak{B}(H)$ with $\pi(x) \leqq \|v\|$, for each $x \in G$. (cf. [18, p. 77].) Then by [18, Corollary 17.6], π is equivalent to a unitary representation, that is, there is an isomorphism $\Psi: H \to H$ such that $\pi': x \mapsto \Psi^{-1}\pi(x)\Psi$ is a continuous representation of G such that each $\pi'(x)$ is unitary. Then, by [13, Proposition 7.8], $\mathfrak{A}' = \overline{\pi(l^1(G))}$ is a strongly amenable Banach algebra. Moreover $\overline{\pi(l^1(G))} = \overline{\Psi^{-1}\pi(l^1(G))\Psi} = \Psi^{-1}\mathfrak{A}\Psi$. Now, if strong amenability was preserved by such a transformation, then we could conclude that $\mathfrak{A}$ is strongly amenable. Unfortunately, this avenue is not open to us.

4. Other Constructions Preserving Amenability.

Having demonstrated that property (**G**) falls short of providing a characterization of amenability, it is natural to ask whether other stability properties of amenability can be used to provide a "constructive" characterization of amenability in Banach algebras.

For this, define a Banach algebra $\mathfrak{A}$ to have *property* (**G**′) if there are closed subalgebras $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \subset \mathfrak{A}_n = \mathfrak{A}$ such that for each $1 \leqq k \leqq n$, $\mathfrak{A}_{k-1}$ is a closed ideal of $\mathfrak{A}_k$ and $\mathfrak{A}_k/\mathfrak{A}_{k-1}$ has property (**G**). A repeated application of [13, Proposition 5.1] demonstrates the amenability of such $\mathfrak{A}$. It is also a simple matter to show, using the fact that each $\mathfrak{A}_k$ factorizes, that each $\mathfrak{A}_k$ is an ideal of $\mathfrak{A}$.

Furthermore, algebras such as $L^1(G)^{\#}$, where G is a nondiscrete locally compact Abelian group, are easily shown to have property (**G**′) while lacking property (**G**). (To show the latter, consider $L^1(G)^{\#}$ as the closed subalgebra $L^1(G) + \mathbb{C}\delta_e$ of $M(G)$. This can be shown to lack property (**G**) by a simple application of Cohen's characterization of homomorphisms $L^1(G_1) \to M(G_2)$, [6, Theorem 1].)

Unfortunately, examples we have already seen are sufficient to show that property (**G**′) is not necessary for amenability. For instance, if $n \geqq 2$, then the Cuntz algebra $\mathcal{O}_n$ is simple – it has no nontrivial ideals, closed or otherwise. Hence the above chain of ideals could only be $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 = \mathcal{O}_n$, and since $\mathcal{O}_n$ lacks property (**G**), it lacks property (**G**′). Also, if H is a group of automorphisms of R^n with $2 \leqq |H| < \infty$, then by [14, Corollary 1.6.2], $L^1_H(\mathsf{R}^n)$ has no subalgebra with property (**G**), and so $L^1_H(\mathsf{R}^n)$ cannot have property (**G**′).

This last example can also be used to show that similar attempts to use other constructions that preserve amenability will also fail. In particular, it is possible to show that if $\{\mathfrak{A}_n\}_{n\in\Delta}$ is a net of amenable closed subalgebras of $\mathfrak{A}$ with union dense in $\mathfrak{A}$, and the approximate diagonals of the $\mathfrak{A}_n$ have a common bound, then $\mathfrak{A}$ is amenable. (This is similar to the construction in [19, Proposition 1.12]. It can be shown to be equivalent.) Thus we can define a property ($\mathbf{G}^\infty$) to be that of having such a net of closed subalgebras, each having property (**G**). As already noted, this cannot occur in $L^1_H(\mathsf{R}^n)$. It can also be shown, by quite different methods, that the Cuntz algebras and many of the closed ideals of commutative group algebras also lack this property. (In fact, the author's PhD thesis presented a characterization of property ($\mathbf{G}^\infty$) in such ideals: $\mathcal{I} \subseteq L^1(G)$ has property ($\mathbf{G}^\infty$) if and only if $X = Z(\mathcal{I}) \in \mathcal{R}_c(\Gamma)$ and $X\Xi \subseteq X$, where Ξ is the component of the identity in Γ.)

Given the examples $L^1_H(G)$, it seems that we will need to consider other constructions, if we are to achieve the goal of obtaining such a characterization of amenability. An obvious place to start is to consider allowing the use of Proposition 2.3, as this is the result that gives us the amenability of the algebras $L^1_H(G)$. However, this is of little use in the non-commutative case, as it provides no guarantee of the amenability of $\mathfrak{A}_H$, when $\mathfrak{A}$ is not commutative. It is not known to the author whether a noncommutative version of Proposition 2.3 does hold.

5. Dense-ranged Homomorphisms of Amenable Banach Algebras.

We are left with the prospect that we cannot characterize amenability of Banach algebras in terms of amenable group algebras. The question arises as to whether there is some other "canonical" class $\mathscr{A}$ of amenable Banach algebras which we could use in place of the amenable group algebras in the definition of property (**G**), to arrive at a characterization of property (**G**). That is:

5.1. Question. Is there some class $\mathscr{A}$ of amenable Banach algebras such that for each amenable Banach algebra $\mathfrak{A}$, there is a $\mathfrak{B} \in \mathscr{A}$ and a dense-ranged continuous homomorphism $\nu\colon \mathfrak{B} \to \mathfrak{A}$?

Evidently, setting $\mathscr{A}$ to be the class of *all* amenable Banach algebras will suffice, but we seek a class considerably smaller. By enlarging $\mathscr{A}$ slightly, we can ensure that $\mathscr{A}$ is closed under taking quotients by closed ideals. Then the above question is equivalent to the one where "homomorphism" is replaced by "monomorphism".

5.2. DEFINITION. Let $\mathfrak{A}$ be a Banach algebra. A *Banach subalgebra* of $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, with its own norm by which it is a Banach algebra, such that the injection $\mathfrak{B} \hookrightarrow \mathfrak{A}$ is continuous.

With this definition, Question 5.1 is asking for a class of amenable Banach algebras $\mathscr{A}$ such that each amenable Banach algebra $\mathfrak{A}$ has a dense Banach subalgebra $\mathfrak{B}$ that is (isomorphic to) a member of $\mathscr{A}$. In any such class $\mathscr{A}$, we must include each amenable Banach algebra $\mathfrak{A}$ which has no dense amenable Banach subalgebras. Define a Banach algebra $\mathfrak{A}$ to be *minimal-amenable* if it has this property, or equivalently, if every dense-ranged homomorphism from an amenable Banach algebra into $\mathfrak{A}$ is onto.

5.3. QUESTION. Which amenable Banach algebras are minimal-amenable?

The only examples known to the author of minimal-amenable Banach algebras are those which are finite-dimensional, and those of the form $C(X)$, where X is a compact F-space. (See [9] for definitions, and [1, Theorem A] for the relevant result.) These examples are not particularly illuminating, in that they are also *minimal*, in that they have no proper dense Banach subalgebras. Also, for such X, $C(X)$ is either finite-dimensional or nonseparable, and so we ask:

5.4. QUESTION. Are there minimal-amenable Banach algebras that are not minimal?

5.5. QUESTION. Are there infinite-dimensional separable minimal-amenable Banach algebras?

A possible answer to each of these questions would be that commutative group algebras are minimal-amenable. The result of [14] can be used to show that if $\mathfrak{A}$ has property **(G)**, then any dense-ranged homomorphism $\mathfrak{A} \to L^1(G)$ is onto, so that any proper dense amenable Banach subalgebra of $L^1(G)$ must lack property **(G)**. It is interesting to note that two standard sources of proper dense Banach subalgebras of group algebras can never yield an amenable algebra. The first of these, Segal algebras, defined as in [20, Section 6.2] lack bounded approximate identities, due to [4, Theorem 1.2]. The second construction is that of Beurling algebras, defined to be $L^1(G, \omega)$, for some submultiplicative weight $\omega\colon G \to \mathrm{R}^+$, as in [20, Section 6.3]. By [11, Theorem 0], such an algebra is amenable if and only if $x \mapsto \omega(x)\omega(x^{-1})$ is bounded. However, since $L^1(G, \omega)$ is assumed to be contained within $L^1(G)$, ω is bounded below, and hence ω is also bounded above. However, this implies $L^1(G,\omega) = L^1(G)$.

We should note that the term "minimal" (or "minimal-amenable", etc) is only supposed to indicate the lack of a certain type of dense subalgebra, and as such, only refers to an ordering (by inclusion) of such dense subalgebras. It is tempting to lift this to an order on the category of Banach algebras (or the category of amenable Banach algebras, etc). Such an order would be defined by $\mathfrak{A} \preceq \mathfrak{B}$ if there is a dense-ranged monomorphism $\mathfrak{A} \to \mathfrak{B}$. However, it is possible to have non-isomorphic Banach algebras $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{A}$. We give an example where both $\mathfrak{A}$ and $\mathfrak{B}$ have property (G).

Define dense Banach subalgebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2$ of $C_0(\mathsf{Z} \times \mathsf{R})$ by

$$\mathfrak{A}_1 = \mathfrak{B}_1 = A(\mathsf{Z} \times \mathsf{R})$$

$$\mathfrak{A}_2 = \{f \in C_0(\mathsf{Z} \times \mathsf{R}): f(n,\cdot) \in A(\mathsf{R})\ (n \in \mathsf{Z})\}$$

$$\mathfrak{A}_3 = \mathfrak{B}_2 = C_0(\mathsf{Z} \times \mathsf{R}).$$

Each of these has carrier space $\mathsf{Z} \times \mathsf{R}$ and $\mathfrak{A}_1 \preceq \mathfrak{A}_2 \preceq \mathfrak{A}_3$. Also, $\mathfrak{B}_1 \cong \mathfrak{B}_1 \oplus \mathfrak{B}_1$ and $\mathfrak{B}_2 \cong \mathfrak{B}_2 \oplus \mathfrak{B}_2$, so that if we define

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3 \qquad \text{and} \qquad \mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$$

$$\text{then} \qquad \mathfrak{B} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \preceq \mathfrak{A} \preceq \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_2 \cong \mathfrak{B}.$$

Suppose $\mathfrak{A} \cong \mathfrak{B}$, so that there is an isomorphism $\nu: \mathfrak{A} \to \mathfrak{B}$. Now, each of $\mathfrak{A}_1$, $\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2$ has carrier space $\mathsf{Z} \times \mathsf{R}$, and so $\nu^*|_{\Phi_{\mathfrak{B}}}$ is a homeomorphism

$$\alpha: (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}) \to (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}).$$

Consider a coset $E_1 = \{n\} \times \mathsf{R} \subseteq \Phi_{\mathfrak{B}_1}$, then $\mathfrak{B}|_{E_1} \cong A(\mathsf{R})$, and so $\mathfrak{A}|_{\alpha(E_1)} \cong A(\mathsf{R})$. However, if $\alpha(E_1) \subseteq \Phi_{\mathfrak{A}_3}$, then $\mathfrak{A}|_{\alpha(E_1)} \cong C_0(\mathsf{R})$. Hence $\alpha(E)$ is either one of the lines in $\Phi_{\mathfrak{A}_1}$ or one of the lines in $\Phi_{\mathfrak{A}_2}$. Similarly, if $E_2 = \{m\} \times \mathsf{R} \subseteq \Phi_{\mathfrak{B}_2}$, then $\alpha(E_2) \subseteq \Phi_{\mathfrak{A}_3}$. Hence $\nu(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \mathfrak{B}_1$ and $\nu(\mathfrak{A}_3) = \mathfrak{B}_2$. For $r = 1, 2$, put $Y_r = \alpha^{-1}(\Phi_{\mathfrak{A}_r}) \subseteq \Phi_{\mathfrak{B}_1}$. Then since the monomorphism $\nu|_{\mathfrak{A}_1}: \mathfrak{A}_1 \to \mathfrak{B}_1$ is a homomorphism of group algebras, $\alpha|_{Y_1}$ is piecewise affine. Thus $Y_1 \in \mathscr{R}(\mathsf{Z} \times \mathsf{R})$ is piecewise-affinely homeomorphic to $\mathsf{Z} \times \mathsf{R}$. By considering the structure of an element of $\mathscr{R}(\mathsf{Z} \times \mathsf{R})$, it is easily shown that $Y_2 = (\mathsf{Z} \times \mathsf{R}) \backslash Y_1 \in \mathscr{R}(\mathsf{Z} \times \mathsf{R})$ is also piecewise-affinely homeomorphic to $\mathsf{Z} \times \mathsf{R}$. Thus $\mathfrak{B}_1|_{Y_2} \cong A(\mathsf{Z} \times \mathsf{R})$, and so $\mathfrak{A}_2 \cong A(\mathsf{Z} \times \mathsf{R})$. This is clearly not the case.

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