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C^∞-BOUNDING SETS AND COMPACTNESS

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Abstract.

In this article it is proved that any set in a real Banach space on which the C^∞-functions are bounded, is relatively compact. In particular, for any real Banach space E, a sequence (x_n) convergences to x in E if and only if $f(x_n) \to f(x)$ for all $f \in C^\infty(E)$.

In the literature, bounding sets have been studied mainly with respect to two function classes: the class of real continuous functions and the class of holomorphic functions. In the first case, bounding sets are for instance related to the Nachbin-Shirota theorem (see [6]). In the second case, bounding sets arise naturally in problems of analytic continuation, envelopes of holomorphy and topologies on spaces of holomorphic mappings [4]. It was proved by Dineen [3] that the non-compact closed set of all unit vectors in l^∞ is bounding with respect to all holomorphic functions on l^∞. This article proves that, contrary to the complex case, all bounding sets in a real Banach space with respect to the C^∞-functions are nothing but relatively compact.

Given a subset A of a Banach space E, we say that A is C^∞-bounding if $\sup_{x \in A} |f(x)| < \infty$ for all $f \in C^\infty(E)$. Contributions to the theory of C^∞-bounding sets have recently been made by the authors of [1] and [2] and by Kriegl-Nel in [7]. In [1] it is proved that the C^∞-bounding sets are relatively compact in the general class of Banach spaces embeddable into $C(K)$, where K is compact and sequentially compact. This class contains all WCG spaces and many others.

Since $E' \subset C^\infty(E) \subset C(E)$ for any Banach space E, the class of C^∞-bounding sets is *a priori* placed between the classes of bounded and relatively compact ones. If A and B are C^∞-bounding sets, also $-A$ and $A \cup B$ are C^∞-bounding. It is therefore of interest to study *symmetric* sets, where A is symmetric if $A = -A$. We start by a result for certain symmetric sequences in the real Banach space l^∞:

LEMMA 1. *For every bounded sequence (a_n) in the real Banach space l^∞, with*

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$\|a_n - a_m\| \geqq 3$ *if* $n \neq m$ *and* $a_{2n+1} = -a_{2n}$ *for all* n, *there is a subsequence* (a_{n_k}) *such that*

$$a_{n_k}^{i_k} - 1 \geqq a_{n_{k+p}}^{i_k} \quad \textit{for all} \quad k, p = 1, 2, \ldots,$$

where $a_n = (a_n^1, a_n^2, \ldots, a_n^i, \ldots)$ *and* (i_k) *is an increasing sequence in* N.

PROOF. Let S_0 be the set of positive even integers and let i_1 be the smallest index for which

$$\sup_{\nu, \mu \in S_0} |a_\nu^{i_1} - a_\mu^{i_1}| > 2.$$

Such i_1 exists since $\|a_n - a_m\| \geqq 3$ if $n \neq m$. Take some $n_1 \in S_0$ such that for all $\nu \in S_1$ either

$$\ominus_1\colon a_{n_1}^{i_1} - 1 \geqq a_\nu^{i_1} \quad \text{or} \quad \oplus_1\colon a_{n_1}^{i_1} + 1 \leqq a_\nu^{i_1},$$

where S_1 is an infinite subset of $S_0 \cap \{n \in \mathrm{N}\colon n \geqq n_1\}$. Assume that the statement $\mathscr{P}(p)$ below holds for $p \geqq 1$.

$\mathscr{P}(p)$: For all k with $1 \leqq k \leqq p$ there are integers $n_k \in S_{k-1}$ and i_k with $0 = i_0 < \ldots < i_{k-1} < i_k$ and $0 = n_0 < \ldots < n_{k-1} < n_k$ and infinite sets S_k with $S_k \subset S_{k-1} \cap \{n\colon n > n_k\}$ such that either $\ominus_k$ or $\oplus_k$ below hold for all $\nu \in S_k$

$$\ominus_k\colon a_{n_k}^{i_k} - 1 \geqq a_\nu^{i_k} \quad \text{or} \quad \oplus_k\colon a_{n_k}^{i_k} + 1 \leqq a_\nu^{i_k}.$$

Since (a_n) is a bounded sequence, there is for each index j with $1 \leqq j \leqq i_p$ at *most a finite number of elements* μ, $\nu \in S_p$ such that $|a_\nu^j - a_\mu^j| > 2$ whenever $\nu \neq \mu$. Therefore as $\|a_\nu - a_\mu\| \geqq 3$ for all ν, $\mu \in S_p$ with $\nu \neq \mu$, there exists an $i_{p+1} > i_p$ with

$$\sup_{\nu, \mu \in S_p} |a_\nu^{i_{p+1}} - a_\mu^{i_{p+1}}| > 2.$$

Obviously there is some $n_{p+1} \in S_p$ such that for all $\nu \in S_{p+1}$ either

$$\ominus_{p+1}\colon a_{n_{p+1}}^{i_{p+1}} - 1 \geqq a_\nu^{i_{p+1}} \quad \text{or} \quad \oplus_{p+1}\colon a_{n_{p+1}}^{i_{p+1}} + 1 \leqq a_\nu^{i_{p+1}},$$

where $S_{p+1} \subset S_p$ is an infinite subset of $\{n\colon n \geqq n_{p+1}\}$.

Since $\mathscr{P}(p + 1)$ holds, it follows by induction that two increasing sequences (n_k) and (i_k) exist, so that either of the conditions $\ominus_k$ or $\oplus_k$ below hold for all k

$$\ominus_k\colon a_{n_k}^{i_k} - 1 \geqq a_{n_{k+p}}^{i_k} \quad \text{for all} \quad p = 1, 2, \ldots,$$

$$\oplus_k\colon a_{n_k}^{i_k} + 1 \leqq a_{n_{k+p}}^{i_k} \quad \text{for all} \quad p = 1, 2, \ldots.$$

Now, if $\ominus_k$ occurs infinitely many times, we pass, if necessary, to a subsequence of $(a_{n_k}^{i_k})$ and obtain the statement. If not, then $\oplus_k$ occurs infinitely often, by which we again, if necessary, restrict to subsequences. We change the sign and use that

$$-a^{i_k}_{n_k+p} = a^{i_k}_{n_k+p+1} \quad \text{for all} \quad p = 0, 1, \ldots$$

to obtain the same statement thus proving the lemma.

THEOREM 2. *In real Banach spaces the C^∞-bounding sets are relatively compact.*

PROOF. Suppose, contrary to the statement, that there exist a real Banach space E and a symmetric C^∞-bounding set $B \subset E$ which is not precompact. Then there is an $\varepsilon > 0$ and a C^∞-bounding sequence (z_n) in B such that $z_{2n+1} = -z_{2n}$ for all n and $\|z_n - z_m\| \geqq \varepsilon$ if $n \neq m$. According to the Hahn-Banach theorem, there exists $l_{mn} \in E'$ with $\|l_{mn}\| = 1$ such that $|l_{mn}(z_n - z_m)| \geqq \varepsilon$ for $m \neq n$. Now the set $T := \{(m,n) \in \mathsf{N}^2 : n \neq m\}$ is countable. Hence we get a well defined continuous linear operator $L: E \to l^\infty;\ z \mapsto \left(\frac{3}{\varepsilon} l_k(z)\right)_{k \in T}$. Thus (a_n), where $a_n = L(z_n)$, is a C^∞-bounding sequence in l^∞ with $a_{2n+1} = -a_{2n}$ for all n. Since $\|a_n - a_m\| \geqq 3$ if $n \neq m$, there is, by Lemma 1, a subsequence (a_{n_k}) and an increasing sequence (i_k) in N such that

$$a^{i_k}_{n_k} - 1 \geqq a^{i_k}_{n_k+p} \quad \text{for all} \quad k, p = 1, 2, \ldots.$$

Let $f: l^\infty \to \mathsf{R}$ be the function that assigns a vector $x = (x^1, x^2, \ldots)$ in l^∞ the value

$$f(x) = h_1(x^{i_1}) + 2g_1(x^{i_1})h_2(x^{i_2}) + 3g_1(x^{i_1})g_2(x^{i_2})h_3(x^{i_3}) + \ldots$$
$$\ldots + kg_1(x^{i_1})g_2(x^{i_2})\ldots g_{k-1}(x^{i_{k-1}})h_k(x^{i_k}) + \ldots,$$

where (h_k) and (g_k) are sequences of non-negative C^∞-functions on R such that

$$h_k(t) = \begin{cases} 1, & \text{for } t = a^{i_k}_{n_k} \\ 0, & \text{for } t \leqq a^{i_k}_{n_k} - \frac{1}{4}, \end{cases}$$

$$g_k(t) = \begin{cases} 1, & \text{for } t \leqq a^{i_k}_{n_k} - 1 \\ 0, & \text{for } t \geqq a^{i_k}_{n_k} - \frac{3}{4}. \end{cases}$$

Take an arbitrary $x_0 = (x^1, x^2, \ldots) \in l^\infty$. If there exists some k with $x^{i_k} > a^{i_k}_{n_k} - \frac{1}{2}$, then for every $y = (y^1, y^2, \ldots)$ with $\sup_\nu |y^\nu| < \frac{1}{4}$

$$f(x_0 + y) = h_1(x^{i_1} + y^{i_1}) + 2g_1(x^{i_1} + y^{i_1})h_2(x^{i_2} + y^{i_2}) + \ldots$$
$$\ldots + kg_1(x^{i_1} + y^{i_1})\ldots g_{k-1}(x^{i_{k-1}} + y^{i_{k-1}})h_k(x^{i_k} + y^{i_k}),$$

by which f is a C^∞-function on l^∞ at the point x_0. On the other hand, if for all k we have that $x^{i_k} \leqq a^{i_k}_{n_k} - \frac{1}{2}$, then $f(x_0 + y) = 0$ for all $y = (y^1, y^2, \ldots)$ with $\sup_\nu |y^\nu| < \frac{1}{4}$. Thus also now f is a C^∞-function on l^∞ at the point x_0, and therefore a C^∞-function on l^∞ since x_0 is arbitrary. By construction, $f(a_{n_k}) = k$ for each k. This is however a contradiction since $L(B)$ is C^∞-bounding and therefore the theorem is proved.

The reader observes that the function f constructed in the proof above is a very special C^∞-function on l^∞ since f locally depends on a finite number of coordinates. Using this we can extend our result in the following way.

Given a real separated locally convex space (lcs) E, let $C_f^\infty(E)$ denote the set of *finitely smooth* functions on E, i.e. $f \in C_f^\infty(E)$ if, for each $x \in E$ there are an open $U \ni x$, a finite set $\{l_1, \ldots, l_n\} \subset E'$ and a smooth function $g: \mathrm{R}^n \to \mathrm{R}$ with $f(x) = g(l_1(x), \ldots, l_n(x))$ for all $x \in U$. Then $C_f^\infty(\mathrm{R}^n) = C^\infty(\mathrm{R}^n)$, but $C_f^\infty(E) \subsetneqq C^\infty(E)$ if E is an infinite-dimensional Banach space. Obviously the set $C_f^\infty(E)$ is an algebra that is closed under composition with continuous linear maps from the left and with smooth functions on the reals from the right. It is then readily checked that we in Theorem 2 actually have proved that each C_f^∞-bounding set in a real Banach space is relatively compact.

The result of Theorem 2 extends easily to locally convex spaces: Let A be a C_f^∞-bounding set in a lcs E. Since E is a subspace of a topological product $\prod_{\iota \in I} E_\iota$ of Banach spaces E_ι, the sets $\mathrm{pr}_\iota(A)$ are C_f^∞-bounding in E_ι for each $\iota \in I$. By Theorem 2, these projected sets are relatively compact in E_ι, hence precompact as these two concepts agree in quasi-complete spaces. Precompactness is preserved by formation of arbitrary products and subspaces [6, p. 65] and therefore Theorem 2 implies the more general

COROLLARY 3. *Any C_f^∞-bounding set in a* lcs *E is precompact and therefore relatively compact if E, in addition, is quasi-complete.*

By considering an example in [5, p. 22], we see that there is a locally convex space with a C_f^∞-bounding set that is not relatively compact. Nevertheless there are a number of situations where the C_f^∞-bounding sets are characterized as relatively compact without use of quasi-completeness. For instance, when E is paracompact and the functions in $C_f^\infty(E)$ induce the topology of the space (e.g. if E admits C_f^∞-smooth partition of unity). Indeed, in this case we obtain using the Tietze extension theorem that every closed C_f^∞-bounding set is countably compact and hence compact by paracompactness of E. We have also the following:

PROPOSITION 4. *Let E be a quasi-complete* lcs. *Then the C_f^∞-bounding sets in $(E, \sigma(E, E'))$ are relatively weakly compact.*

PROOF. Take a C_f^∞-bounding set $A \subset (E, \sigma(E, E'))$ and assume that A is not relatively weakly compact. Then, by [8], there is a $f \in C(E, \sigma(E, E'))$ that is unbounded on A. Pick a sequence (x_n) in A such that $f(x_{n+1}) > f(x_n) + 1$ for every n. Obviously there is for each n a function $g_n \in C_f^\infty(E)$ vanishing on $\{x \in E: |f(x) - f(x_n)| \geqq 1\}$ and with $g_n(x_n) = n$. Thus $g = \sum_{n=1}^\infty g_n$ is a C_f^∞-function on E which is unbounded on A, a contradiction.

Another example of spaces in which the C_f^∞-bounding sets are characterized without quasi-completeness is found in the paper [1] (although stated for C^∞-functions it works for C_f^∞-functions):

PROPOSITION 5. *Each C_f^∞-bounding set in a Lindelöf* lcs *is relatively compact.*

Let E_∞ denote the space E endowed with the weakest topology making all C_f^∞-functions on E continuous. Take a sequence (x_n) which converges to x in E_∞ and let A be the compact set $\{x\} \cup \{x_n: n \in \mathbb{N}\}$ in E_∞. Assume that A is compact in E as well. Thus the identity

$$\text{id}: (A, E) \to (A, E_\infty)$$

is not only a continuous bijection, but also – since E_∞ is Hausdorff – a homeomorphism, by which the sequence (x_n) converges to x in E. We therefore arrive at:

COROLLARY 6. *Let E be a* lcs *that either is quasi-complete or Lindelöf. Then E and E_∞ have the same compact sets. Furthermore $x_n \to x$ in E if and only if $f(x_n) \to f(x)$ for all $f \in C_f^\infty(E)$.*

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