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STABILITY IN OBSTACLE PROBLEMS

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1. Introduction.

We consider a second order degenerate elliptic partial differential equation

$$\nabla \cdot \mathscr{A}(x, \nabla u(x)) = 0 \tag{1.1}$$

with $\mathscr{A}(x, h) \cdot h \approx |h|^p$; for the assumptions on $\mathscr{A}$ see Section 2. A prototype for equation (1.1) is the p-harmonic equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \tag{1.2}$$

which for $p = 2$ reduces to the usual Laplace equation $\Delta u = 0$.

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $\theta \in W^{1,p}(\Omega)$, i.e. θ and its distributional gradient $\nabla\theta$ belong to $L^p(\Omega)$. For $\psi\colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ we write

$$\mathscr{K}_{\psi,\theta} = \{v \in W^{1,p}(\Omega)\colon v \geqq \psi \text{ a.e., } v - \theta \in W_0^{1,p}(\Omega)\}$$

and call a function v a solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem if $v \in \mathscr{K}_{\psi,\theta}$ and if

$$\int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(\varphi - v)\, dx \geqq 0 \tag{1.3}$$

whenever $\varphi \in \mathscr{K}_{\psi,\theta}$. If $\mathscr{K}_{\psi,\theta} \neq \emptyset$, then there is a unique solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem [HKM, Theorem 3.21].

The aforementioned obstacle problems include variational obstacle problems associated with regular variational integrals

$$\int_\Omega F(x, \nabla u)\, dx$$

where $F(x, h) \approx |h|^p$, see [HKM, Chapter 5].

We are interested in stability properties of the solutions u_ψ to the $K_{\psi,\theta}$-obstacle problem for varying ψ and we prove two results which are independent of the exponent p. However, our first result does not belong to this category and it

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depends on the $(1,p)$-quasiuniform converge. We introduce some notation. Let $\psi_i\colon \Omega \to \mathrm{R} \cup \{\pm\infty\}$ be a sequence of functions and $p > 1$. We say that $\psi_i \to \psi$ locally $(1,p)$-quasiuniformly if for each open set $D \Subset \Omega$ and each $\varepsilon > 0$ there is an open set $G \subset D$ such that $\operatorname{cap}_{1,p}(G) < \varepsilon$ and $\psi_i \to \psi$ in $L^\infty(D \setminus G)$. Here $\operatorname{cap}_{1,p}$ refers to the variational Sobolev $(1,p)$-capacity, i.e.

$$\operatorname{cap}_{1,p}(G) = \sup_{\substack{K \subset G \\ K\,\text{compact}}} \inf \int_{\mathrm{R}^n} (|u|^p + |\nabla u|^p)\,dx$$

where the infimum is taken over all $u \in C_0^\infty(\mathrm{R}^n)$ such that $u \geqq 1$ on K. The sequence ψ_i is called locally weakly upper bounded if for each compact set $K \subset \Omega$ there are $M < \infty$ and i_0 such that $\psi_i \leqq M$ a.e. on K for $i \geqq i_0$.

1.4. THEOREM. *Suppose that $\theta \in W^{1,p}(\Omega)$ and that $\psi_i, \psi\colon \Omega \to \mathrm{R} \cup \{\pm\infty\}$ are functions such that* (i) $\psi_i \leqq \theta$ *for all i,* (ii) *the sequence $\psi_i - \psi$ is locally weakly upper bounded and* (iii) $\psi_i \to \psi$ *locally $(1,p)$-quasiuniformly. If u_{ψ_i} and u_ψ are solutions to the $\mathscr{K}_{\psi_i,\theta}$-obstacle and $\mathscr{K}_{\psi,\theta}$-obstacle problems, respectively, then $u_{\psi_i} \to u_\psi$ in $W^{1,p}(\Omega)$.*

We remark, to avoid any possible ambiguity, that in (ii), $\pm\infty - (\pm\infty)$ means 0.

In Theorem 1.4 (ii) and (iii) can be replaced by $\psi_i \to \psi$ in $L^\infty_{\mathrm{loc}}(\Omega)$, see Remark 2.23 (b), and hence Theorem 1.4 has the following version.

1.5. THEOREM. *Suppose that $\theta \in W^{1,p}(\Omega)$ and that $\psi_i\colon \Omega \to \mathrm{R} \cup \{\pm\infty\}$ are such that $\theta \geqq \psi_i$* a.e. *for $i = 1, 2, \ldots$. If $\psi_i \to \psi$ in $L^\infty_{\mathrm{loc}}(\Omega)$, then $u_{\psi_i} \to u_\psi$ in $W^{1,p}(\Omega)$ where u_{ψ_i} and u_ψ are solutions to the $\mathscr{K}_{\psi_i,\theta}$-and $\mathscr{K}_{\psi,\theta}$-obstacle problems, respectively.*

Theorem 1.5 has been proved for example in [HKM, Theorem 3.79] under the condition that $\psi_i \to \psi$ in $W^{1,p}(\Omega)$ and ψ_i is a decreasing sequence.

For the next theorem assume that $\mathscr{S}$ is a family of solutions to obstacle problems in Ω, i.e. for each $u \in \mathscr{S}$ there are $\psi_u\colon \Omega \to \mathrm{R} \cup \{\pm\infty\}$ and $\theta_u \in W^{1,p}(\Omega)$ such that u is a solution to the $\mathscr{K}_{\psi_u,\theta_u}$-obstacle problem. Write

$$\mathscr{O} = \{\psi_u\colon u \in \mathscr{S}\}.$$

1.6. THEOREM. *Suppose that $x_0 \in \Omega$, that there are a neighborhood $\mathscr{U}$ of x_0 and $M < \infty$ such that*

$$\operatorname*{ess\,sup}_{\mathscr{U}} - \operatorname*{ess\,inf}_{\mathscr{U}} u \leqq M \tag{1.7}$$

for all $u \in \mathscr{S}$, and that

(1.8) *the family $\mathscr{O}$ is equicontinuous at x_0.*

Then the family $\mathcal{S}$ is equicontinuous at x_0.

In Theorem 1.6 equicontinuity means a.e. equicontinuity. Thus, for example, $\mathcal{O}$ is equicontinuous at x_0 if for each $\varepsilon > 0$ there is a neighborhood V of x_0 such that

$$\operatorname*{ess\,sup}_V \psi - \operatorname*{ess\,inf}_V \psi \leqq \varepsilon$$

for all $\psi \in \mathcal{O}$.

Continuity of the solution to an obstacle problem has been studied in [KZ]; for Theorem 1.6 the theory of Wiener points for obstacles [KZ] is not needed.

Theorems 1.4 and 1.6 are proved in Sections 3 and 4, respectively. Section 2 contains preliminary considerations. The proof for Theorem 1.4 consists of two parts. We first show that $u_{\psi_i} \to v$ in $W^{1,p}(\Omega)$ and then that $v = u_\psi$. For the first part we employ an improvement of a lemma ([L, Lemma 2.2], [Maz, Lemma 1], [HKM, Lemma 3.73]) which has been used to prove weak and a.e. convergence of the gradients in $L^p(\Omega)$; using the Vitali convergence theorem we show that it is possible to conclude strong convergence in $L^p(G)$. When this paper was completed, a paper of L. Boccardo and F. Murant [BM] containing a similar observation appeared. We also present examples which show that Theorems 1.4–1.6 are best possible.

Our notation is standard. We let

$$\|u\|_p = \|u\|_{L^p(E)} = \left(\int_E |u|^p \, dx\right)^{1/p}$$

denote the usual L^p-norm of a function u. In the Sobolev space $W^{1,p}(\Omega)$ we use the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p.$$

The space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

2. Preliminaries.

We first introduce the assumptions for the equation (1.1). Let $p > 1$ and let Ω be a bounded open set in $\mathbf{R}^n$. We assume that $\mathcal{A}: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is a mapping satisfying the following assumptions for some constants $0 < \alpha \leqq \beta < \infty$:

(2.1) the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbf{R}^n$ and the mapping $\xi \to \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \Omega$;

for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \Omega$

$$\mathscr{A}(x,\xi)\cdot\xi \geqq \alpha|\xi|^p \tag{2.2}$$

$$|\mathscr{A}(x,\xi)| \leqq \beta|\xi|^{p-1} \tag{2.3}$$

$$(\mathscr{A}(x,\xi_1)-\mathscr{A}(x,\xi_2))\cdot(\xi_1-\xi_2) > 0 \quad \text{whenever} \quad \xi_1 \neq \xi_2; \tag{2.4}$$

and

$$\mathscr{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi) \tag{2.5}$$

for $\lambda \in \mathsf{R}$, $\lambda \neq 0$.

For the proofs we need other classes of solutions than solutions to the obstacle problems. Let $\theta \in W^{1,p}(\Omega)$. A function v is called a supersolution of (1.1) with boundary values θ if $v - \theta \in W_0^{1,p}(\Omega)$ and if

$$\int_\Omega \mathscr{A}(x,\nabla v)\cdot\nabla\varphi\,dx \geqq 0 \tag{2.6}$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqq 0$. It is easy to see that (2.6) holds for functions $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geqq 0$, as well. Every solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem is a supersolution with boundary values θ and, conversely, every supersolution v is a solution to the $\mathscr{K}_{v,\theta}$-obstacle problem. Finally v is a solution with boundary values θ if $v - \theta \in W_0^{1,p}(\Omega)$ and if

$$\int_\Omega \mathscr{A}(x,\nabla v)\cdot\nabla\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. For the theory of these solution classes see [HKM]. Since we mostly keep θ fixed, we simply speak about supersolutions and solutions.

We first improve a lemma which has been used in several occasions, see e.g. [HKM, Lemma 3.73], [Maz, Lemma 1], [L, Lemma 2.2], and [G, p. 197]. We show that condition (2.8) below implies the strong convergence for gradients in $L^p(\Omega)$.

2.7. LEMMA. *Suppose that $v_i, v \in W^{1,p}(\Omega)$, $i = 1,2,\ldots$. Then*

$$\lim_{i\to\infty}\int_\Omega (\mathscr{A}(x,\nabla v_i)-\mathscr{A}(x,\nabla v))\cdot\nabla(v_i - v)\,dx = 0 \tag{2.8}$$

if and only if $\nabla v_i \to \nabla v$ in $L^p(\Omega)$.

PROOF. Write

$$I_i = \int_\Omega (\mathscr{A}(x,\nabla v_i)-\mathscr{A}(x,\nabla v))\cdot\nabla(v_i - v)\,dx.$$

Suppose that $\nabla v_i \to \nabla v$ in $L^p(\Omega)$. Now $\mathscr{A}(x,\nabla v_i)$ is a bounded sequence in $L^{p/(p-1)}(\Omega)$ and $\mathscr{A}(x,\nabla v) \in L^{p/(p-1)}(\Omega)$, see (2.3), and since

$$I_i = \int_\Omega \mathscr{A}(x, \nabla v_i) \cdot \nabla(v_i - v)\, dx - \int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(v_i - v)\, dx,$$

we obtain from Hölder's inequality that $I_i \to 0$ as $i \to 0$.

Next assume that $I_i \to 0$. It follows from Young's inequality, see also the proof of Lemma 3.73 in [HKM, p. 80], that ∇v_i is a bounded sequence in $L^p(\Omega)$ and, passing to a subsequence if necessary, we may assume that

$$\nabla v_i \to \nabla v \tag{2.9}$$

a.e. in Ω. For each $\varepsilon > 0$ there is i_0 such that

$$\int_\Omega (\mathscr{A}(x, \nabla v_i) - \mathscr{A}(x, \nabla v)) \cdot \nabla(v_i - v)\, dx < \varepsilon$$

for $i \geqq i_0$. Since the integrand is non-negative, for each measurable set $E \subset \Omega$ we have

$$\int_E (\mathscr{A}(x, \nabla v_i) - \mathscr{A}(x, \nabla v)) \cdot \nabla(v_i - v)\, dx < \varepsilon$$

for $i \geqq i_0$. Then

$$\begin{aligned}
\alpha \int_E |\nabla v_i|^p\, dx &\leqq \int_E \mathscr{A}(x, \nabla v_i) \cdot \nabla v_i\, dx \\
&\leqq \varepsilon + \int_E \mathscr{A}(x, \nabla v) \cdot \nabla(v - v_i)\, dx + \int_E \mathscr{A}(x, \nabla v_i) \cdot \nabla v\, dx \\
&\leqq \varepsilon + \beta \int_E |\nabla v|^{p-1} |\nabla v - \nabla v_i|\, dx + \beta \int_E |\nabla v_i|^{p-1} \nabla v\, dx \\
&\leqq \varepsilon + \beta\, \|\nabla v\|_{L^p(E)}^{p-1} \|\nabla v - \nabla v_i\|_{L^p(\Omega)} + \beta\, \|\nabla v_i\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(E)}
\end{aligned}$$

where we have used Hölder's inequality. Since the sequence ∇v_i is bounded in $L^p(\Omega)$, we see that for each $\varepsilon > 0$ there is $\delta > 0$ such that for all i

$$\int_E |\nabla v_i|^p\, dx < \varepsilon$$

whenever $m(E) < \delta$. This means that $|\nabla v_i|^p$ is uniformly integrable and this together with (2.9) yields that $\nabla v_i \to \nabla v$ in $L^p(\Omega)$, see [HS, p. 203] for the Vitali convergence theorem. Finally, observe that this holds for the original sequence ∇v_i, and not for its subsequence only, since if $\|\nabla v_{i_j} - \nabla v\|_{L^p(\Omega)} \geqq \varepsilon > 0$ in $L^p(\Omega)$ for some subsequence of ∇v_i, then the above proof gives a contradiction. The lemma follows.

For the next lemma assume that $u_i \in W^{1,p}(\Omega)$, $i = 1, 2, \ldots$, and $u \in W^{1,p}(\Omega)$ are such that

$$\|\nabla u_i\|_p \leqq C < \infty \tag{2.10}$$

and

$$\int_\Omega \mathscr{A}(x, \nabla u_i) \cdot \nabla(\varphi_i + u - u_i)\, dx \geqq 0, \tag{2.11}$$

$$\int_\Omega \mathscr{A}(x, \nabla u) \cdot \nabla(\Phi_i + u_i - u)\, dx \geqq 0$$

for some functions φ_i, $\Phi_i \in W^{1,p}(\Omega)$ with

$$\nabla \varphi_i \to 0 \text{ in } L^p(\Omega), \tag{2.12}$$

$$\nabla \Phi_i \to 0 \text{ weakly in } L^p(\Omega), \tag{2.13}$$

2.14. LEMMA. *$\nabla u_i \to \nabla u$ in $L^p(\Omega)$.*

PROOF. We show that

$$I_i = \int_\Omega (\mathscr{A}(x, \nabla u_i) - \mathscr{A}(x, \nabla u)) \cdot \nabla(u_i - u)\, dx \to 0$$

as $i \to \infty$. Then Lemma 2.7 implies that $\nabla u_i \to \nabla u$ in $L^p(\Omega)$.

To this end, inequalities (2.11) yield

$$I_i \leqq \int_\Omega \mathscr{A}(x, \nabla u_i) \cdot \nabla \varphi_i\, dx + \int_\Omega \mathscr{A}(x, \nabla u) \cdot \nabla \Phi_i\, dx. \tag{2.15}$$

Since $\mathscr{A}(x, \nabla u) \in L^{p/(p-1)}(\Omega)$ and $\nabla \Phi_i \to 0$ weakly in $L^p(\Omega)$, the last integral in (2.15) tends to zero as $i \to \infty$. Since $I_i \geqq 0$, it remains to show that

$$\int_\Omega \mathscr{A}(x, \nabla u_i) \cdot \nabla \varphi_i\, dx \to 0 \tag{2.16}$$

as $i \to \infty$. Since ∇u_i is a bounded sequence in $L^p(\Omega)$, we obtain from Hölder's inequality that

$$\left| \int_\Omega \mathscr{A}(x, \nabla u_i) \cdot \nabla \varphi_i\, dx \right| \leqq \beta\, \|\nabla u_i\|_p^{p-1}\, \|\nabla \varphi_i\|_p$$

$$\leqq c\, \|\nabla \varphi_i\|_p$$

where c is independent of i. Since $\nabla \varphi_i \to 0$ in $L^p(\Omega)$, (2.16) follows and the proof is complete.

Lemma 2.18 gives strong convergence of the gradients for solutions to obstacle problems with zero boundary values. Here a result of L.-I. Hedberg is needed.

2.17. LEMMA ([H]). *Let $\varphi \in W_0^{1,p}(\Omega)$. Then for each compact set $K \subset \Omega$ and for each $\varepsilon > 0$ there is a function $\eta \in C_0^\infty(\Omega)$ such that $0 \leqq \eta \leqq 1$, $\eta = 1$ on K, and $\|\nabla(1-\eta)\varphi)\|_p < \varepsilon$.*

2.18. LEMMA. *Suppose that $\varphi \in W_0^{1,p}(\Omega)$ and that $\theta_i\colon \Omega \to \mathsf{R} \cup \{\pm\infty\}$ is a locally weakly upper bounded sequence of functions such that for $i = 1.2, \ldots, \varphi \geqq \theta_i$* a.e. *and $\theta_i \to 0$ locally $(1, p)$-quasiuniformly. Then the solutions u_{θ_i} to the $\mathscr{K}_{\theta_i, 0}$-obstacle problem satisfy $\nabla u_{\theta_i} \to 0$ in $L^p(\Omega)$.*

PROOF. Since $\varphi \in \mathscr{K}_{\theta_i, 0}$, the family $\mathscr{K}_{\theta_i, 0}$ is non-empty. Hence the solutions u_{θ_i} exist. Since $u_{\theta_i} \geqq 0$ ([HKM, Lemma 3.18]), we may assume that $\theta_i \geqq 0$.

Let $D \Subset \Omega$ be an open set and $t > 0$. Since $\theta_i \to 0$ locally $(1, p)$-quasiuniformly, there is an open set $G \subset D$ and a function $u \in W_0^{1,p}(\Omega)$ such that (i) $0 \leqq u \leqq 1$, (ii) $u = 1$ $(1, p)$-quasieverywhere on G, (iii) $\theta_i \to 0$ uniformly in $D \backslash G$, and

$$\text{(iv)} \qquad \|\nabla u\|_p < t;$$

for this construction see [HKM, Corollary 4.13] and [HKM, p. 49]. Note that a capacity function can be cut off from $W^{1,p}(\mathsf{R}^n) = W_0^{1,p}(\mathsf{R}^n)$ to $W_0^{1,p}(\Omega)$ because $D \Subset \Omega$. That $u = 1$ $(1, p)$-quasieverywhere on G means that $u = 1$ on G except of a set of $(1, p)$-capacity zero. This implies that $u = 1$ a.e. on G, see [HKM, Lemma 2.10].

After these preliminaries we prove the lemma. Let $\varepsilon > 0$ and choose a function $\eta \in C_0^\infty(\Omega)$ as in Lemma 2.17 such that

$$\|\nabla((1-\eta)\varphi)\|_p \leqq \varepsilon. \tag{2.19}$$

Next let D be an open set with $\operatorname{spt} \eta \subset D \Subset \Omega$. Since θ_i is a locally weakly upper bounded sequence, there are i_0 and $M \in (0, \infty)$ such that

$$\theta_i \leqq M \quad \text{a.e. on} \quad D \tag{2.20}$$

for $i \geqq i_0$. Choose $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \|\nabla \eta\|_p < \varepsilon \tag{2.21}$$

and $G \subset D$ and $u \in W_0^{1,p}(\Omega)$ as in (i)–(iv) for $t = \varepsilon/M$. Then pick $i_0' \geqq i_0$ so large that for $i \geqq i_0'$

$$\theta_i \leqq \varepsilon_1 \quad \text{on} \quad D \backslash G. \tag{2.22}$$

Consider the function

$$v = (1-\eta)\varphi + \varepsilon_1 \eta + Mu.$$

Now $v \in W_0^{1,p}(\Omega)$ and from (2.20) and (2.22) it follows that $v \in \mathscr{K}_{\theta_i,0}$ for $i \geqq i_0'$. Indeed, $v \geqq \varphi \geqq \theta_i$ in $\Omega \backslash D$, $v \geqq (1-\eta)\theta_i + \theta_i \eta = \theta_i$ in $D \backslash G$ for $i \geqq i_0'$ and $v \geqq Mu \geqq \theta_i$ in G for $i \geqq i_0$. Hence

$$\int_\Omega \mathscr{A}(x, \nabla u_{\theta_i}) \cdot \nabla(v - u_{\theta_i})\, dx \geqq 0$$

and this yields for $i \geqq i_0'$

$$\begin{aligned} \|\nabla u_{\theta_i}\|_p &\leqq \left(\frac{\beta}{\alpha}\right) \|\nabla v\|_p \\ &\leqq \left(\frac{\beta}{\alpha}\right) \|\nabla((1-\eta)\varphi)\|_p + \varepsilon_1 \|\nabla \eta\|_p + M \|\nabla u\|_p) \\ &\leqq 3\left(\frac{\beta}{\alpha}\right)\varepsilon \end{aligned}$$

where the last inequality follows from (2.19), (2.21), and from (iv) with $t = \varepsilon/M$. Thus $\nabla u_{\theta_i} \to 0$ in $L^p(\Omega)$ as required.

2.23. REMARKS. (a) Lemmas 2.14 and 2.18 and their proofs remain as stated in any open set $\Omega \subset \mathsf{R}^n$. The proof of Lemma 2.7 needs a slight adjustment for an unbounded open set $\Omega \subset \mathsf{R}^n$; sets $E \subset \Omega$ outside a large ball require a separate treatment in the uniform integrability.

(b) If in Lemma 2.18 it is assumed that $\theta_i \to 0$ in $L^\infty_{\text{loc}}(\Omega)$, then the conclusion holds. In fact, for $i \geqq i_0$, $\theta_i \leqq \varepsilon$ a.e. in D and the function $v = (1-\eta)\varphi + \varepsilon\eta$ will do. After this observation the proof for Theorem 1.5 is the same as for Theorem 1.4.

3. Proof for Theorem 1.4.

First we make some preliminary observations. Let u be a solution of (1.1) in Ω with boundary values θ. Then u is a solution to the $\mathscr{K}_{-\infty,\theta}$-obstacle problem and hence $u_{\psi_i}, u_\psi \geqq u$ a.e., see [HKM, Lemma 3.22]. This means that u_{ψ_i} and u_ψ are also solutions to the $\mathscr{K}_{\max(\psi_i,u),\theta}$-and $\mathscr{K}_{\max(\psi,u),\theta}$-obstacle problems, respectively. Thus we can replace ψ_i and ψ by $\max(\psi_i, u)$ and $\max(\psi, u)$. Observe that after this replacement we still have that $\psi_i \to \psi$ locally $(1,p)$-quasiuniformly and that $\psi_i - \psi$ is a locally weakly upper bounded sequence; note that $\psi_i \leqq \theta$ a.e. implies $\psi_i < \infty$ a.e. As a boundary function we can use, instead of θ, the function u_θ which is a solution to the $\mathscr{K}_{\theta,\theta}$-obstacle problem. Then $u_\theta \geqq u_{\psi_i}$ and $u_\theta \geqq u_\psi$ and replacing θ by u_θ we may assume that

$$u \leqq \psi_i \leqq u_{\psi_i} \leqq \theta, \qquad u \leqq \psi \leqq u_\psi \leqq \theta \tag{3.1}$$

a.e. in Ω. Observe that $\theta - u \in W_0^{1,p}(\Omega)$.

Choosing $v = u_{\psi_i}$ and $\varphi = \theta$ in (1.3) we see that

$$\|\nabla u_{\psi_i}\|_p \leqq C$$

where $C < \infty$ is independent of i. By the Poincaré inequality the sequence u_{ψ_i} is bounded in $L^p(\Omega)$ as well. Thus we may assume, passing to a subsequence if necessary, that $u_{\psi_i} \to v$ weakly in $L^p(\Omega)$ for some $v \in W^{1,p}(\Omega)$ and $\nabla u_{\psi_i} \to \nabla v$ weakly in $L^p(\Omega)$. Since $W_0^{1,p}(\Omega)$ is weakly closed, $v - \theta \in W_0^{1,p}(\Omega)$. By the Sobolev imbedding theorem [A, Theorem 6.2, Part IV] we may also assume that

$$u_{\psi_i} \to v \quad \text{in} \quad L^p(\Omega) \tag{3.2}$$

and that

$$u_{\psi_i} \to v \quad \text{a.e. in } \Omega. \tag{3.3}$$

Since $u_{\psi_i} \geqq \psi_i$ a.e. and since $\psi_i \to \psi$ locally $(1, p)$-quasiuniformly and hence a.e. in Ω, we have $v \geqq \psi$ a.e.

Next we will prove that

$$\nabla u_{\psi_i} \to \nabla v \quad \text{in} \quad L^p(\Omega). \tag{3.4}$$

From (3.2) it then follows that $u_{\psi_i} \to v$ in $W^{1,p}(\Omega)$. Finally we will show that $v = u_\psi$ and this completes the proof because now the original sequence, and not only its subsequence, must converge to u_ψ in $W^{1,p}(\Omega)$.

To prove (3.4) we reduce the problem to zero boundary values. Let φ_i be a solution to the $\mathcal{K}_{\psi_i - \psi, 0}$-obstacle problem; note that $\theta - u \in \mathcal{K}_{\psi_i - \psi, 0}$ and hence a solution exists. Now $\psi_i - \psi \leqq \theta - u \in W_0^{1,p}(\Omega)$ a.e. in Ω and $\psi_i - \psi \to 0$ locally $(1, p)$-quasiuniformly. Morever, the sequence $\psi_i - \psi$ is locally weakly upper bounded in Ω; observe that ψ_i and ψ are a.e. finite by (3.1). Lemma 2.18 yields

$$\nabla \varphi_i \to 0 \quad \text{in} \quad L^p(\Omega). \tag{3.5}$$

We let $\Phi_i = v - u_{\psi_i}$. Then

$$\int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla(\Phi_i + u_{\psi_i} - v)\, dx = 0. \tag{3.6}$$

On the other hand,

$$\varphi_i + v \geqq \psi_i - \psi + v \geqq \psi_i - \psi + \psi = \psi_i \tag{3.7}$$

a.e. in Ω and hence

$$\int_\Omega \mathcal{A}(x, \nabla u_{\psi_i}) \cdot \nabla(\varphi_i + v - u_{\psi_i})\, dx \geqq 0 \tag{3.8}$$

because u_{ψ_i} is the solution to the $\mathcal{K}_{\psi_i, \theta}$-obstacle problem and

$\varphi_i + v - \theta \in W_0^{1,p}(\Omega)$. From (3.5), (3.6), and (3.8) together with Lemma 2.14 it now follows that $\nabla u_{\psi_i} \to \nabla v$ in $L^p(\Omega)$.

It remains to show that $v = u_\psi$. Since a solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem is unique, it suffices to show that

$$\int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(\eta - v)\, dx \geqq 0 \tag{3.9}$$

for all $\eta \in \mathscr{K}_{\psi,\theta}$. Fix $\eta \in \mathscr{K}_{\psi,\theta}$ and write $u_i = \eta + \varphi_i$ where φ_i, as above, is the solution to the $\mathscr{K}_{\psi_i - \psi, 0}$-obstacle problem. From (3.5) we obtain

$$\nabla u_i \to \nabla \eta \quad \text{in} \quad L^p(\Omega) \tag{3.10}$$

and this implies (3.9). Indeed, passing to a subsequence if necessary, we may assume that

$$\mathscr{A}(x, \nabla u_{\psi_i}) \cdot \nabla(u_i - u_{\psi_i}) \to \mathscr{A}(x, \nabla v) \cdot \nabla(\eta - v) \tag{3.11}$$

a.e. in Ω and for all measurable sets $E \subset \Omega$ we obtain

$$\begin{aligned}
&\int_E |\mathscr{A}(x, \nabla u_{\psi_i}) \cdot \nabla(u_i - u_{\psi_i})|\, dx \\
&\leqq \beta \int_E |\nabla u_{\psi_i}|^{p-1} |\nabla(u_i - u_{\psi_i})|\, dx \\
&\leqq \beta\, \|\nabla u_{\psi_i}\|_{L^p(E)}^{p-1} (\|\nabla u_i\|_{L^p(\Omega)} + \|\nabla v_{\psi_i}\|_{L^p(\Omega)} \\
&\leqq \beta\, \|\nabla u_{\psi_i}\|_{L^p(E)}^{p-1} (\|\nabla \eta\|_{L^p(\Omega)} + \|\nabla \varphi_i\|_{L^p(\Omega)} + \|\nabla u_{\psi_i}\|_{L^p(\Omega)}) \\
&\leqq C\beta\, \|\nabla(u_{\psi_i})\|_{L^p(E)}^{p-1} \\
&< \varepsilon
\end{aligned}$$

whenever we choose $m(E) < \delta = \delta(\varepsilon)$. This means that the functions $\mathscr{A}(x, \nabla u_{\psi_i}) \cdot \nabla(u_i - u_{\psi_i})$ are uniformly integrable over Ω and together with (3.11) this implies

$$\begin{aligned}
\lim_{t \to \infty} &\int_\Omega \mathscr{A}(x, \nabla u_{\psi_i}) \cdot \nabla(u_i - u_{\psi_i})\, dx \\
&= \int_\Omega \mathscr{A}(x, \nabla v) \cdot \nabla(\eta - v)\, dx.
\end{aligned} \tag{3.12}$$

Since $u_i - \theta \in W_0^{1,p}(\Omega)$ and $u_i \geqq \eta + \psi_i - \psi \geqq \psi_i$, u_i belongs to $\mathscr{K}_{\psi_i,\theta}$ and thus

$$\int_\Omega \mathscr{A}(x, \nabla u_{\psi_i}) \cdot \nabla(u_i - u_{\psi_i})\, dx \geqq 0. \tag{3.13}$$

From (3.12) and (3.13) we thus conclude that (3.9) holds. The proof is complete.

3.14. REMARKS. (a) The local $(1, p)$-quasiuniform convergence is the weakest possible in Theorem 1.4. This can be seen by choosing a compact set $K \subset \Omega$ with positive $(1, p)$-capacity and $m(K) = 0$. Then choose compact sets $K_1 = \overline{K + B(1/i)}$ (the $1/i$-inflations of K_i). Now $K_1 \supset K_2 \supset \dots$ and $\cap K_i = K$. Let $\varphi_i = \chi_{K_i}$ and $\psi = \chi_K$. Then ψ_i does not converge to ψ locally $(1, p)$-quasiuniformly and $u_{\psi_i} \not\to u_\psi = 0$ in $W^{1,p}(\Omega)$ where u_{ψ_i} is a solution of the $\mathcal{K}_{\psi_i, 0}$-obstacle problem. This example, with suitably chosen K, also shows that no local $(1, q)$-quasiuniform convergence, $q < p$, suffices in Theorem 1.4.

(b) In general, Theorem 1.5 is also the best possible. No L^s-convergence, $1 \leqq s < \infty$, for the obstacles ψ_i is enough in Theorem 1.5. To see this choose $p > n$ and let $\Omega = B(0, 1)$. Choose a sequence $\psi_i \in C_0^\infty(B(0, 1))$ such that $\psi_i(0) = 1$ and $\psi_i \searrow 0$ in $L^s(B(0, 1))$. Then each solution u_i of the $\mathcal{K}_{\psi_i, 0}$-obstacle problem in $B(0, 1)$ satisfies $u_i(0) = 1$ and $u_i \,|\, B(0, 1/2) \geqq c > 0$ where c is independent of i. On the other hand, the solution to the $\mathcal{K}_{0,0}$-obstacle problem is $u = 0$ and hence $u_i \not\to u$ in $W^{1,p}(B(0, 1))$.

(c) The condition $\psi_i \leqq \theta$ in Theorem 1.4 is also necessary since otherwise it is easy to construct a sequence ψ_i such that $\psi_i \to 0$ uniformly in Ω but the solutions u_i to the $\mathcal{K}_{\psi_i, 0}$-obstacle problem satisfy

$$\|\nabla u_i\|_p \to \infty$$

as $i \to \infty$. Hence again the sequence u_i cannot converge in $W^{1,p}(\Omega)$ to the solution $u = 0$ of the $\mathcal{K}_{0,0}$-obstacle problem.

4. Proof for Theorem 1.6.

Let $x_0 \in \Omega$ be as in Theorem 1.6 and let $\varepsilon > 0$. Fix a ball $B = B(x_0, r)$ such that $B(x_0, 4r) \subset\subset \Omega$,

$$\operatorname*{ess\,sup}_B u - \operatorname*{ess\,inf}_B u \leqq M \tag{4.1}$$

and

$$\operatorname*{ess\,sup}_{B(x_0, 4r)} \psi_u - \operatorname*{ess\,inf}_{B(x_0, 4r)} \psi_u \leqq \varepsilon \tag{4.2}$$

for all $u \in \mathcal{S}$.

Fix $u \in \mathcal{S}$. We consider two cases. First assume that

$$\operatorname*{ess\,inf}_B u \geqq \operatorname*{ess\,sup}_B \psi_u + \varepsilon. \tag{4.3}$$

Now u is a solution of (1.1) in B. Indeed, let $\varphi \in C_0^\infty$ and let $t > 0$ be so small that $|t\varphi| \leqq \varepsilon$. Then $u + t\varphi \in \mathcal{K}_{\psi_u, \theta_u}$ by (4.3) and hence

$$t\int_B \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx \geqq 0. \tag{4.4}$$

Changing the sign of t we obtain (4.4) with the reverse inequality. Thus

$$\int_B \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx = 0 \tag{4.5}$$

for all $\varphi\in C_0^\infty(B)$ and this means that u is a solution of (1.1) in B. Now [HKM, Theorem 6.6] implies

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x_o,s)} u - \operatorname*{ess\,inf}_{B(x_o,s)} u &\leqq 2^\kappa\left(\frac{s}{r}\right)^\kappa\left(\operatorname*{ess\,sup}_B u - \operatorname*{ess\,inf}_B u\right)\\ &\leqq 2^\kappa\left(\frac{s}{r}\right)^\kappa M\end{aligned}$$

where we have also used (4.1) and $\kappa=\kappa(p,\beta/\alpha,n)>0$. Choosing $s=s(p,\beta/\alpha,n,M,r,\varepsilon)\in(0,r]$

$$2^\kappa\left(\frac{s}{r}\right)^\kappa M\leqq\varepsilon$$

we see that

$$\operatorname*{ess\,sup}_{B(x_o,s)} u - \operatorname*{ess\,inf}_{B(x_o,s)} u\leqq\varepsilon.$$

Next assume that

$$\operatorname*{ess\,inf}_B u < \operatorname*{ess\,sup}_B \psi_u+\varepsilon. \tag{4.7}$$

Let $m=\operatorname{ess\,sup}_B\psi_u$. We may assume that $m=0$; note that $m=-\infty$ is not possible in case (4.7) and $m=\infty$ is always impossible. This renormalization does not affect condition (4.1). Now $u-\varepsilon$ is a solution to the $\mathscr{K}_{\psi_u-\varepsilon,u-\varepsilon}$-obstacle problem in B and by (4.2), $\psi-\varepsilon\leqq 0$ a.e. in B. Thus [HKM, Theorem 3.34] implies

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x_o,r/2)}(u-\varepsilon)&\leqq\operatorname*{ess\,sup}_{B(x_o,r/2)}(u-\varepsilon)^+\\ &\leqq c\left(\fint_B|u-\varepsilon|^q\,dx\right)^{1/q}\end{aligned}\tag{4.8}$$

where $q=n(p-1)/2(n-p)$ if $p<n$ and $q=p$ if $p\geqq n$. Here $c=c(p,B/\alpha,n)<\infty$ and $\fint$ denotes the mean value integral, i.e.

$$\fint_B=\frac{1}{m(B)}\int.$$

Next from [HKM, Theorem 3.59] we obtain

$$\left(\fint_B (u+\varepsilon)^q\,dx\right)^{1/q} \leqq c\operatorname*{ess\,inf}_B (u+\varepsilon) \tag{4.9}$$

where c is as above; note that by (4.2)

$$u+\varepsilon \geqq \psi_u + \varepsilon \geqq 0 \text{ a.e. in } \quad B(x_0, 4r)$$

and hence $u+\varepsilon$ is a non-negative supersolution in $B(x_0, 4r)$. Since $-2\varepsilon < \psi - \varepsilon < u - \varepsilon < u + \varepsilon$, we obtain $|u-\varepsilon|^q \leqq \max((2\varepsilon)^q, |u+\varepsilon|^q) \leqq (2\varepsilon)^q + |u+\varepsilon|^q$. Now (4.8), (4.9), and (4.7) yield

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_0,r/2)} u &\leqq c\operatorname*{ess\,inf}_B u + c\varepsilon \\ &\leqq c\operatorname*{ess\,sup}_B \psi_u + c\varepsilon \\ &\leqq c\varepsilon \end{aligned} \tag{4.10}$$

because $\operatorname{ess\,sup}_B \psi_u = m = 0$. Since $u \geqq \psi_u$ a.e., we see that

$$\operatorname*{ess\,inf}_{B(x_0,r/2)} u \geqq \operatorname*{ess\,inf}_B \psi_u \geqq -\varepsilon$$

and now (4.10) yields

$$\operatorname*{ess\,sup}_{B(x_0,r/2)} u - \operatorname*{ess\,inf}_{B(x_0,r/2)} u \leqq c\varepsilon + \varepsilon = (c+1)\varepsilon \tag{4.11}$$

where $c = c(p, n, \beta/\alpha) < \infty$. Since $u \in \mathscr{S}$ was arbitrary, inequalities (4.6) and (4.11) show that $\mathscr{S}$ is equicontinuous at x_0. The theorem follows.

4.12. REMARKS. (a) Condition (1.7) in Theorem 1.6 is satisfied in many cases. In particular, it holds when the families $\mathscr{O}$ and $\{\theta_u : u \in \mathscr{S}\}$ are uniformly bounded. Simple examples show that condition (1.7) is necessary for Theorem 1.6.

(b) It follows from (1.7) and from [HKM, Lemma 3.47] that for each $u \in \mathscr{S}$

$$\int_{B(x_0,r/2)} |\nabla u|^p\,dx \leqq c$$

where $c = c(p, \beta/\alpha, n, r, M) < \infty$ and $B(x_0, r) \subset \Omega$ is such that (1.6) holds for $\mathscr{U} = B(x_0, r)$. By Sobolev's imbedding theorem for $p > n$ each function $u \in \mathscr{S}$ is uniformly Hölder continuous in $B(x_0, r/2)$ and hence condition (1.8) is not needed in Theorem 1.6. Thus for $p > n$ condition (1.7) alone implies the equicontinuity of the family $\mathscr{S}$. If $p \leqq n$, then there exist non-continuous supersolutions of (1.1) and hence some control for obstacles is needed in order to obtain equicontinuity for solutions.

(c) Michael and Ziemer proved in [MZ], see also [HKM, Theorem 3.67], that a solution to the $\mathscr{K}_{\psi,\theta}$-obstacle problem is continuous provided that ψ is continuous. Theorem 1.6 has a somewhat different character.

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