MATH. SCAND. 74 (1994), 293–312

ON THE LOCATION OF ZEROS OF SOLUTIONS OF $f'' + A(z)f = 0$ WHERE $A(z)$ IS ENTIRE

SHENGJIAN WU

Abstract.

We investigate the distribution of zero-sequence of solutions of $f'' + Af = 0$, where A is polynomial or transcendental entire, near some rays. Results are obtained concerning the rays near which the exponent of convergence of zeros of the solutions attains its maximal value.

1. Introduction and main results.

Since 1982 there have been many papers on the oscillation theory of the solutions of the differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function. In this paper we shall investigate the distribution of zeros of solutions of (1.1). We first consider the case where $A(z)$ in (1.1) is a polynomial of degree $n \geqq 1$. It follows from the Wiman-Valiron theory that any nontrivial solution of (1.1) is an entire function of order $\frac{n+2}{2}$ [2, Th. 1]. The first general reslt on the exponent of convergence of the zero-sequence of the solutions is the following theorem which was due to Bank and Laine.

THEOREM A [2, Th. 1). *Let $A(z)$ be a polynomial of degree $n \geqq 1$. If f_1 and f_2 are two linearly independent solutions of* (1.1), *then at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is* $\frac{n+2}{2}$.

By generalizing a result of Hellerstein, Shen and Williamson [10], Gundersen [5, Th. 1] proved a stronger result that the conclusion of Theorem A still holds if the zero-sequence is replaced by the nonreal one.

In order to state our results, we need give some definitions.

Received April 16, 1993.

Let $g(z)$ be an entire function in the plane and let $\arg z = \theta \in \mathrm{R}$ be a ray. We denote, for each $\varepsilon > 0$, the exponent of convergence of zero-sequence of $g(z)$ in the angular region $\Omega(\theta - \varepsilon, \theta + \varepsilon) = \{z \mid \theta - \varepsilon \leqq \arg z \leqq \theta + \varepsilon, |z| > 0\}$ by $\lambda_{\theta,\varepsilon}(g)$, and by $\lambda_\theta(g) = \lim_{\varepsilon \to 0} \lambda_{\theta,\varepsilon}(g)$. We also denote the order of growth of $g(z)$ by $\sigma(g)$. We are interested in the distribution of those rays for which $\lambda_\theta(g) = \sigma(g)$. Our first result that concerns the case where A in (1.1) is a polynomial is the following:

THEOREM 1. *Let $A(z)$ be a polynomial of degree $n \geqq 1$ and let f_1 and f_2 be two linearly independent solutions of* (1.1). *If for some real number θ_0*

$$\overline{\lim_{r \to \infty}} \frac{\log \log |E(re^{i\theta_0})|}{\log r} = \frac{n+2}{2}, \tag{1.2}$$

where $E = f_1 f_2$, then there exist θ_1 and θ_2 with $\theta_1 \leqq \theta_0 \leqq \theta_2$ such that $\theta_2 - \theta_1 = \dfrac{2\pi}{n+2}$ and $\lambda_{\theta_1}(E) = \lambda_{\theta_2}(E) = \dfrac{n+2}{2}$.

Since E is of order $\dfrac{n+2}{2}$ [2, Le., A], a routine application of the Phragmen-Lindelöf principle implies that there certainly exists θ such that (1.2) holds. Thus we have the following:

COROLLARY 1. *Let $A(z)$ be a polynomial of degree $n \geqq 1$, and let f_1 and f_2 be two linearly independent solutions of* (1.1). *Then there exist two rays* $\arg z = \theta_1, \theta_2$ *with* $\theta_2 - \theta_1 = \dfrac{2\pi}{n+2}$ *such that* $\max(\lambda_{\theta_1}(f_1), \lambda_{\theta_1}(f_2)) = \max(\lambda_{\theta_2}(f_1), \lambda_{\theta_2}(f_2)) = \dfrac{n+2}{2}$.

Since $\dfrac{2\pi}{n+2} < \pi$ for $n \geqq 1$, Corollary 1 implies Gundersen's result (Theorem 1 in [5]).

We next turn to the case where A in (1.1) is a transcendental entire function of finite order. It is well known that any non-trivial solution of (1.1) is an entire function of infinite order. Let f_1 and f_2 be two linearly independent solutions of (1.1) and let $E = f_1 f_2$. Then $\lambda(E) = +\infty$ is equivalent to $\sigma(E) = +\infty$ [2, Le. B], where $\lambda(E)$ denotes the exponent of convergence of zero-sequence of E. Unlike the case of polynomial, when A is transcendental, the distribution of the rays $\arg z = \theta$ for which $\lambda_\theta(E) = +\infty$ largely depends on the growth of E itself along the rays. If we denote for any $\alpha < \beta$,

$$\Omega(\alpha, \beta) = \{z \mid \alpha \leqq \arg z \leqq \beta, |z| > 0\};$$

$$\Omega(\alpha, \beta, r) = \{z \mid z \in \Omega(\alpha, \beta), |z| < r\};$$

and for an entire function $g(z)$ in the plane,

$$M(r, \Omega(\alpha, \beta), g) = \sup_{\alpha \leqq \theta \leqq \beta} |g(re^{i\theta})|,$$

we may state our next result in the following form.

THEOREM 2. *Let $A(z)$ be a transcendental entire function of finite order in the plane and let f_1, f_2 be two linearly independent solutions of* (1.1). *Set $E = f_1 f_2$. Then $\lambda_\theta(E) = +\infty$, if and only if*

$$\overline{\lim_{r \to \infty}} \frac{\log\log M(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), E)}{\log r} = +\infty \tag{1.3}$$

for any $\varepsilon > 0$.

Especially, if

$$\overline{\lim_{r \to \infty}} \frac{\log\log |E(re^{i\theta})|}{\log r} = +\infty, \tag{1.4}$$

then we have $\lambda_\theta(E) = +\infty$.

Although Theorem 2 provides no information concerning the distribution of zeros of solutions of (1.1) in terms of the entire function $A(z)$, it is possible to get some further results in some cases. We next consider some applications of Theorem 2. Our starting point is the following theorem due to Bank, Laine and Langley.

THEOREM B [3, Th. 1]. *Let $A(z)$ be a transcendental entire function of finite order ρ with the following property: there exists a set $H \subseteq \mathrm{R}$ of measure zero, such that for each real number $\theta \in \mathrm{R} \backslash H$, either*

(1.5) (i) $$r^{-N} |A(re^{i\theta})| \to \infty \text{ as } r \to +\infty, \text{ for each } N > 0,$$

or

(1.6) (ii) $$\int_0^\infty r |A(r)e^{i\theta}| \, dr < +\infty,$$

or

(iii) *there exist positive real numbers K and b, and a nonnegative real number n (all possibly depending on θ), such that $(n+2)/2 < \rho$, and*

$$|A(re^{i\theta})| \leqq Kr^n \quad \text{for all} \quad r \geqq b. \tag{1.7}$$

Then if f_1 and f_2 are linearly independent solutions of

$$f'' + Af = 0,$$

we have

$$\max(\lambda(f_1), \lambda(f_2)) = +\infty.$$

By using Theorem 2, we can prove

THEOREM 3. *Suppose that* $A(z)$ *satisfies the conditions of Theorem* B. *If* f_1 *and* f_2 *are linearly independent solutions of* $f'' + Af = 0$ *and* $\Omega(\alpha, \beta)$ *is an angular region with* $\beta - \alpha > \dfrac{\pi}{\rho}$ *such that there exists a ray* $\arg z = \theta \in (\alpha, \beta)$ *with*

$$\overline{\lim_{r \to \infty}} \frac{\log\log|A(re^{i\theta})|}{\log r} = \rho, \tag{1.8}$$

then there exists at least one ray $\arg z = \theta_0 \in (\alpha, \beta)$ *such that*

$$\max(\lambda_{\theta_0}(f_1), \lambda_{\theta_0}(f_2)) = +\infty.$$

From the Phragment-Lindelöf principle, those θ's such that (1.8) holds always form a union of intervals. Thus we have the following:

COROLLARY 2. *Under the assumption of Theorem* B, *if* $\rho = \sigma(A) > \frac{1}{2}$ *and* f_1 *and* f_2 *are two linearly independent solutions of* $f'' + Af = 0$, *then there exist at least two rays* $\arg z = \theta_1, \theta_2$ *with* $0 < \theta_2 - \theta_1 \leqq \dfrac{\pi}{\rho}$ *such that*

$$\max\{\lambda_{\theta_1}(f_1), \lambda_{\theta_1}(f_2)\} = \max\{\lambda_{\theta_2}(f_1), \lambda_{\theta_2}(f_2)\} = +\infty\cdot$$

Especially, if $\rho > 1$, *then at least one of* f_1 *and* f_2 *has the property that the exponent of convergence of its nonreal zero-sequence is infinite.*

Recently there have also been some results concerning (1.1) with $A = \sum_{j=q}^{m} Q_j \exp(jP)(-\infty < q \leqq m < +\infty)$, where Q_j and P are polynomials (see [1] and [9]). In this direction, we have the following result.

COROLLARY 3. *Let* $J \geqq 1$, *and let* $P_1, \ldots, P_J$ *be nonconstant polynomials whose degrees are* $d_1, \ldots, d_J$ *respectively, and suppose that for* $i \neq j$,

$$\deg(P_i - P_j) = \max(d_i, d_j).$$

Set

$$A(z) = \sum_{j=1}^{J} B_j(z) e^{P_j(z)}$$

where, each j, $B_j(z)$ *is an entire function, not identically zero, of order strictly less than* d_j. *If* f_1 *and* f_2 *are linearly independent solutions of* $f'' + (A + Q)f = 0$, *where* $Q(z)$ *is a polynomial whose degree* m *satisfies* $\dfrac{m+2}{2} \leqq \sigma(A) = \max(d_j)$, *then there* *exists at least one ray* $\arg z = \theta$ *in every angular region* $\Omega(\alpha, \beta)$ *of opening larger than* $\dfrac{\pi}{\sigma(A)}$ *such that*

$$\max(\lambda_\theta(f_1), \lambda_\theta(f_2)) = +\infty.$$

If $J = 1$ in Corollary 3, we can prove a stronger result. For a polynomial

$$P(z) = (x + iy)z^n + \dots + a_0$$

with x, y real, we define, for each real θ,

$$\delta(P, \theta) = x \cos n\theta - y \sin n\theta.$$

Then we can state our result as follows.

THEOREM 4. *Suppose that* $A(z) = B(z)e^{P(z)} \not\equiv 0$, *where* $B(z)$ *is an entire function of order strictly less than the degree of the polynomial* $P(z)$. *If* f_1 *and* f_2 *are two linearly independent solutions of* $f'' + (A + Q)f = 0$, *where* $Q(z)$ *is a polynomial with degree* m *satisfies* $\dfrac{m+2}{2} < \sigma(A)$, *then for any* θ *satisfying* $\delta(P, \theta) = 0$ *we have*

$$\max(\lambda_\theta(f_1), \lambda_\theta(f_2)) = +\infty.$$

2. Preliminaries.

We shall assume that the reader is familiar with the standard notation of Nevenlinna theory (see [4] or [6]). Our proofs require the Nevanlinna characteristic for an angle (see [4], [14]): If $0 < \beta - \alpha \leqq 2\pi$ and $k = \dfrac{\pi}{\beta - \alpha}$ and $g(z)$ is meromorphic on the angular domain $\Omega(\alpha, \beta)$, we denote

$$A_{\alpha\beta}(r, g) = \frac{k}{\pi}\int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right)\left\{\log^+ |g(te^{i\alpha})| + \log^+ g(te^{i\beta})|\right\}\frac{dt}{t};$$

$$B_{\alpha\beta}(r, g) = \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin k(\theta - \alpha)\, d\theta;$$

$$C_{\alpha\beta}(r, g) = 2 \sum_{1 < |b_v| < r} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}}\right) \sin k(\beta_v - \alpha);$$

$$D_{\alpha\beta}(r, g) = A_{\alpha\beta}(r, g) + B_{\alpha\beta}(r, g);$$

$$S_{\alpha\beta}(r, g) = A_{\alpha\beta}(r, f) + B_{\alpha\beta}(r, g) + C_{\alpha\beta}(r, g),$$

where $|b_v| = |b_v|e^{i\beta_v}$ $(v = 1, 2, \dots,)$ are the poles of $g(z)$ in $\Omega(\alpha, \beta)$, counting multiplicities. If we only consider the distinct poles of g, we denote the corresponding angular counting function by $\bar{C}_{\alpha\beta}(r, g)$.

For a positive function $\varphi(r)$, $r \in (0, \infty)$, the order of $\varphi(r)$ is defined by

$$\varlimsup_{r\to\infty} \frac{\log \varphi(r)}{\log r}.$$

Especially the order of $S_{\alpha\beta}(r, g)$ is denoted by $\sigma_{\alpha\beta}(g)$.

3. Lemmas required for the proof of Theorem 1.

LEMMA 1. *Suppose that $g(z)$ ($\not\equiv$ const) is meromorphic in the plane and that $\Omega(\alpha, \beta)$ is an angular domain, where $0 < \beta - \alpha \leqq 2\pi$. Then*

(i) [4, Chap. 1] *for any complex number $a \neq \infty$*

$$S_{\alpha\beta}\left(r, \frac{1}{g-a}\right) = S_{\alpha\beta}(r, g) + \varepsilon(r, a), \tag{3.1}$$

where $\varepsilon(r, a) = O(1)$ $(r \to \infty)$;

(ii) [4, P. 138] *for any $r < R$*

$$A_{\alpha\beta}\left(r, \frac{g'}{g}\right) \leqq K\left\{\left(\frac{R}{r}\right)^k \int_1^R \frac{\log^+ T(t, g)}{t^{1+k}}\,dt + \log^+ \frac{r}{R-r} + \log\frac{R}{r} + 1\right\}, \tag{3.2}$$

and

$$B_{\alpha\beta}\left(r, \frac{g'}{g}\right) \leqq \frac{4k}{r^k} m\left(r, \frac{g'}{g}\right), \tag{3.3}$$

where $k = \dfrac{\pi}{\beta - \alpha}$ and K is a positive constant not depending on r and R.

LEMMA 2 [13, 7, P. 193]. *Suppose that $\Omega(\alpha, \beta)$ and $\Omega(\alpha', \beta')$ are two angular domains such that $\alpha < \alpha' < \beta' < \beta$ and that $g(z)$ is analytic on $\Omega(\alpha, \beta)$. If*

$$\varlimsup_{r\to\infty} \frac{\log\log M(r, \Omega(\alpha', \beta'), g)}{\log r} \equiv \rho(\Omega(\alpha', \beta'), g) > \frac{\pi}{\beta - \alpha}, \tag{3.4}$$

then we have for every a with at most one exception

$$\varlimsup_{r\to\infty} \frac{\log n(\Omega(\alpha, \beta, r), g = a)}{\log r} \geqq \rho(\Omega(\alpha', \beta'), g), \tag{3.5}$$

where $n(\Omega(\alpha, \beta, r), g = a)$ denotes the roots of the equation $g(z) = a$, counting multiplicities, in the sector $\Omega(\alpha, \beta, r)$.

LEMMA 3 [11, *Chap.* 7.4]. *Let $A(z) = a_n z^n + \ldots + a_0$ be a polynomial with $a_n = |a_n| e^{i\alpha_n} \neq 0$ $(0 \leqq \alpha_n < 2\pi)$. Define $\theta_k = \dfrac{\alpha_n + 2k\pi}{n+2}$ for $k = 0, 1, \ldots, n+1$, and* *fix* $\varepsilon > 0$. *If* f *is a solution to* $f'' + Af = 0$, *only finitely many of the zeros of* f *lie outside* $\bigcup_{k=0}^{n+1} \Omega(\theta_k - \varepsilon, \theta_k + \varepsilon)$. *If for some* k, f *has infinitely many zeros in* $\Omega(\theta_k - \varepsilon, \theta_k + \varepsilon)$, *then*

$$n(\Omega(\theta_k - \varepsilon, \theta_k + \varepsilon, r), f = 0) = (1 + o(1))\sqrt{|a_n|}\, r^{\frac{n+2}{2}} \Big/ \frac{\pi(n+2)}{2}. \tag{3.6}$$

4. Proof of Theorem 1.

Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$, where A is a polynomial of degree $n \geqq 1$. Suppose that

$$\overline{\lim_{r\to\infty}} \frac{\log\log|E(re^{i\theta_0})|}{\log r} = \frac{n+2}{2}, \tag{4.1}$$

where $\theta_0 \in \mathrm{R}$ and $E = f_1 f_2$. It follows from the Phragmen-Lindelöf principle that there exists an interval $[\theta_1', \theta_2']$ containing θ_0 such that for all $\theta \in [\theta_1', \theta_2']$ we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log|E(re^{i\theta})|}{\log r} = \frac{n+2}{2}. \tag{4.2}$$

By using Lemma 3, we need only to prove that there exist two rays $\arg z = \theta_1$, θ_2 with $\theta_0 \in (\theta_1, \theta_2)$ such that $\lambda_{\theta_1}(E) = \lambda_{\theta_2}(E) = \frac{n+2}{2}$ and $\theta_2 - \theta_1 \leqq \frac{2\pi}{n+2}$. If this is not true, then there must exist an angular domain $\Omega(\theta_1, \theta_2)$ satisfying the following properties:

(a) $\theta_2 - \theta_1 > \frac{2\pi}{n+2}$;

(b) there exists a ray $\arg z = \theta_3 \in (\theta_1, \theta_2)$ such that (4.2) holds for θ_3;

(c) $\lambda_\theta(E) < \frac{n+2}{2}$ for all $\theta \in [\theta_1, \theta_2]$.

From (c), the definition of $\lambda_\theta(E)$ and the fact that finitely many zeros of f only lie outside of the critical sectors described in Lemma 3, we deduce that

$$\overline{\lim_{r\to\infty}} \frac{\log n(\Omega(\theta_1 + \varepsilon, \theta_2 - \varepsilon, r), E = 0)}{\log r} < \frac{n+2}{2}, \tag{4.3}$$

for every $\varepsilon > 0$.

In order to obtain a contradiction, we choose a fixed $\varepsilon_0 > 0$ such that $\theta_2 - \theta_1 - 6\varepsilon_0 > \frac{2\pi}{n+2}$ and $\theta_3 \in (\theta_1 + 3\varepsilon_0, \theta_2 - 3\varepsilon_0)$. From this choice of ε_0 we have

$$\overline{\lim_{r\to\infty}}\frac{\log\log M(r,\Omega(\theta_1+3\varepsilon_0,\theta_2-3\varepsilon_0),E)}{\log r} \tag{4.4}$$
$$\geqq \overline{\lim_{r\to\infty}}\frac{\log^+\log^+|E(re^{i\theta_3})|}{\log r}$$
$$=\frac{n+2}{2}>\frac{\pi}{\theta_2-\theta_1-4\varepsilon_0}.$$

By using Lemma 2, we have for all $a\in C$ with at most one exception

$$\overline{\lim_{r\to\infty}}\frac{\log n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,r),E=a)}{\log r}=\frac{n+2}{2}. \tag{4.5}$$

Taking a fixed $a\in \mathrm{C}$ such that (4.5) holds, we deduce from (4.5) that there exists a sequence (r_n) of real numbers with $r_n\to+\infty(n\to\infty)$ such that for every $\varepsilon>0$ we have

$$n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,r_n),E=a)\geqq r_n^{\frac{n+2}{2}-\varepsilon}$$

for all sufficiently large n.

Suppose that $a_v=|a_v|\,e^{i\alpha_v}(v=1,2,\ldots)$ are the roots of $E=a$, counting multiplicities, in $\Omega(\theta_1+\varepsilon_0,\theta_2-\varepsilon_0)$. To compute $\sigma_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(E)$, we first observe that $\theta_1+2\varepsilon_0<\alpha_v<\theta_2-2\varepsilon_0$ implies for $k=\dfrac{\pi}{\theta_2-\theta_1-2\varepsilon_0}$ the inequalities

$$k\varepsilon_0<k(\alpha_v-\theta_1-\varepsilon_0)<\pi-k\varepsilon_0,$$

hence

$$\sin k(\alpha_v-\theta_1-\varepsilon_0)\geqq\sin(k\varepsilon_0). \tag{4.6}$$

Moreover, we write a sum below as a Stieltjes-integral:

$$\sum\left(\frac{1}{|a_v|^k}-\frac{|a_v|^k}{(2r_n)^{2k}}\right)=\sum\frac{1}{|a_v|^k}-\sum\frac{|a_v|^k}{(2r_n)^{2k}}$$
$$=\int_1^{r_n}\frac{dn(t)}{t^k}-\frac{1}{(2r_n)^{2k}}\int_1^{r_n}t^k\,dn(t),$$

where a short-hand notation $n(t)=n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,t),E=a)$ will be used. Application of Lemma 1 (i), the formula (4.6) and the partial integration of the above Stieltjes-integrals now results in

$$
\begin{aligned}
(4.7)\quad S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(2r_n, E) &= S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(2r_n, \frac{1}{E-a}\right) + O(1) \\
&\geqq C_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(2r_n, \frac{1}{E-a}\right) + O(1) \\
&= 2\sum_{1<|a_v|<2r_n}\left(\frac{1}{|a_v|^k} - \frac{|a_v|^k}{(2r_n)^{2k}}\right)\sin k(\alpha_v - \theta_1 - \varepsilon_0) + O(1) \\
&\geqq 2\sum_{\substack{1<|a_v|<r_n \\ \theta_1+2\varepsilon_0<a_v<\theta_2-2\varepsilon_0}}\left(\frac{1}{|a_v|^k} - \frac{||a_v|^k}{(2r_n)^{2k}}\right)\sin(k\varepsilon_0) + O(1) \\
&\geqq 2\sin(k\varepsilon_0)\left\{k\int_1^{r_n}\frac{n(t)}{t^{1+k}}dt + \frac{n(r_n)}{r_n^k}\right. \\
&\qquad \left. - \frac{(r_n)^k(r_n)}{(2r_n)^{2k}} + \frac{k}{(2r_n)^{2k}}\int_1^{r_n} t^{k-1}n(t)\,dt\right\} + O(1) \\
&\geqq \left(1 - \frac{1}{2^{2k}}\right)\frac{\sin(k\varepsilon_0)}{r_n^k}n(r_n) + O(1) \\
&\geqq r_n^{\frac{n+2}{2}-k-2\varepsilon}.
\end{aligned}
$$

Therefore we have

$$
\varlimsup_{r\to\infty}\frac{\log S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r, E)}{\log r} \geqq \frac{n+2}{2} - k - 2\varepsilon.
$$

As ε can be arbitrary small, $\sigma_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(E)$ is at least $\dfrac{n+2}{2} - k > 0$.

On the other hand, in [2, P. 354], Bank and Laine proved that

$$
E^2 = c^2\left(\left(\frac{E'}{E}\right)^2 + 2\left(\frac{E''}{E}\right) - 4A\right)^{-1}, \tag{4.8}
$$

where $c \neq 0$ is the Wronskian of f_1 and f_2.

By using Lemma 1 (ii) in which we set $R = 2r$ and the fact that E is of finite order, we deduce that

$$
\begin{aligned}
&D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E'}{E}\right) = O(1), \\
&D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E''}{E}\right) \leqq D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E'}{E}\right) + D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E''}{E'}\right) + O(1) \\
&\qquad = O(1)
\end{aligned}
$$

and

$$S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r,A) = O(1).$$

Thus we have

$$S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r,E) = O\left(\bar{C}_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r,\frac{1}{E}\right) + O(1)\right). \tag{4.9}$$

(4.7) and (4.9) show that the order of $\bar{C}_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r,\frac{1}{E}\right)$ is at least $\frac{n+2}{2} - k > 0$.

Therefore, by Lemma 3, there must be a critical ray $\arg z = \theta_k \in (\theta_1 + \varepsilon_0, \theta_2 - \varepsilon_0)$ with infinitely many zeros around that ray. Hence, by Lemma 3 again, we know that the order of $n(\Omega(\theta_1 + \varepsilon_0, \theta_2 - \varepsilon_0, r), E = 0)$ is $\frac{n+2}{2}$. This contradicts (4.3), proving theorem 1.

5. Discussion of Theorem 1.

REMARK 1. Theorem 1 is sharp. In fact, consider the equation

$$f'' - zf = 0. \tag{5.1}$$

According to a result of Hille [11, chap. 7.4], there exist three pairwise independent solutions $f_k(z)$ $(k = 1,2,3)$ to (5.1) such that for $z \notin \Omega\left(\frac{\pi + 2k\pi}{3} - \varepsilon, \frac{\pi + 2k\pi}{3} + \varepsilon\right)$ and $|z|$ sufficiently large

$$f_k(z) = (1 + o(1))(-z)^{-\frac{1}{4}} \exp\left(\tfrac{2}{3} e^{\frac{i\pi}{2}} (-1)^{k+1} i z^{\frac{3}{2}} (1 + o(1))\right).$$

It is seen that $f_k(z) \to 0$ *in* $\Omega\left(\frac{\pi + 2(k+1)\pi}{3} + \varepsilon, \frac{\pi + 2(k+2)\pi}{3} - \varepsilon\right)$ as $|z|$ tends infinity and $\lambda(f_k) = \frac{3}{2}$. Thus, as Hille observed, $\lambda_\theta(f_k) = \frac{3}{2}$ only when $\theta = \frac{\pi + 2k\pi}{3}$. Therefore $\lambda_\theta(f_k f_{k+1}) = \frac{3}{2}$ only for $\theta = \frac{\pi + 2k\pi}{3}$ and $\frac{\pi + 2(k+1)\pi}{3}$.

REMARK 2. Let θ_k be as defined in Lemma 3 and let f_1 and f_2 be two linearly independent solutions of (1.1). It follows from Lemma 3 that we have in fact proved that there exists an integer k such that $\lambda_{\theta_k}(f_1 f_2) = \lambda_{\theta_{k+1}}(f_1 f_2) = \frac{n+2}{2}$.

It is also easily seen from the proof of Theorem 1 that if $\lambda_{\theta_1}(f_1 f_2) = \frac{n+2}{2}$ for some $\theta_1 \in \mathrm{R}$, then we can find $\theta_2 \in \mathrm{R}$ such that $\lambda_{\theta_2}(f_1 f_2) = \frac{n+2}{2}$ and the magnitude of $\Omega(\theta_1, \theta_2)$ or $\Omega(\theta_2, \theta_1)$ is $\frac{2\pi}{n+2}$.

REMARK 3. When $A(z) = \frac{P(z)}{Q(z)}$ is rational with $n = \mathrm{di}(A) = \text{degree } P - \text{degree } Q \geqq 1$ in (1.1), using our methods with obvious modifications we can prove that the conclusion of Theorem 1 remains true provided that $E = f_1 f_2$ is transcendental [8, Th. 1].

6. Lemmas required for the proofs of Theorem 2–4.

To prove Theorem 2 and Theorem 3, we need some estimates, restricted in an angle, for the logarithmic derivative of an entire function. The first lemma in this section is due to A. Mokhon'ko.

LEMMA 4 [12]. *Let $z = r\exp(i\varphi)$, $r_0 + 1 < r$ and $\alpha < \varphi < \beta$, where $0 < \beta < \alpha \leqq 2\pi$. If $g(z)$ is meromorphic in the angular region $\Omega(\alpha, \beta)$ and $\sigma_{\alpha\beta}(g)$ is finite, then there exist $K_1 > 0$ and $M_1 > 0$ depending only on g and $\Omega(\alpha, \beta)$, and not depending on z, such that*

$$\left|\frac{g'(z)}{g(z)}\right| \leqq K_1 r^{M_1} (\sin k(\varphi - \alpha))^{-2} \tag{6.1}$$

for all $z \notin D_1$, where $k = \frac{\pi}{\beta - \alpha}$ and D_1 is an R-set, that is, a countable union of discs whose radii have finite sum.

As an application we may estimate the growth of $\frac{g''}{g}$ where g is regular in an angle.

LEMMA 5. *Let $z = r\exp(i\varphi)$, $r_0 + 1 < r$ and $\alpha \leqq \varphi \leqq \beta$, where $0 < \beta - \alpha \leqq 2\pi$. If $g(z)$ is regular in $\Omega(\alpha, \beta) \cap (|z| \geqq r_0)$ and $\sigma_{\alpha\beta}(g)$ is finite, then for every $\varepsilon \in \left(0, \frac{\beta - \alpha}{2}\right)$ except for a set of ε with linear measure zero, there exist $K > 0$ and $M > 0$ depending only on g, ε and $\Omega(\alpha, \beta)$, and not depending on z such that*

$$\left|\frac{g''(z)}{g(z)}\right| \leqq K r^M (\sin k(\varphi - \alpha) \sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2} \tag{6.2}$$

for all $z \in \Omega(\alpha + \varepsilon, \beta - \varepsilon)$ *outside an R-set D, where* $k = \dfrac{\pi}{\beta - \alpha}$ *and* $k_\varepsilon = \dfrac{\pi}{\beta - \alpha - 2\varepsilon}$.

PROOF. Since $\sigma_{\alpha\beta}(g)$ is finite, it follows from Lemma 2 that for every $\varepsilon > 0$ there exists $K_\varepsilon (0 < K_\varepsilon < +\infty)$ such that

$$\overline{\lim_{r \to \infty}} \frac{\log \log M(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), g)}{\log r} < K_\varepsilon. \tag{6.3}$$

From (6.3) we have

$$\log |g(re^{i(\alpha+\varepsilon)})| < r^{K_\varepsilon + 1},$$
$$\log |g(re^{i(\beta-\varepsilon)})| < r^{K_\varepsilon + 1}$$

and

$$\log |g(re^{i\theta})| < r^{K_\varepsilon + 1}$$

for all large r and all $\theta \in [\alpha + \varepsilon, \beta - \varepsilon]$. Noting that $g(z)$ is regular in $\Omega(\alpha, \beta)$, we deduce from the definition of the Nevanlinna angular characteristic that $\sigma_{\alpha+\varepsilon, \beta-\varepsilon}(g)$ is finite.

Let D_1 be the R-set in Lemma 4. Then the set of ε for which the rays $\arg z = \alpha + \varepsilon$ or $\beta - \varepsilon$ meet D_1 infinitely often (i.e., meet infinitely many discs in D_1) has measure zero. Suppose that ε is a number such that $0 < \varepsilon < \dfrac{\beta - \alpha}{2}$ and $\arg z = \alpha + \varepsilon$ and $\beta - \varepsilon$ meet D_1 at most finitely many times. For such ε, we have

$$\begin{aligned} S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g') &= D_{\alpha+\varepsilon, \beta-\varepsilon}(r, g') \\ &\leqq D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) + D_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1) \\ &= D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) + S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1). \end{aligned} \tag{6.4}$$

If $|z| = r$ does not meet D_1, by using Lemma 4 and from the definition of $D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \dfrac{g'}{g}\right)$, we have

$$D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) = O(1). \tag{6.5}$$

Combining (6.4) and (6.5), we deduce that

(6.6) $$S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g') \leqq S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1)$$

for all r except for a set of r with finite linear measure. This implies that $\sigma_{\alpha+\varepsilon, \beta-\varepsilon}(g')$ is finite.

Applying Lemm 4 to $g'(z)$ and $\Omega(\alpha + \varepsilon, \beta - \varepsilon)$, if $z = re^{i\varphi}$, $r_0 + 1 < r$ and $\alpha + \varepsilon < \varphi < \beta - \varepsilon$, then there exists $K_2 > 0$ and $M_2 > 0$ depending only on $g'(z)$ and $\Omega(\alpha + \varepsilon, \beta - \varepsilon)$ such that

$$\left|\frac{g''(z)}{g'(z)}\right| \leqq K_2 r^{M_2}(\sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2}$$

for all $z \notin D_2$, where D_2 is an R-set. Thus if $z \notin D_1 \cup D_2$, we have

$$\begin{aligned}\left|\frac{g''(z)}{g(z)}\right| &\leqq \left|\frac{g''(z)}{g'(z)}\right|\left|\frac{g'(z)}{g(z)}\right| \\ &\leqq K_1K_2r^{M_1+M_2}(\sin k(\varphi - \alpha)\sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2}.\end{aligned}$$

Using K, M, D instead of K_1K_2, $M_1 + M_2$, $D_1 \cup D_2$, we obtain (6.2).

7. Proof of Theorem 2.

Suppose that $f(z)$ is a nontrivial solution to $f'' + Af = 0$. Then

(7.1) $$\frac{f''}{f} \equiv -A.$$

We apply Wiman-Valiron theory to (7.1). Hence there exists a set $D \subset [1, \infty)$ of finite logarithmic measure such that if $r \notin D$ and z is a point on $|z| = r$ at which $|f(z)| = M(r, f)$, then

(7.2) $$\left|\frac{f''(z)}{f(z)}\right| = \left(\frac{v(r)}{r}\right)^2 (1 + \eta(z)) = |A(z)| \leqq M(r, A),$$

where $\eta(z) \to 0$ (as $|z| \to \infty$) and $v(r)$ denotes the central index of f. Thus we have [4, pp. 360–361]

(7.3) $$v(r) \leqq 4r(M(2r, A))^{\frac{1}{2}}$$

for all sufficiently large r. (7.3) implies that the order of $\log T(r, f)$ is at most $\sigma(A)$.

Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$ and $E = f_1f_2$. The above argument implies that $\sigma(\log T(r, E)) \leqq \sigma(A)$, since

$$\begin{aligned}T(r, E') &= m(r, E') \\ &\leqq 2m(r, f_1) + 2m(r, f_2) + m\left(r, \frac{f_1'}{f_1}\right) + m\left(r, \frac{f_2'}{f_2}\right) + O(1),\end{aligned}$$

we deduce that $\sigma(\log T(r, E')) \leqq \sigma(A)$. Thus if ε is sufficiently small, we deduce from Lemma 1 (ii) in which we set $R = 2r$ that

$$A_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O\left(\int_1^{2r} \frac{\log^+ T(t, E)}{t^{1+\frac{\pi}{2\varepsilon}}}\, dt\right)$$

$$= O\left(\int_1^{2r} \frac{t^{\sigma(A)+1}}{t^{1+\frac{\pi}{2\varepsilon}}}\, dt\right)$$

$$= O(1).$$

Since [6, P. 36]

$$m\left(r, \frac{E'}{E}\right) = O(\log^+ T(2r, E) + \log r)$$

$$= O(r^{\sigma(A)+1}),$$

we deduce from (3.3) that

$$B_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O\left(r^{-\frac{\pi}{2\varepsilon}} m\left(r, \frac{E'}{E}\right)\right)$$

$$= O(r^{\sigma(A)+1-\frac{\pi}{2\varepsilon}})$$

$$= O(1).$$

Therefore we have

(7.4) $$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O(1).$$

Similarily we have

(7.5) $$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E''}{E}\right) \leqq D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) + D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E''}{E'}\right) + O(1)$$

$$= O(1)$$

and

(7.6) $$D_{\theta-\varepsilon,\theta+\varepsilon}(r, A) = O(1),$$

for any $\theta \in \mathrm{R}$.

From the Nevanlinna theory it follows from (7.4), (7.5), (7.6) and (4.8) that

(7.7) $$S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) = O\left(\bar{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{E}\right) + O(1)\right)$$

for all sufficiently small $\varepsilon > 0$.

Now suppose that $\theta_0 \in \mathrm{R}$ such that for any sufficiently small $\varepsilon > 0$

$$\overline{\lim_{r\to\infty}} \frac{\log\log M\left(r, \Omega\left(\theta_0 - \frac{\varepsilon}{3}, \theta_0 + \frac{\varepsilon}{3}\right), E\right)}{\log r} = +\infty,$$

by using Lemma 2, we can find a complex number a such that

$$\overline{\lim_{r\to\infty}} \frac{\log n(\Omega(\theta_0 - \frac{2}{3}\varepsilon, \theta_0 + \frac{2}{3}\varepsilon, r), E = a)}{\log r} = +\infty.$$

As we did in the proof of (4.7), we deduce that $\sigma_{\theta_0 - \varepsilon, \theta_0 + \varepsilon}(r, E)$ is infinite. It follows from (7.7) that $\lambda_{\theta_0}(E) = +\infty$.

On the other hand, if there exists $\varepsilon_0 > 0$ such that

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0), E)}{\log r} \leqq K < +\infty,$$

as we did in the proof of Lemma 5, we know that $\sigma_{\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0}(E)$ must be finite. As in the proof of (4.7) we deduce that the order of $n(\Omega(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0, r), E = 0)$ is finite. Since $\lambda_{\theta_0,\varepsilon}(E) \leqq \lambda_{\theta_0,\varepsilon_0}(E)$ for any $\varepsilon (0 < \varepsilon \leqq \varepsilon_0)$, therefore $\lambda_{\theta_0}(E) < +\infty$. The proof of Theorem 2 is completed.

8. Proof of Theorem 3.

Observe first that if $\rho \leqq \frac{1}{2}$, then $\beta - \alpha > 2\pi$. Since $\lambda(E) = +\infty$ [15], we see easily from the definition of $\lambda_\theta(E)$ that there exists at least one ray $\arg z = \theta_0$ such that $\lambda_{\theta_0}(E) = +\infty$. In the following we assume that $\rho > \frac{1}{2}$.

Suppose that $\Omega(\alpha, \beta)$ is an arbitrary angular domain with $\beta - \alpha > \dfrac{\pi}{\rho}$ and that there exists a ray $\arg z = \theta_0$ such that $\alpha < \theta_0 < \beta$ and (1.8) holds. It follows from the Phragmen-Lindelöf principle that there exists an interval $[\theta_1, \theta_2]$ containing θ_0 such that (1.8) holds for all $\theta \in [\theta_1, \theta_2]$. So we may suppose $[\theta_1, \theta_2] \subset (\alpha, \beta)$. Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$. If there is no ray $\arg z = \theta$ with $\alpha < \theta < \beta$ such that $\lambda_\theta(E) = +\infty$, where $E = f_1 f_2$, we shall derive a contradiction.

We choose a fixed $\varepsilon_0 > 0$ such that $\beta - \alpha - 4\varepsilon_0 > \dfrac{\pi}{\rho}$ and $4\varepsilon_0 < \theta_2 - \theta_1$. From Theorem 2 we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0), E)}{\log r} < +\infty. \tag{8.1}$$

Consequently we deduce from (8.1) that $\sigma_{\alpha+\varepsilon_0,\beta-\varepsilon_0}(E)$ is finite. Now we claim that there exists θ_1', θ_2' with $\alpha + \frac{3}{2}\varepsilon_0 < \theta_1' < \alpha + 2\varepsilon_0$ and $\beta - 2\varepsilon_0 < \theta_2' < \beta - \frac{3}{2}\varepsilon_0$ such that $\sigma_{\theta_1',\theta_2'}(E)$ is at least $\rho - \dfrac{\pi}{\theta_2' - \theta_1'}$.

In fact, by using Lemma 4 and 5, we may choose a fixed $\varepsilon \in (\varepsilon_0, \frac{5}{4}\varepsilon_0)$ such that

$$\left|\frac{E'}{E}(re^{i\varphi})\right| \leqq Kr^M(\sin k(\varphi - \alpha - \varepsilon_0))^{-2} \tag{8.2}$$

$$\left|\frac{E''}{E}(re^{i\varphi})\right| \leqq Kr^M(\sin k_\varepsilon(\varphi - \alpha - \varepsilon)\sin k(\varphi - \alpha - \varepsilon_0))^{-2} \tag{8.3}$$

for all $z = re^{i\varphi} \in \Omega(\alpha + \varepsilon, \beta - \varepsilon)$ outside an R-set D, where $k = \dfrac{\pi}{\beta - \alpha - 2\varepsilon_0}$, $k_\varepsilon = \dfrac{\pi}{\beta - \alpha - 2\varepsilon}$, and K and M are constants depending only on $\Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0)$, E and ε, and not depending on r and φ. Since the set of θ for which the ray $\arg z = \theta$ meets D infinitely often (i.e., meets infinitely many discs of D) has measure zero, we can find two rays $\arg z = \theta_1', \theta_2'$ with $\alpha + \frac{3}{2}\varepsilon_0 < \theta_1' < \alpha + 2\varepsilon_0$ and $\beta - 2\varepsilon_0 < \theta_2' < \beta - \frac{3}{2}\varepsilon_0$ such that they only meet finitely many discs in D. So if r is sufficiently large,

$$\{z = re^{i\theta} \mid \theta = \theta_1' \text{ or } \theta_2'\} \cap D = \phi. \tag{8.4}$$

From

$$4A = \left(\frac{E'}{E}\right)^2 - 2\left(\frac{E''}{E}\right) - \frac{c^2}{E^2},$$

where $c = W(f_1, f_2) \neq 0$, we have

$$\begin{aligned} S_{\theta_1'\theta_2'}(r, E) &= S_{\theta_1'\theta_2'}\left(r, \frac{1}{E}\right) + O(1) \\ &\geqq \tfrac{1}{2} S_{\theta_1'\theta_2'}\left(r, \frac{c^2}{E^2}\right) + O(1) \\ &\geqq \tfrac{1}{2} D_{\theta_1'\theta_2'}\left(r, \frac{c^2}{E^2}\right) + O(1) \\ &= \tfrac{1}{2} D_{\theta_1'\theta_2'}\left(r, 4A + 2\left(\frac{E''}{E}\right) - \left(\frac{E'}{E}\right)^2\right) + O(1) \\ &\geqq \tfrac{1}{2}\left(D_{\theta_1'\theta_2'}(r, A) - 2D_{\theta_1'\theta_2'}\left(r, \frac{E'}{E}\right) - D_{\theta_1'\theta_2'}\left(r, \frac{E''}{E}\right)\right) + O(1) \end{aligned} \tag{8.5}$$

$$= \tfrac{1}{2}S_{\theta'_1\theta'_2}(r, A) - D_{\theta'_1\theta'_2}\left(r, \frac{E'}{E}\right) - \tfrac{1}{2}D_{\theta'_1\theta'_2}\left(r, \frac{E''}{E}\right) + O(1).$$

From the choice of θ'_1 and θ'_2, as we did in the proof of (4.6), we deduce that $\sin k(\varphi - \alpha - \varepsilon_0) \geqq \sin \dfrac{k\varepsilon_0}{4}$ for all $\varphi \in [\theta'_1, \theta'_2]$. If $\varphi \in [\theta'_1, \theta'_2]$ and $z = re^{i\varphi}$ lie outside D, we deduce from (8.2) and (8.3) that

$$\left|\frac{E'}{E}(re^{i\varphi})\right| \leqq K\left[\sin\left(\frac{k\varepsilon_0}{4}\right)\right]^{-2} r^M \tag{8.6}$$

and

$$\left|\frac{E''}{E}(re^{i\varphi})\right| \leqq K\left[\sin\left(\frac{k\varepsilon_0}{4}\right)\sin\left(\frac{k_\varepsilon\varepsilon_0}{4}\right)\right]^{-2} r^M. \tag{8.7}$$

From (8.4), (8.6), (8.7) and definition of $D_{\theta'_1,\theta'_2}$, we have

$$D_{\theta'_1\theta'_2}\left(r, \frac{E'}{E}\right) + D_{\theta'_1\theta'_2}\left(r, \frac{E''}{E}\right) = O\left(\int_1^r \frac{\log t}{t^{1+\frac{\pi}{\theta'_1\theta'_2}}}dt + \frac{\log r}{r^{1+\frac{\pi}{\theta'_1\theta'_2}}}\right) = O(1) \tag{8.8}$$

for all r except for a set of r with finite linear measure. It follows from (8.5) and (8.8) that

$$S_{\theta'_1\theta'_2}(2r, E) \geqq \tfrac{1}{6}S_{\theta'_1\theta'_2}(r, A), \tag{8.9}$$

for all larger r.

We next show that the order of $S_{\theta'_1\theta'_2}(r, A)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. In fact, since there exists a ray $\arg z = \theta \in [\alpha + 3\varepsilon_0, \beta - 3\varepsilon_0] \subset (\theta'_1, \theta'_2)$ such that (1.8) holds, we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\alpha + 3\varepsilon_0, \beta - 3\varepsilon_0), A)}{\log r} = \rho > \frac{\pi}{\beta - \alpha - 4\varepsilon_0}. \tag{8.10}$$

By using Lemma 2, we deduce that there exists a complex number a such that the order of $n\{\Omega(\alpha + 2\varepsilon_0, \beta - 2\varepsilon_0, r), A = \alpha\}$ is ρ. As in the proof of (4.7), we deduce that $\sigma_{\theta'_1\theta'_2}(A)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. From (8.9) we know that $\sigma_{\theta'_1\theta'_2}(E)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. The claim is proved.

From the claim we must have

(8.11) $$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_1', \theta_2'), E)}{\log r} \geqq \rho,$$

otherwise by direct calculation we deduce that $\sigma_{\theta_1'\theta_2'}(r, E)$ is less than $\rho - \dfrac{\pi}{\theta_2' - \theta_1'}$.

On the other hand, since $A(z)$ satisfies the condition (i), (ii) and (iii) of the Theorem B and $\sigma_{\alpha+\varepsilon_0, \beta-\varepsilon_0}(E)$ is finite, by using Lemma 4 and 5 and in the similarity to the proof of Theorem 1 in [3], we deduce that for every $\theta \in [\alpha + \frac{5}{4}\varepsilon_0, \beta - \frac{5}{4}\varepsilon_0]$ except for a set of θ with linear measure zero there exists $r(\theta) > 0$ such that

$$\log^+ |E(re^{i\theta})| < 0(r^{\rho-\varepsilon}) \quad r > r(\theta), \varepsilon = \varepsilon(\theta) > 0.$$

It is from (8.1) that the Phragmen-Lindelöf principle is applicable in $\Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0)$. Therefore we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_1 + \frac{3}{2}\varepsilon_0, \theta_2 - \frac{3}{2}\varepsilon_0), E)}{\log r} \leqq \rho - \varepsilon$$

for some $\varepsilon > 0$. This contradicts (8.11). Theorem 3 is completely proved.

9. Proof of Corollary 3.

It was shown in [3, Le. 5] that $A(z)$ satisfies the assumption of Theorem 3 and that for some j, $A(re^{i\theta}) = B_j(re^{i\theta})e^{P_j(re^{i\theta})}(1 + o(1))$ as $r \to \infty$ with $z = re^{i\theta}$ outside a fixed R-set. Thus every angular domain $\Omega(\alpha, \beta)$ with $\beta - \alpha > \dfrac{\pi}{\rho}$ where $\rho = \sigma(A) =$ degree P_j, must contain a ray $\arg z = \theta \in (\alpha, \beta)$ such that (1.8) holds. The corollary follows.

10. Proof of Theorem 4.

Suppose that degree $P = n$ and that $\delta(\theta_0, P) = 0$ for $\theta_0 \in R$. Then $\delta(\theta, P) < 0$ for all $\theta \in \left(\theta_0 - \dfrac{\pi}{n}, \theta_0\right)$ or $\left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$. We assume $\delta(\theta, P) < 0$ for $\theta \in \left(\theta_0 - \dfrac{\pi}{n}, \theta_0\right)$ (the case $\delta(\theta, P) < 0$ for $\theta \in \left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$ can be similarly treated). When $z = re^{i\theta} \in \Omega\left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$ and lies outside an R-set D, we have [3, Le. 3]

$$|A(z)| \geqq \exp(\tfrac{1}{2}\delta(P,\theta)r^n). \tag{10.1}$$

Thus for any $\varepsilon > 0$, there exists $\theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 + \varepsilon\right)$ such that

$$\overline{\lim_{r\to\infty}} \frac{\log\log(|A(re^{i\theta}) + Q(re^{i\theta})|)}{\log r} = n.$$

It follows from Theorem 3 that $\lambda_{\theta_1}(E) = +\infty$ for some $\theta_1 \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 + \varepsilon\right)$, where $E = f_1 f_2$.

Since the order of $B(z)$ is strictly less than n, there exists $\sigma < n$ such that

$$M(r,B) = \max_{0 \leqq \theta \leqq 2\pi} |B(re^{i\theta})| < \exp(r^\sigma).$$

If $\theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right)$,

$$\begin{aligned} |A(re^{i\theta}) + Q(re^{i\theta})| &\leqq M(r,B)\exp(\tfrac{1}{2}\delta(P,\theta)r^n) + r^m \\ &\leqq M(r,B)\exp(-Kr^n) + r^m \\ &\leqq \exp(-Kr^n) + r^m \end{aligned}$$

where $K > 0$ is a constant depending only on ε.

It follows from [3, Lemma 2] that there exists $b > 0$ such that every solution f of $f'' + (A + Q)f = 0$ satisfies

$$\log^+ |f(re^{i\theta})| \leqq Kr^{\frac{m}{2}+1}$$

for all $\theta \in \left[\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right]$ and for all $r > b$, where K is a constant depending only on ε.

Thus we have

$$\log^+ |E(re^{i\theta})| \leqq 2Kr^{\frac{m}{2}+1} \leqq r^{n-\varepsilon},$$

for all $\theta \in \left[\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right]$. This implies that

$$\overline{\lim_{r\to\infty}} \frac{\log\log M\left(r, \Omega\left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right), E\right)}{\log r} \leqq n - \frac{\varepsilon}{2}.$$

It follows from Theorem 2 that there is no ray $\arg z = \theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right)$ such that $\lambda_\theta(E) = +\infty$. So we must have $\lambda_{\theta_0 - \varepsilon, \theta_0 + \varepsilon}(E) = +\infty$ for any $\varepsilon > 0$. Therefore $\lambda_{\theta_0}(E) = +\infty$. The proof of Theorem 4 is completed.

ACKNOWLEDGEMENT. The author wish to thank Professor S. Bank, Professor I. Laine and the referee for their valuble suggestions and comments.

REFERENCES

1. S. Bank, *On determining the location of complex zeros of solutions of certain differential equations*, Acta Math. Pura ed Appl. 51 (1988), 67–96.
2. S. Bank and I. Laine, *On the oscillation theory of $f'' + Af = 0$ where A is entire*, Trans. Amer. Math. Soc. 273 (1982), 351-363.
3. S. Bank, I. Laine and J. Langley, On the frequency of zeros of solutions of second order linear differential equations, Resultate Math., 10 (1986), 8-24.
4. A. A. Goldberg and I. V. Ostrovskii, *The distribution of values of meromorphic functions*, (in Russian), Izdat Nauk Moscow, 1970.
5. G. G. Gundersen, *On the real zeros of solutions of $f'' + A(z) = 0$ where $A(z)$ is entire*, Ann. Acad. Sci. Fenn. Ser. AI Math 11 (1986), 275–294.
6. W. K. Hayman, Meromorphic Functions, Oxford, 1964.
7. W. K. Hayman and L. Yang, *Growth and values of functions regular in an angle*, Proc. London Math. Soc. (3) 44 (1982), 193–214.
8. S. Hellerstein and J. Rossi, *Zeros of meromorphic solutions of second order linear differential equations*, Math. Z. 192 (1986), 603–612.
9. S. Hellerstein, S. and J. Rossi, *On the distribution of solutions of second-order differential equations*, Complex Variables 13 (1989), 99–109.
10. S. Hellerstein, L. C. Shen and J. Williamson, *Real zeros of derivatives of meromorphic functions and solutions of second order differential equations*, Trans. Amer. Math. Soc. 285 (1984), 759–776.
11. E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley publishing Company, Reading, Massachusetts-Menlo Park, Califormin-London-Don Mills, Ontario, 1969.
12. A. Z. Mokhon'ko, *An estimate of the modulus of the logarithmic derivative of a function which is meromorphic in an angular region, and its application*, Ukrainskii matematicheskii zhurnal, 41, (6) (1989), 839–843, English translation of Ukrain Math. 1990, 722–725.
13. R. Nevanlinna, *Untersuchungen über den Picardschen Satz*, Acta. Soc. Sci. Fenn. 50 No. 6 (1924).
14. R. Nevanlinna, *Über die Eigenschaften meromorpher Funktionen in einem Winkelraum*, Acta Soc. Sci. Fenn. 50 (1925), 1–45.
15. J. Rosssi, *Second order differential equation with transcendental coefficients*, Proc. Amer. Math. Soc. 97 (1986), 61–66.

DEPARTMENT OF MATHEMATICS
PEKING UNIVERSITY
BEIJING 100871
P.R. CHINA