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ON n-SUM GROUPS

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An innocent example in [3, p. 82] asks the student to prove that no group can be the set-theoretic union of two of its proper subgroups. Of course, as the example of $C_2 \times C_2$ shows, a group can indeed be the sum, i.e. the set-theoretic union, of three of its proper subgroups, which leads naturally to the

DEFINITION. A group G is said to be *an n-sum group* if it can be written as the sum of n of its proper subgroups and of no smaller number. We then write $\sigma(G) = n$.

It is the object of this note to consider the properties of $\sigma(G)$. Some previous work is to be found in [1] and [2].

It is obvious that a cyclic group can never be so written, since no subgroup containing a generator can be proper. Equally obviously, any finite non-cyclic group is the sum of the proper subgroups generated by single elements. We assume without further mention that all our groups are of this type. For convenience we shall write $\sigma(G) = \infty$ if G is cyclic.

We then write $G = \sum_{r=1}^{n} H_r$, where for each r, H_r is a proper subgroup, which can be assumed to be maximal where convenient. We suppress suffices where there is no risk of confusion, and also usually assume the indices $i(H_r) = i_r$ to be non-decreasing, so that the subgroups are arranged in non-increasing order.

THEOREM 1. *If* $G = \sum_{r=1}^{n} H_r$ *then* $|G| \leqq \sum_{r=2}^{n} |H_r|$, *with equality if and only if* (a) $H_1 H_r = G$, $r \neq 1$ *and* (b) $H_r \cap H_s \subset H_1$, $r \neq s$.

PROOF. The number of elements of H_r not contained within H_1 equals

$$\begin{aligned}|H_r| - |H_1 \cap H_r| &= |H_r|\{1 - |H_1|/|H_1 H_r|\}\\ &\leqq |H_r|\{1 - |H_1|/|G|\}\end{aligned}$$

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and so

$$|G| \leqq |H_1| + \{1 - |H_1|/|G|\} \sum_{r=2}^{n} |H_r|$$

from which the inequality follows. Equality occurs if and only if there is both equality at each stage above, i.e. $H_1 H_r = G, r \geqq 2$, and if no two subgroups have any common element not also contained in H_1, i.e. $H_r \cap H_s \subset H_1, r \neq s$.

If the subgroups are ordered as above, we obtain immediately

LEMMA 1. *If* $\sigma(G) = n$, *then* $i_2 \leqq n - 1$.

LEMMA 2. *If* $N \lhd G$, *then* $\sigma(G) \leqq \sigma(G/N)$.

For, if $G/N = \sum H$, then $G = \sum HN$.

COROLLARY. $\sigma(H \times K) \leqq \min\{\sigma(H), \sigma(K)\}$.

Thus for any given n-sum group, we can construct new n-sum groups, and so are led to the

DEFINITION. A group G is said to be *a primitive n-sum group* if $\sigma(G) = n$ and G has no normal subgroup N for which $\sigma(G/N) = n$.

Since every maximal subgroup of G contains the Frattini subgroup Φ as a subgroup and $\Phi \lhd G$, it follows that any primitive n-sum group has $|\Phi| = 1$.

The case $n = 3$ has been dealt with in [1], where in the present notation it is proved that the only primitive 3-sum group is $C_2 \times C_2$.

LEMMA 3. *If p is prime,* $C_p \times C_p$ *is a primitive* $(p + 1)$*-sum group.*

For every proper subgroup has index p, and so $\sigma \geqq (p + 1)$ by Lemma 1. If G is generated by elements a and b then G is the sum of the $p + 1$ cyclic subgroups generated by ab^r, $0 \leqq r \leqq p - 1$ and by b. That G is primitive follows on observing that every factor group of G is cyclic.

THEOREM 2. *If G is a non-cyclic p-group, then* $\sigma(G) = p + 1$, *and G is a primitive* $(p + 1)$*-sum group only if it is* $C_p \times C_p$.

PROOF. (1) Since every proper subgroup has index p at least, $\sigma(G) \geqq p + 1$ by Lemma 1.

(2) If G is non-cyclic and abelian, then it is the direct product of two abelian p-groups, and so G has a factor group isomorphic to $C_p \times C_p$. Thus $\sigma(G) \leqq p + 1$ by Lemma 2 and 3.

(3) If $|G| = p^k$ with $k \geqq 2$, we prove $\sigma(G) \leqq p + 1$ by induction on k. For $k = 2$, G is $C_p \times C_p$ and the result follows from Lemma 3. For $k \geqq 3$, if G is abelian the

result is shown above, whereas otherwise $G/Z(G)$ is a smaller non-cyclic p-group, and so the result follows from the inductive hypothesis.

(4) It is evident from the above that the only primitive case is $C_p \times C_p$.

LEMMA 4. *If* $(|H|, |K|) = 1$ *then* $\sigma(H \times K) = \min\{\sigma(H), \sigma(K)\}$.

PROOF. $G = H \times K$ is cyclic if and only if both H and K are cyclic, and then there is nothing to prove. Otherwise suppose $\sigma(G) = n$. Since $(|H|, |K|) = 1$, any subgroup of G is of the form $X \times Y$ where $X \subset H$, $Y \subset K$, and for any maximal subgroup of G, either $X = H$ and Y is a maximal subgroup of K or vice-versa. Thus

$$G = \sum_{r=1}^{p} H \times Y_r + \sum_{s=1}^{q} X_s \times K = G_1 + G_2.$$

say, where $p + q = n$; $p \geqq 0$; $q \geqq 0$.

We shall show that one of p and q must vanish. For if $q \neq 0$, then $G_1 \neq G$ and so there exists an element $(h_1, k_1) \notin G_1$. Then also $(h, k_1) \notin G_1$ for any $h \in H$, whence $(h, k_1) \in G_2$ for every $h \in H$. But then $(h, k) \in G_2$ for every $h \in H$ and $k \in K$, so that $G_2 = G$, whence $p = 0$.

Now if $p = 0$, then $G = G_2 = \left(\sum_{s=1}^{n} X_s\right) \times K$, and so $H = \sum_{s=1}^{n} X_s$ whence $\sigma(H) \leqq n = \sigma(G)$. Similarly if $q = 0$ then $\sigma(K) \leqq n = \sigma(G)$. In either case, the result follows from the corollary to Lemma 2.

THEOREM 3. *If G is a non-cyclic nilpotent group then $\sigma(G) = p + 1$, where p is the smallest prime for which the Sylow p-sum group is non-cyclic. The only nilpotent primitive $(p + 1)$-subgroup is $C_p \times C_p$.*

PROOF. Since G is the direct product of its Sylow subgroups, it follows from Lemma 4 that $\sigma(G) = \min\{\sigma(S)\}$ for the Sylow subgroups S of G, and the result then follows by Theorem 2.

LEMMA 5. *If $G = \sum_{r=1}^{n} H_r$ where $\sigma(G) = n$, and L is a subgroup of all save possibility one of the subgroups H_r, then it is a subgroup of them all.*

For, if $L \subset H_r$ for each $r \neq$ k, let $a \in H_k$, but $a \notin H_r$ for $r \neq k$. Then also $aL \cap H_r = \emptyset$ for $r \neq k$, and so $aL \subset H_k$. Thus $L \subset H_k$.

THEOREM 4. *If G is a primitive n-sum group, then either $G \sim C_p \times C_p$ for some prime p, or else $|Z(G)| = 1$.*

PROOF. If G is abelian the result follows by Theorem 3. Suppose if possible that G is non-abelian, with non-trivial centre, and let p denote a prime factor of $|Z(G)|$.

Let u be an element of $Z(G)$ of order p, and U be the subgroup generated by u. Then $U \lhd G$ and so if $\sigma(G) = n$ and $G = \sum_{r=1}^{n} H_r$ with each H_r maximal, since G is primitive there exists at least one H_r of which U is not a subgroup. By the previous lemma, there must exist at least two such, H and K, say.

Since $U \lhd G$, HU is a subgroup of G which contains H as a proper subgroup, and so since H was assumed maximal, $HU = G$, i.e. $G = \sum_{i=0}^{p-1} Hu^i$. Also since $u^i \in Z(G)$, $u^iH = Hu^i$ and so $H \lhd G$ and $i(H) = p$. Similarly for K. Also since the elements of H and of U commute and $G = HU$, $G \sim H \times U$ and so since G was assumed to be a primitive n-sum group, $\sigma(G) = n < \sigma(H)$. Since both H and K are maximal and are normal subgroups of index p, $X = H \cap K$ is a normal subgroup of index p^2. Then $X \lhd H$ and since $|H/X| = p$, $H/X \sim C_p$, and so $H = \sum_{j=0}^{p-1} Xv^j$ where $v \in H$, $v \notin X$ and $v^p \in X$. Since $u \in Z(G)$ we obtain

$$\begin{aligned} G &= \sum_{i=0}^{p-1} Hu^i \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} Xu^iv^j \\ &= \sum_{j=0}^{p-1} Xv^j + \sum_{\rho=0}^{p-1} \left\{ \sum_{\tau=0}^{p-1} X(uv^{\rho})^{\tau} \right\} \\ &= H + \sum_{\rho=0}^{p-1} B_\rho, \quad \text{say} \end{aligned}$$

and it is easily seen that for $0 \leqq \rho \leqq p - 1$, B_ρ is a subgroup of G of index p. It follows immediately that $\sigma(G) \leqq p + 1$. Also for $\rho > 0$, $U \not\subset B_\rho$, and so just as before $G \sim U \times B_\rho$ and then $\sigma(B_\rho) > n$. Now since $G = \sum_{r=1}^{n} H_r$, $B_\rho = G \cap B_\sigma = \sum_{r=1}^{n} H_r \cap B_\rho$ and so unless $H_r \cap B_\rho = B_\rho$ we should find that $\sigma(B_\rho) \leqq n$, which is impossible. Thus for each $\rho \geqq 1$, there must exist a suitable r with $B_\rho \subset H_r$. But since $i(B_\rho) = p$ and H_r is a proper subgroup, it follows that $B_\rho = H_r$. Thus the $p - 1$ different values of ρ give $p - 1$ different values of r. Thus in the representation $G = \sum_{r=1}^{n} H_r$ there are at least p different terms, viz., H and $B_1, B_2, \ldots, B_{p-1}$, and these between them do not contain U. Thus $n \geqq p + 1$, and so finally $\sigma(G) = n = p + 1$. But then $G/X \sim C_p \times C_p$ and so $\sigma(G/X) = \sigma(G)$, which is impossible, since G was assumed to be a primitive n-sum group. Thus if G is not abelian, $|Z(G)| = 1$, which concludes the proof.

REMARK. If G is a non-abelian group with non-trivial centre, we do not assert that $\sigma(G) = \sigma(G/Z(G))$, but merely that G must have some normal subgroup N with $\sigma(G) = \sigma(G/N)$. This is illustrated by the example $S_3 \times C_2$, which is a 3-sum group as can be seen by factoring out its Sylow 3-subgroup; but factoring out the centre would leave S_3 which is a 4-sum group. However, the search for new primitive n-sum groups can be restricted to groups with trivial centre.

LEMMA 6. *If H is a maximal subgroup of G then either H has precisely $i(H)$ conjugates in G or $H \lhd G$ and $i(H)$ is prime.*

For, $N(H)$, the normaliser of H in G is a normal subgroup of G containing H. Since H is assumed maximal, either $N(H) = H$ or $N(H) = G$. In the former case, H has precisely $i(H) = i(N(H))$ distinct conjugates; in the latter $H \lhd G$.

If $H \lhd G$, let x denote any element of G not contained in H. Since H is maximal, H and x generate G and since $xH = Hx$, every element of G must be of the form hx^i for some $h \in H$ and integer i. Let ρ be the least positive integer with $x^\rho \in H$. Then $G = \sum_{i=1}^{\rho} Hx^i$, and so $\rho = i(H)$. But then $i(H)$ cannot be composite for if $\rho = \lambda\mu$, then $y = x^\lambda$ satisfies $y \notin H$, and so H and y would generate a proper subgroup of G containing H as a proper subgroup, contradicting the supposition of maximality.

THEOREM 5. (1) *There are no 2-sum groups.*

(2) *G is a 3-sum group if and only if it possesses at at least two subgroups of index 2. The only primitive 3-sum group is $C_2 \times C_2$.*

(3) *G is a 4-sum group if and only if $\sigma(G) \neq 3$ and G has at least two subgroups of index 3. The only primitive 4-sum groups are $C_3 \times C_3$ and S_3.*

(4) *G is a 5-sum group if and only if $\sigma(G) \neq 3$ or 4 and G has a maximal subgroup of index 4. The only primitive 5-sum group is the alternating group A_4.*

PROOF. (1) Follows from Lemma 1.

(2) If $\sigma(G) = 3$, then, by Lemma 1, $i_1 = i_2 = 2$, and so G must have two subgroups of index 2. Conversely, if G has two subgroups H_1 and H_2 of index 2, they are both normal subgroups of G and hence so is $N = H_1 \cap H_2$. But $i(N) = 4$ and so $G/N \sim C_2 \times C_2$, which is itself a 3-sum group.

(3) If $\sigma(G) = 4$, then by Lemma 1, $i_2 \leqq 3$, and so by the above $i_2 = 3$. Then by Theorem 1, it follows that $i_3 = i_4 = 3$. Conversely, suppose that $\sigma(G) \neq 3$ and that G has two subgroups A and B each of index 3. Then *either* both A and B are normal subgroups of G, *or* at least one is not normal. If $A \lhd G$, $B \lhd G$, then with $X = A \cap B$ we find $X \lhd G$ and $G/X \sim C_3 \times C_3$.

If $A \ntriangleleft G$, let X denote the maximal subgroup of A which is normal in G. Then G/X is isomorphic to a subgroup of the symmetric group on the right cosets of A, i.e. to a subgroup of S_3. But X is also a subgroup of the other two conjugates of A, and so G/X cannot be cyclic, since it contains three subgroups of the same order. Thus $G/X \sim S_3$, which concludes the discussion of this case.

(4) If $\sigma(G) = 5$, then by Lemma 1, $i_2 \leqq 4$, and by the above $i_2 \neq 2$. If $i_2 = 3$, then by Theorem 1, $i_3 \leqq 4$, and $i_3 = 3$ is impossible by the above. Conversely, suppose that $\sigma(G) \neq 3$ or 4 and that G possesses a maximal subgroup B of index 4. Then by Lemma 6, B has precisely 4 conjugates. Let X denote the maximal subgroup of B which is normal in G. Then G/X is isomorphic to a subgroup of S_4, the symmetric group on the right cosets of B. Also B/X has index 4 in G/X, and so $|G/X| = 8$ or 12 or 24 with G/X non-cyclic since it contains four subgroups of the same order as B/X. Now $|G/X| = 8$ is impossible, since then $\sigma(G/X) = 3$ by Theorem 2, and $|G/X| = 24$ would give $G/X \sim S_4$ and this too is impossible, for S_4 has a factor group S_3 and so $\sigma(S_4) \leqq \sigma(S_3) = 4$, by Lemma 2. The only remaining case is $|G/X| = 12$, and since the only subgroup of order 12 of S_4 is A_4, and as is easily verified $\sigma(A_4) = 5$, the result follows.

After the above theorem it is natural to suppose that 6-sum groups could similarly be characterised by the existence of two subgroups of index 5, but this cannot be so, for both A_5 and S_5 contain five such, yet we find

LEMMA 7. $\sigma(A_5) = 10$, $\sigma(S_5) = 16$.

PROOF. The alternating group on the five symbols 1, 2, 3, 4 and 5, contains:
1 element of order 1
15 elements of order 2, (12) (34) etc.,
20 elements of order 3, (123) etc., forming 10 inverse pairs
24 elements of order 5, (12345) etc., forming six sets of four generators of cyclic subgroups.

Now no proper subgroup X containing an element of order 5 can also contain one of order 3, since otherwise $|X| \geqq 15$, impossible since A_5 is simple. Similarly, X cannot two elements of order 5 which are not powers of each other. Thus six proper subgroups are required to contain all the elements of order 5, none of which can contain a single element of order 3. They can however be chosen so that they contain between them all the elements of order 2, for example by chosing X_1 to be the subgroup generated by (12345) and (13) (45) with order 10.

Now let Y be a proper subgroup containing an element of order 3. Then $|Y| = 3, 6$ or 12, and so Y contains 2 or 8 elements of order 3. But the elements of order 3 in Y can involve at most four of the five symbols, since otherwise if say Y contained (123) and (145) it would also contain (12345), of order 5. Thus to contain the 10 elements (123), (124), (125), (134), (135), (145), (234), (235), (245) and (345) at least four subgroups Y are required. But four clearly suffice, since we may take the four alternating groups on the symbols (1, 2, 3, 4), {1, 2, 3, 5}, {1, 2, 4, 5} and {1, 3, 4, 5}. Thus $\sigma(A_5) = 10$, and of course A_5, being simple is then a primitive 10-sum group.

The proof for S_5 is similar; the details are omitted.

LEMMA 8. *If* $\sigma(G) = n$ *and if* $G = \sum_{r=1}^{m} H_r + \sum_{r=m+1}^{n} K_r$, *where each subgroup is maximal, where* $H_r \lhd G$ *and where all the subgroups* H_r *have distinct orders, then* $|G| \leqq \sum_{r=m+1}^{n} |K_r|$.

PROOF. By Lemma 6, $i(H_r) = p_r$, say where p_r is prime, and by hypothesis, the primes p_r are distinct. It then follows without difficulty that the index of the intersection of any subset of the subgroups H_r is simply the product of their indices, and so if $D = \sum_{r=1}^{m} H_r$, then D contains precisely

$$\sum |H_r| - \sum\sum |H_r \cap H_s| + \sum\sum\sum |H_r \cap H_s \cap H_t| - \dots$$
$$= |G| \{\sum 1/p_r - \sum\sum 1/(p_r p_s) + \sum\sum\sum 1/(p_r p_s p_t) - \dots\}$$
$$= |G| \left\{1 - \prod_{r=1}^{m} (1 - 1/p_r)\right\}$$

distinct elements.

Now let k_r denote the number of elements of K_r which lie outside D. Since $H_s \lhd G$ and K_r is maximal, $|H_s \cap K_r| = |K_r|/p_s$. Next since $H_s \cap H_t \cap K_r$ is a subgroup of each of $H_s \cap K_r$ and $H_t \cap K_r$ it follows that $|H_s \cap H_t \cap K_r|$ divides $|K_r|/p_s$ and $|K_r|/p_t$ and hence $|K_r|/p_s p_t$. On the other hand

$$|H_s \cap H_t \cap K_r| = \frac{|K_r| \cdot |H_s \cap H_t|}{|K_r(H_s \cap H_t)|}$$
$$= \frac{|K_r| \cdot |G|}{p_s p_t |K_r(H_s \cap H_t)|}$$
$$\geqq |K_r|/p_s p_t,$$

and so $|H_s \cap H_t \cap K_r| = |K_r|/p_s p_t$.

The argument can be extended, and we obtain

$$k_r = |K_r|\{1 - \sum 1/p_s + \sum\sum 1/(p_s p_t) - \ldots\}$$
$$= |K_r| \prod_{s=1}^{m} (1 - 1/p_s).$$

Thus we have

$$|G| \leqq \text{number of elements in } D + \sum_{r=m+1}^{n} k_r$$
$$= |G| - |G| \cdot \prod_{s=1}^{m} (1 - 1/p_s) + \sum_{r=m+1}^{n} |K_r| \cdot \prod_{s=1}^{m} (1 - 1/p_s),$$

which yields the required result.

COROLLARY. *If k is the least integer such that G has more than one maximal subgroup of index k, then $\sigma(G) \geqq k + 1$*

LEMMA 9. *If $\sigma(G) = 6$ then $i_1 = 2$ or 5 and $i_r = 5$ for $2 \leqq r \leqq 6$.*

PROOF. By Theorem 5, G has no maximal subgroup of index 4, and at most one of index 2 and one of index 3. Thus $i_3 \geqq 5$. Then by the previous lemma we cannot have $i_1 = 2$ and $i_2 = 3$, and so $i_r = 5$ for $2 \leqq r \leqq 6$. To conclude the proof we must merely show that $i_1 = 3$ is impossible.

Suppose if possible that $i_1 = 3$. Then $H_1 \lhd G$ and by Theorem 1, $H_2 \cap H_3 \subset H_1$ and so $3 = i_1$ divides $i(H_2 \cap H_3)$ as does $5 = i_2$. But we also have $|H_2 \cap H_3| = |H_2| \cdot |H_3|/|H_2 H_3| \geqq |G|/25$ and so $i(H_2 \cap H_3) = 15$. Since $H_1 \lhd G$, $i(H_1 \cap H_2) = i_1 i_2 = 15$, and so $H_1 \cap H_2 = H_2 \cap H_3 = H_2 \cap H_4$ similarly. Thus $X = H_1 \cap H_2$ is the intersection of any pair of the six subgroups. Since $H_1 \lhd G$ it then follows that $X \lhd H_2$ and similarly $X \lhd H_3$ whence $X \lhd G$ since H_2 and H_3 generate G. But then G/X is the sum of six proper subgroups which is impossible as G/X must be cyclic since it has order 15. This concludes the proof.

LEMMA 10. *If G is a primitive 6-sum group and $i_1 = 2$, then either $G \sim D_{10}$, the dihedral group of order 10, or $G \sim W$, a group of order 20 defined by $a^5 = b^4 = e$, $ba = a^2 b$.*

PROOF. Since now $i_1 = 2$, $i_r = 5$, $2 \leqq r \leqq 6$, we find just as before that $i(H_1 \cap H_r) = 10$, $r \neq 1$ and $i(H_r \cap H_s) = 10$ or 20 for each $r \neq 1$, $s \neq 1$ and $r \neq s$. There are now two cases:

Case 1. Suppose that there exist r, s with $r \neq 1$ and $r \neq s$ for which $i(H_r \cap H_s) = 10$. Then $X = H_1 \cap H_r = H_r \cap H_s = H_1 \cap H_s$ and since $|X| = \frac{1}{2}|H_2|$, $X \lhd H_2$ and similarly $X \lhd H_3$, whence $X \lhd G$. Then $|G/X| = 10$ and G/X cannot be cyclic since it contains two distinct subgroups H_2/X and H_3/X of order 2. Thus $G/X \sim D_{10}$, and since $\sigma(D_{10}) = 6$ and G was assumed primitive, $|X| = 1$, i.e. $G \sim D_{10}$.

Case 2. Suppose that $i(H_r \cap H_s) = 20$ for every $r \neq 1, s \neq 1, r \neq s$. By Theorem 1, $H_2 \cap H_3 \subset H_1$. Let $B_r = H_1 \cap H_r$ and $X = B_2 \cap B_3$. Then B_2 has index 5 in H_1 and so is a maximal subgroup of H_1 and $X \lhd B_2$ as X has index 2 in B_2. Similarly $X \lhd B_3$ another maximal subgroup of H_1 and so $X \lhd H_1$. Hence the normaliser $N(X) \supset H_1$ and so $N(X) = H_1$ or G. Suppose if possible that $N(X) = H_1$. Then X would have precisely two conjugates in G, X and Y, say. Now $X \lhd H_1$ and so $X \ntriangleleft H_2$ otherwise $X \lhd G$ since both H_1 and H_2 are maximal in G, and so $\exists b \in H_2$ with $bXb^{-1} \neq X$ and so $bXb^{-1} = Y$. Since $X \subset H_2$ this implies $Y \subset H_2$ and similarly $Y \subset H_3$ whence $Y \subset H_2 \cap H_3 = X$, which is impossible. Thus $N(X) = G$, i.e. $X \lhd G$, and so $|G/X| = 20$. Here $K = G/X$ cannot be cyclic since it contains at least two subgroups H_2/X and H_3/X each of order 4. We shall show that $\sigma(K) = 6$ and that $K \sim W$. Certainly $\sigma(K) \geqq 6$ by Lemma 2, and K contains the subgroup $K_1 = H_1/X$ of index 2. By Theorem 5, this is the only subgroup of index 2. It contains all elements of K whose order is a multiple of 5, since the Sylow 5-subgroup F is unique in any group of order 20. Thus all the 10 remaining elements of K have order dividing 4, and so there must be five Sylow 2-subgroups which between them contain them all. Thus $\sigma(K) \leqq 6$, and so in fact $\sigma(K) = 6$.

Now $K/F \sim C_4$ otherwise $\sigma(K) = 3$. Let a be a generator of F and K/F be generated by Fb, with $a^5 = b^4 = e$. Since $F \lhd K$, $ba \in Fb$. Clearly $ba \neq b$, and $ba \neq ab$, otherwise K would be cyclic. Also if $ba = a^4 b$ then it is easily verified that $L = \{e, b^2\} \lhd K$ with $K/L \sim D_{10}$ a case we have considered already. If $ba = a^2 b$ then $K = W$. Finally if $ba = a^3 b$, then if $c = b^3$ we find $ca = a^2 c$ and again $K = W$. This concludes the proof.

LEMMA 11. *If G is a non-cyclic group of order dividing* 24, *then* $\sigma(G) \leqq 5$.

PROOF. After Theorem 2, we need only consider groups of orders 6, 12 or 24. The only non-cyclic group of order 6 is S_3 which is a 4-sum group. Consider a group G of order 12. If its Sylow 2-subgroup of index 3 is not unique, then $\sigma(G) \leqq 4$ by Theorem 5. Suppose then that the Sylow 2-subgroup is unique. If the Sylow 3-subgroup is also unique, the result follows by Theorem 3. Otherwise, there exist precisely four Sylow 3-subgroups, which together with the Sylow 2-subgroup cover G.

Finally, consider a group G of order 24. As above, it suffices to assume that its Sylow 2-subgroup, T, is unique, and that there are just four Sylow 3-subgroups, which cannot be maximal subgroups by Lemma 6. Let K denote a maximal subgroup of G containing one of the Sylow 3-subgroups. If $i(K) = 4$ then $\sigma(G) \leqq 5$ by Theorem 5. If $i(K) = 2$, and G had another subgroup of index 2, then $\sigma(G) = 3$. If K is the only such, then T and K between them contain all elements of G with orders 1, 2, 3, 4, and 8, but do not contain all elements of G. Thus there are elements of G outside both K and T, one of which has order 6, and generates a subgroup, L, of index 4. If L is maximal then $\sigma(G) \leqq 5$ as before, and otherwise

L would lie in a maximal subgroup of index 2, distinct from K, contrary to supposition.

LEMMA 12. *The only primitive 6-sum group with* $i_1 = 5$ *is* $C_5 \times C_5$.

PROOF. Let $G = \sum H_r$ be a representation of such a group as the sum of six subgroups each with index 5. Then by Theorem 1, $X = H_r \cap H_s$ has index 25 and is independent of r and s. Since G is assumed primitive, X contains no subgroup normal in G apart from $\{e\}$.

Let Y_r be the largest subgroup of H_r which is normal in G. Since $i_1 = 5$, $|G/Y_1|$ divides $120 = 5!$, and so $k = |H_1|/|Y_1|$ divides 24. Let $Z_1 = X \cap Y_1$. For any $h_2 \in H_2$, $h_2 Z_1 h_2^{-1}$ is contained within H_2 and within Y_1 and so within their intersection, Z_1. Thus $Z_1 \lhd H_2$ and similarly $Z_1 \lhd H_3$ and since H_2 and H_3 are maximal in G, $Z_1 \lhd G$. But $Z_1 \subset X$ and so $|Z_1| = 1$. Thus $|X| \cdot |Y_1| = |XY_1| \cdot |X \cap Y_1| = |XY_1|$. But XY_1 is a subgroup of H_1 properly containing the subgroup X of index 5 in H_1. Thus $XY_1 = H_1$ and so $|Y_1| = 5$.

Now since $k|24$, $5 \| |H_1|$ and so Y_1 is a Sylow 5-subgroup of H_1. Since $Y_1 \lhd G$, $Y_1 \lhd H_1$ and so Y_1 is the only Sylow 5-subgroup of H_1 which therefore has precisely four elements of order 5. The same then holds similarly for the other subgroups H_r and so G has precisely 24 elements of order 5 and none of order 25. Thus F, the Sylow 5-subgroup of G, is of the form $C_5 \times C_5$, is unique and therefore a normal subgroup of G and has index k, dividing 24.

It then follows that $G/F \sim C_k$, otherwise by the previous lemma, $\sigma(G) < 6$. If $k = 1$ then $G = F$ as required. Suppose if possible that $k > 1$. Then if Y_1 and Y_2 are generated by a and b respectively, the other Ys are generated by ab, ab^2, ab^3 and ab^4 in some order, where $ba = ab$ and $a^5 = b^5 = e$. Let Fc generate G/F. Then without loss of generality $c^k = e$, since if this not be the case, we can generate G/F by Fd where $d = c^5$ and then $d^k = e$. Now since $Y_1 \lhd G, ca = a^r c, 1 \leqq r \leqq 4$, and similarity $cb = b^s c$ and $c(ab) = (ab)^t c$. Then $(ab)^t c = a^r b^s c$ which implies that $r = s = t$. Now if $r = 1$ then $G \sim F \times C_k$ impossible since G was assumed a primitive 6-sum group. If $r = 4$, then $c^2 a = ac^2$ and $c^2 b = bc^2$ and so $c^2 \in Z(G)$, impossible by Theorem 4 unless $k = 2$ in which case the subgroup A generated by a satisfies $A \lhd G$ and $G/A \sim D_{10}$ and so again G would not be a primitive 6-sum group. Finally if $r = 2$ or 3, we find similarly that $c^4 \in Z(G)$, impossible unless $k = 2$ or 4 by Theorem 4. Now $k \neq 2$ since it is found that $c^2 a = a^4 c^2$, and for $k = 4$ we find that $A \lhd G$ with $G/A \sim W$. This concludes the proof.

Summarising the results above we obtain.

THEOREM 6. *The only primitive* 6-*sum groups are* $C_5 \times C_5$, D_{10} *and* W.

Theorem 3 above states that for a non-cyclic nilpotent group, $G, \sigma(G) = p + 1$, where p is the least prime for which G possesses at least two maximal subgroups of index p. As Lemma 7 shows, this is certainly not true of groups in general, but we shall show that the condition nilpotent can be relaxed to supersoluble. We proceed as follows:

LEMMA 13. *If G is not cyclic, p a prime, $X \lhd G$, $X \sim C_p$ and G/X is cyclic, then* $\sigma(G) = p + 1$.

PROOF. Let $G/X \sim C_m$. If G is abelian, then $G \sim C_p \times C_m$, and since G is not cyclic $p \mid m$ and so $\sigma(G) = p + 1$ by Theorem 3.

If G is not abelian, let a generate X and Xb generate G/X. Then $b^m \in X$ and m is the least positive integer with this property. If $b^m \neq e$, then b would have order mp, impossible since G was assumed not to be cyclic. Also $bab^{-1} \in bXb^{-1} = X$, and so $bab^{-1} = a^r$ for some integer r satisfying $1 \leqq r < p$, i.e.

$$G = \{a^i b^j \mid a^p = b^m = e; \quad bab^{-1} = a^r\}.$$

It follows that $r \neq 1$, since G was supposed non-abelian. Also $a = b^m a b^{-m} = a^{r^m}$ and so $r^m \equiv 1 \pmod p$. Now it is easily seen that $(a^i b)^k = a^{i(1+r+\dots+r^{k-1})} b^k$ and so $a^i b$ cannot have order less than m. On the other hand

$$1 + r + r^2 + \dots + r^{m-1} = \frac{r^m - 1}{r - 1} \equiv 0 \pmod p,$$

and so $a^i b$ has order m precisely. Thus for $i = 0, 1, \dots, p-1$ $a^i b$ generates a subgroup X_i of order m, i.e. index p in G. Thus each X_i is maximal and must occur in the representation of G as a sum of proper subgroups. But $\sum_{i=0}^{p-1} X_i$ still does not contain the element $a \in G$, and so $\sigma(G) \geqq p + 1$.

To show that $\sigma(G) \leqq p + 1$, suppose that r belongs to exponent k modulo p. Then $k \mid m$, and $k \neq 1$ since $r \neq 1$. Suppose first of all that $k = m$. Then for $M < m$, $1 + r + \dots + r^{M-1} \not\equiv 0 \pmod p$ and so every element of the form $a^N b^M$ with $M > 0$ occurs as $(a^i b)^M$ for some i, i.e. lies in one of the X_i, and since all the powers of a lie in the subgroup generated by a, $\sigma(G) \leqq p + 1$.

Finally, if $k < m$, then $b^k a b^{-k} = a^{r^k} = a$, and so the subgroup B generated by b^k being a subgroup of $Z(G)$ is normal in G. Then we find that $G/B \sim H$, where

$$H = \{\alpha^i \beta^j \mid \alpha^p = \beta^k = e; \beta\alpha\beta^{-1} = \alpha^r\},$$

where now r belongs to exponent k, and so $\sigma(G) \leqq \sigma(H) \leqq p + 1$.

LEMMA 14. *Let G be a non-cyclic supersoluble group of order $\prod_{i=1}^{r} p_i^{\alpha(i)}$ with $p_1 < p_2 < \dots < p_r$ prime and each $\alpha(i) > 0$. Then $\sigma(G) \leqq p_r + 1$.*

PROOF. Consider a chief series $G = B_0 \supset B_1 \supset \dots \supset B_R = \{e\}$ where

$R = \sum_{i=1}^{r} \alpha(i)$ and in which B_i/B_{i+1} has prime order for each i. Then G/B_1 is cyclic, but G/B_R is not. Let k denote the least integer for which G/B_{k+1} is not cyclic. Then $1 \leqq k \leqq R - 1$ and $G/B_k \sim (G/B_{k+1})/(B_k/B_{k+1})$ where G/B_k is cyclic, G/B_{k+1} is not, and $B_k/B_{k+1} \sim C_p$ for some prime factor p of $|G|$. Then by Lemmas 2 and 13, $\sigma(G) \leqq \sigma(G/B_{k+1}) = p + 1 \leqq p_r + 1$.

LEMMA 15. *With G as in Lemma* 14, *if G has at least two subgroups of index* $p, \sigma(G) \leqq p + 1$.

PROOF. (1) If G has two subgroups A, B of index p both of which are normal, then they are maximal and so $X = A \cap B$ is normal with index p^2. Moreover G/X cannot be cyclic since it possesses two subgroups A/X and B/X of order p. Thus $G/X \sim C_p \times C_p$ and so $\sigma(G) \leqq \sigma(G/X) = p + 1$.

(2) If at least one of the two subgroups, A say, of index $p = p_m$ is not normal, then it must have precisely p conjugates. If $m = r$, the result follows from the previous lemma. If $m < r$, then A is a supersoluble group of order $|G|/p$ having a subgroup H of order $p_r^{\alpha(r)}$. But by Philip Hall's Theorem, G has precisely one subgroup of this order, whence $H \lhd G$ and $H \subset A^c$ for every conjugate A^c of A. Thus G/H is a supersoluble group which is not cyclic since it has at least p subgroups A^c/H all of index p. We proceed in like fashion until finally we arrive at a non-cyclic supersoluble group K of order $\prod_{i=1}^{m} p_i^{\alpha(i)}$. Then $\sigma(G) \leqq \sigma(G/H) \leqq \ldots \leqq \sigma(K) \leqq p_m + 1$ by Lemma 14, which concludes the proof.

LEMMA 16. *With G as in Lemma* 14, *if G has at most one subgroup of each of the indices* $p_1, p_2, \ldots, p_s$ *then* $s < r$ *and* $\sigma(G) \geqq p_{s+1} + 1$.

PROOF. Let $\sigma(G) = n$, and suppose that $G = \sum_{i=1}^{m} H_i + \sum_{i=m+1}^{n} K_i$, where $0 \leqq m \leqq s$, and where H_i are subgroups with indices chosen from the set $\{p_1, p_2, \ldots, p_s\}$. Then by Lemma 8, $s < r$ and $|G| \leqq \sum_{i=m+1}^{n} |K_i| \leqq (n - m)|G|/p_{s+1}$ and so $n \geqq m + p_{s+1}$. The result follows if $m \neq 0$; if $m = 0$ it follows by using Lemma 1 instead of Lemma 8.

THEOREM 7. *Let G denote a supersoluble group. If for every prime factor p of* $|G|$, *G possesses at most one subgroup of index p, then G is cyclic. Otherwise if p is the least prime for which G possesses more than one subgroup of index p, then* $\sigma(G) = p + 1$, *and then* $i_r = p$ *for* $r \geqq 2$.

PROOF. Except for the last statement, the result follows from Lemmas 15 and 16. For the last part, with m defined as in the proof of Lemma 16, we find that

$\sigma(G) = p + 1 \geqq m + p$, and so $m = 0$ or 1. Thus, at most one of the summands can be normal in G, whence $i_r \geqq p$ with at most one exception. But by Lemma 1, $i_2 \leqq p$, and the result easily follows.

We can now find all primitive $(p + 1)$-sum supersoluble groups.

THEOREM 8. *Let G be a primitive $(p + 1)$-sum supersoluble group. Then either $G \sim C_p \times C_p$ or G is a metacyclic group*

$$\{a^i b^j \mid a^p = b^N = e;\ bab^{-1} = a^r\}$$

of order pN, where $N \mid (p - 1)$ and r belongs to exponent N modulo p.

PROOF. With $|G|$ as in the statement of Lemma 14, since G is a primitive $(p + 1)$-sum group, as in the proof of Lemma 15, $p \nless p_r$, and so $p = p_r$. If $G = B_0 \supset B_1 \supset \ldots \supset B_R = \{e\}$ is a chief series with $p_1 = |B_0/B_1| \leqq |B_1/B_2| \leqq \ldots \leqq |B_{R-1}/B_R| = p_r = p$, let k be defined as in the proof of Lemma 14. Then

$$p + 1 = p_r + 1 = \sigma(G) \leqq \sigma(G/B_{k+1}) = p + 1,$$

and so $\sigma(G) = \sigma(G/B_{k+1})$. Since G is supposed primitive, $B_{k+1} = \{e\}$, i.e. $k = R - 1$, and so G/B_{R-1} is cyclic. Let $X = B_{R-1}$. Then $X \lhd G$, $X \sim C_p$ and G/X is cyclic, of order N say.

The conclusion then follows as in Lemma 13, for as G is supposed a primitive $(p + 1)$-sum group, r must belong to exponent N modulo p.

The next logical step would be to consider soluble groups, and here it seems entirely plausible to conjecture that for such a group G, $\sigma(G) = 1 + c$ where c is a suitable chief factor, and so a prime power. Unfortunately, we have not been able to complete a proof of this, but some partial results follow.

LEMMA 17. *If G is not cyclic, p a prime, $X \lhd G$, $X \sim (C_p)^N$ no proper subgroup of X is normal in G and G/X is cyclic, then $\sigma(G) = p^N + 1$.*

PROOF. Let $G/X \sim C_m$ be generated by Xc. Then m is the least positive integer with $c^m \in X$. If $c^m \neq e$, let $d = c^m$. Then d has order p and generates a subgroup D, with $D \lhd G$ since $D \subset Z(G)$. Since $D \subset X$ and X has no proper subgroup normal in G, it follows that $D = X$, i.e. $N = 1$, and now Lemma 13 yields the result.

If $N \neq 1$, then $c^m = e$. Thus for any $x \in X$, $c^m x c^{-m} = x$. Consider any fixed $x \in X$, $x \neq e$ and let μ denote the least positive integer with $c^\mu x c^{-\mu} = x$. Obviously $\mu \mid m$. Now since $X \lhd G$, $x_i = c^i x c^{-1} \in X$ for each i. The elements x_i with $0 \leqq i < \mu$ are distinct and they generate a subgroup Y contained in X which is normal in G, since X is abelian, $G = \sum X c^i$ and $c^j x_i c^{-j} = x_{i+j}$. But by hypothesis, no non-trivial subgroup of X can be normal in G, and so since $Y \neq \{e\}$ it follows that $Y = X$. Thus $d = c^\mu$ commutes with every element of X and also of course with every power of c, and so $d \in Z(G)$. If $\mu < m$, then $d \neq e$, and so again D, the subgroup generated by d, is a normal subgroup of G contained within X, whence $D = X$ and $N = 1$, which is impossible. Thus $\mu = m$. It follows that no element except e of X can commute with any element of the cyclic group C generated by c, except e.

For $x \in X$, $x \neq e$, let $T = C \cap (xCx^{-1})$. If $t \in T$ then $t = c^r = xc^s x^{-1}$ and then $c^{r-s} = c^{-s}xc^s x^{-1} \in G'$. But since G/X is abelian, $G' \subset X$. Thus c^{r-s} lies in both C and X and so $r = s$. Thus x and c^r commute, whence $t = e$, *i.e.* C and xCx^{-1} have no element other than e in common. Since this applies for any $x \in X$, $x \neq e$, it follows that no two of the p^N conjugates of C have any element other than e in common, and so they contain between them precisely $1 + (m - 1)p^N$ distinct elements of G. The remaining $p^N - 1$ elements all lie in X and so $\sigma(G) \leqq p^N + 1$.

On the other hand, C is a maximal subgroup of G, for if $C \subset C_1$, $C \neq C_1$ then C_1 would have to contain at least one $x \in X$, $x \neq e$, since $G = \sum Xc^i$. Thus, since no proper subgroup of X is normal in G, C_1 would have to contain all of X, and then $C_1 = G$. Thus each of the p^N conjugate subgroups xCx^{-1} is maximal. Moreover each one is cyclic, and therefore no one of them can be omitted in a representation $G = \sum H$ otherwise its generator would not lie in any H. As before these p^N subgroups do not suffice, since no element of X other than e is yet included. Thus $\sigma(G) \geqq p^N + 1$, and combining the two inequalities yields the result.

COROLLARY. *There exists a group G with $\sigma(G) = p^N + 1$ for every prime power p^N.*

PROOF. All that remains is to construct G satisfying the conditions of the lemma. We observe first of all that there exists an automorphism $\varphi: X \to X$ which leaves invariant no proper subgroup of X. For let X be represented as the additive group of the field $GF(p^N)$ and let t denote a generator of the multiplicitive group. Define $\varphi(x) = xt$. Then φ is an automorphism on X and if Y is any subgroup of X other than $\{0\}$ then $\varphi(Y) = Y$ implies that for any $y \in Y$, $yt \in Y$, and then $yt^2, yt^3, \ldots$ all belong to Y. Since t has multiplicitive order $p^N - 1$ this implies $Y = X$.

Now define the group G of order $p^N(p^N - 1)$ by adjoining to X the element ξ of order $p^N - 1$, with $\xi x \xi^{-1} = \varphi(x)$ for any $x \in X$. Then this group satisfies the conditions.

THEOREM 9. *Let G be a non-cyclic soluble group. Then $\sigma(G) \leqq c + 1$, where c is a suitable chief factor of G. In particular if $|G| = \prod_{i=1}^{r} p_i^{\alpha(i)}$ then $\sigma(G) \leqq 1 + \max_{1 \leqq i \leqq r} \{p_i^{\alpha(i)}\}$.*

PROOF. Let $G = B_0 \supset B_1 \supset \ldots \supset B_R = \{e\}$ be a chief series. Then each of the factor groups B_r/B_{r+1} is elementary abelian.

If G/B_1 is not cyclic, then $G/B_1 \sim (C_p)^N$ for some $N \geqq 2$ and then $\sigma(G) \leqq \sigma(G/B_1) = p + 1 < p^N + 1 = c + 1$.

If G/B_1 is cyclic, since G/B_R is not cyclic, let r with $R > r \geqq 1$ be the largest r with G/B_r cyclic. Then G/B_{r+1} is not cyclic, and so $C_m \sim G/B_r \sim (G/B_{r+1})/(B_r/B_{r+1}) = H/X$, say. Now $X = B_r/B_{r+1} \sim (C_p)^N$ with $N \geqq 1$, and no proper subgroup of X is normal in H, otherwise there would exist a proper subgroup of B_r normal in G, which properly contained B_{r+1}. By Lemma 17, $\sigma(H) = p^N + 1$, and so $\sigma(G) \leqq \sigma(G/B_{r+1}) = \sigma(H)$ by Lemma 2 yields the result, since $p^N = |B_r/B_{r+1}|$ is a chief factor.

Some open questions remain. In the first place, it remains to prove or disprove the conjecture that for a soluble group G, $\sigma(G) = 1 + c$, for a chief factor c. If true, it is to be expected that c be the smallest chief factor for which G has two maximal subgroups of index c.

Another conjecture is that no 7-sum groups exist. Certainly such a group could not be supersoluble, and although it may be shown in a rather laborious way that it could not even be soluble, a great deal of detail would be required to complete the discussion. Details are omitted, as there seems no obvious way to extend this to soluble groups in general.

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