MATH. SCAND. 75 (1994), 101–110

C^*-ALGEBRAS OF DYNAMICAL SYSTEMS ON THE NON-COMMUTATIVE TORUS

CARLA FARSI and NEIL WATLING

Abstract.

Here we prove that the C^*-algebras of dynamical systems on $\mathscr{A}_\rho$ associated to trace two affine automorphisms are classified by K-theoretical invariants. We also prove a partial classification result for the crossed products associated to general affine automorphisms induced by SL(2, Z) and compute their fixed point subalgebras.

As noted by Brenken and Watatani, SL(2, Z) has a natural representation on the automorphism group of the rotation algebra $\mathscr{A}_\rho$ [3], [14] with Watatani also computing the entropy of the dynamical systems arising from this action. The fixed point subalgebras associated to these automorphisms of $\mathscr{A}_\rho$ can also be explicitly classified [5] with parabolic matrices giving rise to 'trivial' subalgebras [6]. Here we wish to consider a slight generalization of the above automorphisms of $\mathscr{A}_\rho$, which we call affine transformations. These are analogues of affine transformations on compact groups. For example, affine rotations of T^n and affine quasi rotations of T^2 were considered by Riedel and Rouhani in [11] and [12] respectively. More precisely:

DEFINITION 1. Let $\mathscr{A}_\rho$ be the universal C^*-algebra generated by two unitaries U and V satisfying $VU = \rho UV$, with $\rho = e^{2\pi i\theta}$, $0 \leqq \theta < 1$. An affine transformation of $\mathscr{A}_\rho$ is an automorphism $\phi_{A,\lambda_1,\lambda_2}: \mathscr{A}_\rho \to \mathscr{A}_\rho$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathsf{Z})$, $\lambda_i \in \mathsf{T}$, $i = 1, 2$, and

$$\phi(U) = \lambda_1 U^a V^c, \ \phi(V) = \lambda_2 U^b V^d.$$

ϕ is said to be in standard form if $\lambda_i = 1$, $i = 1, 2$ and we will write ϕ_A instead of $\phi_{A,1,1}$. We will also write $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathsf{Z}$ instead of $\mathscr{A}_\rho \rtimes_{\phi_{A,\lambda_1,\lambda_2}} \mathsf{Z}$, $\mathscr{A}_\rho^{A,\lambda_1,\lambda_2}$ instead of $\mathscr{A}_\rho^{\phi_{A,\lambda_1,\lambda_2}}$, $\mathscr{A}_\rho \rtimes_A \mathsf{Z}$ instead of $\mathscr{A}_\rho \rtimes_{\phi_A} \mathsf{Z}$ and $\mathscr{A}_\rho^A$ instead of $\mathscr{A}_\rho^{\phi_A}$.

Received October, 1992; in revised form March 1, 1993 and June 15, 1993.

The fixed point subalgebras $\mathscr{A}_\rho^{A,\lambda_1,\lambda_2}$ corresponding to the above automorphisms are essentially the same as those derived from $\mathrm{SL}(2,\mathbf{Z})$ (Proposition 21) and when the trace is not 2, the crossed product C^*-algebra of $\mathscr{A}_\rho$ by any affine transformation is isomorphic to the crossed product C^*-algebra of $\mathscr{A}_\rho$ by the affine transformation associated to it in standard form (Proposition 3), so that these crossed products are isomorphic to those induced by the standard action of $\mathrm{SL}(2,\mathbf{Z})$. It is very difficult problem to determine the isomorphism classes of these C^*-algebras, even the conjugacy classes of $\mathrm{SL}(2,\mathbf{Z})$, clearly relevant to the problem, are not easily computable. We make some remarks concerning them and prove a partial classification result for crossed products associated to matrices of $\mathrm{SL}(2,\mathbf{Z})$ conjugate to some basic types (Theorem 12). When the trace is two the situation is more tractable since these algebras correspond to C^*-algebras generated by twisted cocycles on a discrete group which is a generalization of the discrete Heisenberg group. For $A \neq I_2$, we compute their K-theory and the range of any trace on $K_0(\mathscr{A}_\rho \rtimes \mathbf{Z})$ and show that these, together with the twist [13], are complete isomorphism invariants (Theorem 20) as is true for the Heisenberg group ([10] and [13]) which is a special case. Note that if $A = I_2$, $\mathscr{A}_\rho \rtimes_{I_2,\lambda_1,\lambda_2} \mathbf{Z}$ is a three dimensional non-commutative torus, so that the tracial range and the twist are not sufficient isomorphism invariants [4]. We begin by recalling the definition of twist and describing some properties of crossed product C^*-algebras of $\mathscr{A}_\rho$ by affine transformations $\phi_{A,\lambda_1,\lambda_2}$ with Trace $(A) \neq 2$.

Definition 2 ([13]). Let A be a unital C^*-algebra with tracial states τ such that all tracial states agree on $K_0(A)$ and [1] generates a free direct summand of $K_0(A)$. The twist of A is defined to be zero unless $\{x \in K_0(A) | \tau_*(x) \in \mathbf{Q}\} \cong \mathbf{Z}^2$, in which case it is the distance of $\tau_*(e)$ from $\mathbf{Z}$, where e is the other generator of $\{x \in K_0(A) | \tau_*(x) \in \mathbf{Q}\}$. The twist is an isomorphism invariant.

Proposition 3. *If* $A \in \mathrm{SL}(2,\mathbf{Z})$ *with* $\mathrm{Trace}(A) \neq 2$, *then* $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_A \mathbf{Z}$, *for all* $\lambda_i \in \mathbf{T}$.

Proof. Let $\phi(U) = \lambda_1 U^a V^c$, $\phi(V) = \lambda_2 U^b V^d$. Define $U' = \lambda_1{}^\alpha \lambda_2{}^\beta U$ and $V' = \lambda_1{}^\gamma \lambda_2{}^\delta V$, with α, β, γ and $\delta \in \mathbf{Q}$. Then U' and V' generate $\mathscr{A}_\rho$ and ϕ corresponds to $\phi'(U') = \lambda U'^a V'^c$, $\phi'(V') = \mu U'^b V'^d$, where $\lambda = \lambda_1^{\alpha(1-a)-\gamma c+1} \lambda_2^{\beta(1-a)-\delta c}$ and $\mu = \lambda_1^{\gamma(1-d)-\alpha b} \lambda_2^{\delta(1-d)-\beta b+1}$. It is straightforward to check that α, β, γ and δ can be chosen such that $\lambda = \mu = 1$ provided $\det\begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix} \neq 0$, that is, $\mathrm{Trace}(A) \neq 2$.

Lemma 4. *If* $\mathrm{Trace}(A) \neq 2$ *and* A *is conjugate to* B *in* $\mathrm{SL}(2,\mathbf{Z})$, *then* $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_B \mathbf{Z}$.

PROOF. Assume $K^{-1}AK = B$, $K = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix}$ in $\mathrm{SL}(2, \mathbf{Z})$ and define the automorphism δ of $\mathscr{A}_\rho$ by $\delta(U) = U^{k_{1,1}}V^{k_{2,1}}$, $\delta(V) = U^{k_{1,2}}V^{k_{2,2}}$. Then by Proposition 3 $\delta \circ \phi_A \circ \delta^{-1}$ is an automorphism of $\mathscr{A}_\rho$ such that $\mathscr{A}_\rho \rtimes_{\delta\circ\phi_A\circ\delta^{-1}} \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_B \mathbf{Z}$.

LEMMA 5. *If* $A \in \mathrm{SL}(2, \mathbf{Z})$ *with* $\mathrm{Trace}(A) \neq 2$, *then* $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{A^T} \mathbf{Z}$.

PROOF. Since A^T is conjugate to A^{-1} in $\mathrm{SL}(2, \mathbf{Z})$ (By the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$) the conclusion follows from Lemma 4 as $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{A^{-1}} \mathbf{Z}$.

On the other hand it would be very interesting to determine if $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}$ implies A is conjugate to B or B^T in $\mathrm{SL}(2, \mathbf{Z})$. This is probably too strong however: if we restrict to $\rho = 1$, that is, the commutative case $C(\mathbf{T}^2) \rtimes_A \mathbf{Z}$, where infact A is a hyperbolic ($|\mathrm{Trace}(A)| > 2$) element of $\mathrm{GL}(2, \mathbf{Z})$, it is already known that the entropy of $\phi_A (= \log \frac{1}{2}(|\mathrm{Trace}(A)| + \sqrt{|\mathrm{Trace}(A)|^2 - 4}))$ is an isomorphism invariant together with $\mathrm{Trace}(A)$ if $\det(A) = 1$ [8]. As for $\rho \neq 1$, Watatani [14] has shown the entropy of ϕ_A, which is an outer conjugacy invariant, has the same form as above so another possibility is $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}$ implies $|\mathrm{Trace}(A)| = |\mathrm{Trace}(B)|$ or even $\mathrm{Trace}(A) = \mathrm{Trace}(B)$.

PROPOSITION 6. *Let* $A, B \in \mathrm{SL}(2, \mathbf{Z})$ *with* $\mathrm{Trace}(A)$, $\mathrm{Trace}(B) \neq 2$. *Then* $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}$ *implies* $\rho' = \rho^{\pm 1}$ *and* $\mathrm{Trace}(B) = \mathrm{Trace}(A)$ or $\mathrm{Trace}(B) = 4 - \mathrm{Trace}(A)$.

PROOF. To simplify the proof we shall denote A by C_1 and B by C_2. Note that $\mathrm{rank}(C_i - I_2) = 2$, so $\ker(C_i - I_2) = 0$ and $K_0(\mathscr{A}_\rho) \cong \mathbf{Z}^2$ by Pimsner-Voiculescu sequence, see for example [1]. Also for any tracial state τ^i on $\mathscr{A}_\rho \rtimes_{C_i} \mathbf{Z}$ (a tracial state on the crossed product $\mathscr{A}_\rho \rtimes_{C_i} \mathbf{Z}$ can be constructed by extending the one on $\mathscr{A}_\rho$ which is $\mathbf{Z}$-invariant), $\tau^i_*(K_0(\mathscr{A}_\rho \rtimes_{C_i} \mathbf{Z})) = \tau^i_*(K_0(\mathscr{A}_\rho))$, $i = 1, 2$ ([1] Section 10.10). Moreover all tracial states agree on $\dot{K}_0$. Since $\mathscr{A}_\rho \rtimes_{C_1} \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_{C_2} \mathbf{Z}$, then $\mathbf{Z} + \theta\mathbf{Z} = \mathbf{Z} + \theta'\mathbf{Z}$, where $\rho = e^{2\pi i\theta}$, $\rho' = e^{2\pi i\theta'}$, $0 \leqq \theta, \theta' < 1$. If ρ and ρ' are of infinite order, this clearly implies $\rho' = \rho^{\pm 1}$, while if ρ and ρ' are of finite order the same conclusion holds by using the twist.

Now, by Pimsner-Voiculescu, $K_1(\mathscr{A}_\rho \rtimes_{C_i} \mathbf{Z}) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_\alpha \oplus \mathbf{Z}_\beta$ for some α and β in $\mathbf{N}$, since $(C_i - I_2)$ is equivalent to the diagonal form $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with α dividing β (i.e. D can be obtained from $(C_i - I_2)$ by elementary row and column operations). Therefore $\mathrm{Det}(C_i - I_2) = \pm\alpha\beta$. But $\mathrm{Det}(C_i - I_2) = 2 - \mathrm{Trace}(C_i)$ so $2 - \mathrm{Trace}(C_2) = \pm(2 - \mathrm{Trace}(C_1))$, that is, $\mathrm{Trace}(C_2) = \mathrm{Trace}(C_1)$ or $\mathrm{Trace}(C_2) = 4 - \mathrm{Trace}(C_1)$.

REMARK 7. (1) Note the above proposition shows that if we restrict to matrices with trace greater than 2 (respectively less than 2), the trace is an isomorphism invariant.

(2) If Trace(A) = Trace(B) and (Trace$(A) - 2$) is prime then $\mathcal{A}_\rho \rtimes_A \mathbf{Z}$ and $\mathcal{A}_{\rho'} \rtimes_B \mathbf{Z}$ have the same K-theory.

REMARK 8. It can also be shown that if ρ and ρ' are of infinite order and $A^n = 1$, then $\mathcal{A}_\rho \rtimes_A \mathbf{Z}_n \cong \mathcal{A}_{\rho'} \rtimes_A \mathbf{Z}_n$ if and only if $\rho' = \rho^{\pm 1}$ [2], [7]. If ρ and ρ' are of finite order $\mathcal{A}_\rho \rtimes_A \mathbf{Z}_n \cong \mathcal{A}_{\rho'} \rtimes_A \mathbf{Z}_n$ if and only if ρ and ρ' have the same order, see for example [5].

It would also be interesting to determine complete isomorphism invariants for $\mathcal{A}_\rho \rtimes_A \mathbf{Z}$. For Trace $(A) \neq 2$ we only have a partial result while we have a complete classification for Trace$(A) = 2$, $A \neq I_2$. If we consider the conjugacy classes of elements A of SL$(2, \mathbf{Z})$ we have the following.

PROPOSITION 9 ([9] PG. 44–47). *Let* $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $W = XY^{-1}$, $Z = (XY^{-1})^T$.

Then: (1) *Every* $A \in \mathrm{SL}(2, \mathbf{Z})$ *with* $|\mathrm{Trace}(A)| \leqq 2$ *is conjugate in* SL$(2, \mathbf{Z})$ *to* $\pm X$, $\pm Y$, $\pm Y^{-1}$, $\pm W^\alpha$ *or* $\pm Z^\beta$, *with* $\alpha, \beta \in \mathbf{N}$.

(2) *Every* $A \in \mathrm{SL}(2, \mathbf{Z})$ *with* $|\mathrm{Trace}(A)| \geqq 3$ *is conjugate in* SL$(2, \mathbf{Z})$ *to*

$$\pm W^{\alpha_1} Z^{\beta_1} \dots W^{\alpha_s} Z^{\beta_s}, \text{ with } \alpha_i, \beta_i \in \mathbf{N} \setminus \{0\}.$$

Moreover two such elements are conjugate in SL$(2, \mathbf{Z})$ *if and only if they are cyclic permutations of one another. Note we shall use the notation* $\sim$ *for conjugacy.*

This describes the conjugacy classes in principle. Unfortunately as $|\mathrm{Trace}(A)|$ grows so do the possibilities for the α_i, β_i's and it becomes increasingly more complicated to concretely describe the conjugacy classes of SL$(2, \mathbf{Z})$ using $|\mathrm{Trace}(A)|$ as a parameter. However it is possible to make some comments for small values of the trace.

REMARK 10. By using the isomorphism $U \to V$, $V \to U$ from $\mathcal{A}_\rho$ onto $\mathcal{A}_{\rho^{-1}}$, Proposition 3 and Lemma 5 we can see that $\mathcal{A}_\rho \rtimes_A \mathbf{Z} \cong \mathcal{A}_{\rho^{-1}} \rtimes_A \mathbf{Z}$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$ with Trace$(A) \neq 2$ is conjugate in SL$(2, \mathbf{Z})$ to $B = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ or B^T. For small values of $|\mathrm{Trace}(A)|$ this can be shown to be true, while if $|\mathrm{Trace}(A)| \geqq 3$ we can assume, from Proposition 9, $A \sim \pm W^{\alpha_1} Z^{\beta_1} \dots W^{\alpha_s} Z^{\beta_s}$ then, $B \sim \pm Z^{\beta_s} W^{\alpha_s} \dots Z^{\beta_1} W^{\alpha_1}$, and $B^T \sim \pm Z^{\alpha_1} W^{\beta_1} \dots Z^{\alpha_s} W^{\beta_s}$, so we require one of the latter two to be a cyclic permutation of the first. The first examples of A not having this property arise for Trace $(A) = 15$. In particular $A =$

$\begin{pmatrix} 3 & -5 \\ -7 & 12 \end{pmatrix} = W^2Z^2WZ$ or $A = \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix} = W^2ZWZ^2$ are such examples.

REMARK 11. If we restrict to $|\mathrm{Trace}(A)| < 2$ it is easy to check that the K-theory, the tracial range and the twist are complete isomorphism invariants. As for $|\mathrm{Trace}(A)| \geqq 3$, a description of the conjugacy classes for elements of $\mathrm{SL}(2,\mathbf{Z})$ with $3 \leqq |\mathrm{Trace}(A)| \leqq 6$ is given by ($\varepsilon = \pm 1$, $T = 3, \ldots, 6$)

$$\mathrm{Trace}(A) = \varepsilon T\colon \varepsilon\begin{pmatrix} 1 & -k \\ -(T-2)/k & T-1 \end{pmatrix},$$

where k runs over the positive divisors of $T - 2$. By applying Lemmas 4 and 5 together with Proposition 6 we see that the K-theory (specifically $K_1(\mathscr{A}_\rho \rtimes_A \mathbf{Z})$), the tracial range and $|\mathrm{Trace}(A)|$ (or the entropy of ϕ_A) are complete invariants for $3 \leqq |\mathrm{Trace}(A)| \leqq 6$. However this already breaks down for $\mathrm{Trace}(A) = 7$, for example, if $A = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -5 & 6 \end{pmatrix}$.

The following theorem characterizes the isomorphism classes of crossed products $\mathscr{A}_\rho \rtimes_A \mathbf{Z}$ when A is conjugate to some special types of matrices in $\mathrm{SL}(2,\mathbf{Z})$.

THEOREM 12. *Let $A, B \in \mathrm{SL}(2,\mathbf{Z})$. If A and B are conjugate in $\mathrm{SL}(2,\mathbf{Z})$ to matrices of the form $W^\alpha Z^\beta$, for some $\alpha, \beta \in \mathbf{N}\setminus\{0\}$ with $\alpha|\beta$ or $\beta|\alpha$, then $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}$ if and only if they have the same K-theory, tracial range and twist.*

PROOF. We only prove sufficiency since necessity is obvious, as the K-theory, tracial range and twist are isomorphism invariants [1], [13]. Conversely, since $K_1(\mathscr{A}_\rho \rtimes_A \mathbf{Z}) \cong K_1(\mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_\alpha \oplus \mathbf{Z}_\beta$, for some $\alpha, \beta \in \mathbf{N}\setminus\{0\}$ with $\alpha|\beta$, then $A, B \sim \begin{pmatrix} 1 & -\beta \\ -\alpha & 1+\alpha\beta \end{pmatrix}$ or its transpose. By Lemma 5 we can assume $A, B \sim \begin{pmatrix} 1 & -\beta \\ -\alpha & 1+\alpha\beta \end{pmatrix}$. Also, by Proposition 6, we must have $\rho' = \rho^{\pm 1}$. If $\rho' = \rho$ we are done, so assume $\rho' = \rho^{-1}$. Now $\begin{pmatrix} 1 & -\beta \\ -\alpha & 1+\alpha\beta \end{pmatrix} \sim \begin{pmatrix} 1+\alpha\beta & -\beta \\ -\alpha & 1 \end{pmatrix}$ (by $Z^{-\beta}$), hence by Remark 10 $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho^{-1}} \rtimes_A \mathbf{Z} \cong \mathscr{A}_{\rho'} \rtimes_B \mathbf{Z}$ and the theorem follows.

We will now consider matrices with trace two, the only case not mentioned so far, for which it is possible to give complete isomorphism invariants. As a consequence of Proposition 9 we have:

LEMMA 13 ([9]). *Every $A \in \mathrm{SL}(2,\mathbf{Z})$ with $\mathrm{Trace}(A) = 2$ is conjugate in $\mathrm{SL}(2,\mathbf{Z})$ to $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, some $m \in \mathbf{Z}$.*

LEMMA 14. *Let $\phi_{A,\lambda_1,\lambda_2}$ be an affine transformation of $\mathscr{A}_\rho$ with $A \sim \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ in* $SL(2,Z)$, $m \neq 0$. *Then* $K_0(\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} Z) \cong Z^3$, $K_1(\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} Z) \cong Z^3 \oplus Z_m$. *Moreover* $K_0(\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} Z)$ *is generated by* $K_0(\mathscr{A}_\rho)$ *and any x such that* $\langle \partial(x) \rangle = \mathrm{Ker}(1-(\phi_A)_1)$, *where* ∂ *is the connecting homomorphism* $\partial : K_0(\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} Z) \to K_1(\mathscr{A}_\rho)$ *in the Pimsner-Voiculescu exact sequence associated to* $\phi_{A,\lambda_1,\lambda_2}$.

PROOF. Straightforward using the Pimsner-Voicelescu exact sequence.

REMARK 15. A simple consequence of Lemma 14 (see also the proof of Proposition 6) is that if $A \in SL(2,Z)$ with $\mathrm{Trace}(A) = 2$ then $\mathscr{A}_\rho \rtimes_A Z$ and $\mathscr{A}_{\rho'} \rtimes_B Z$ can be isomorphic only if $\mathrm{Trace}(B) = 2$.

LEMMA 16. *Let* $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in SL(2,Z)$, $\lambda_i \in T$, $i = 1, 2$, *and suppose there exists* $K = (k_{i,j})$ *in* $SL(2,Z)$ *such that* $K^{-1}AK = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = M, m \neq 0$. *Then* $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} Z \cong \mathscr{A}_\rho \rtimes_{M,\lambda,1} Z$, *with*

$$\lambda = \lambda_1^{k_{1,1}} \lambda_2^{k_{2,1}} \rho^\kappa, \kappa = \left\{ \frac{ack_{1,1}}{2}(k_{1,1}-1) + \frac{bdk_{2,1}}{2}(k_{2,1}-1) + k_{1,1}k_{2,1}bc \right\} \in Z.$$

PROOF. Define the automorphism δ of $\mathscr{A}_\rho$,

$$\delta(U) = \delta_1 U^{k_{1,1}} V^{k_{2,1}},\ \delta(V) = U^{k_{1,2}} V^{k_{2,2}} \text{ where } \delta_1 = \lambda_1^{\frac{k_{1,2}}{m}} \lambda_2^{\frac{k_{2,2}}{m}} \rho^\nu,$$

$$\nu = \frac{1}{m}\left\{ \frac{ack_{1,2}}{2}(k_{1,2}-1) + \frac{bdk_{2,2}}{2}(k_{2,2}-1) + k_{1,2}k_{2,2}bc \right\} - \left\{ \frac{k_{1,1}k_{2,1}}{2}(m-1) + k_{2,1}k_{1,2} \right\}.$$

By using the equality $VU = \rho UV$ it is simple to check that $\phi_{M,\lambda,1} \circ \delta = \delta \circ \phi_{A,\lambda_1,\lambda_2}$.

The following lemma is essentially contained in [10].

LEMMA 17. *Let* $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2,Z)$, $g = \det(G)$ *and* $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $m \in Z \setminus \{0\}$, $M' = \begin{pmatrix} 1 & mg \\ 0 & 1 \end{pmatrix}$. *Then,*

(1) $\mathscr{A}_\rho \rtimes_{M,\lambda,1} Z \cong \mathscr{A}_{(\rho^\alpha \lambda^\gamma)^g} \rtimes_{M,(\rho^\beta \lambda^\delta)^g,1} Z$.

(2) $\mathscr{A}_\rho \rtimes_{M,\lambda,1} Z \cong \mathscr{A}_{(\rho^\alpha \lambda^\gamma)} \rtimes_{M',(\rho^\beta \lambda^\delta),1} Z$.

PROOF. The crossed product $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z}$ can be characterized as the universal C^*-algebra generated by three unitaries U, V and Ω satisfying,

$$VU = \rho UV,\ \Omega^* U \Omega = \lambda U,\ \Omega^* V \Omega = U^m V.$$

Now use the transformations:

$$u = \delta_1 U^g, \quad v = V^\alpha \Omega^{-\gamma}, \quad \omega = \Omega^\delta V^{-\beta}, \quad \text{for case (1)},$$

$$u = \delta_2 U, \quad v = V^\alpha \Omega^{-\gamma}, \quad \omega = \Omega^\delta V^{-\beta}, \quad \text{for case (2)},$$

where δ_1 and δ_2 are chosen such that $w^* v w = u^m v$ and $\omega^* v \omega = u^{gm} v$ respectively.

For $\lambda = \rho = m = 1$ the relations above characterize the discrete Heisenberg group [10], [13].

COROLLARY 18. *Let* $A \in \mathrm{SL}(2, \mathbf{Z})$ *with* $A \sim M, M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathbf{Z}\backslash\{0\}$. *Then,*

(1) $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_M \mathbf{Z}$.

(2) $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{M^{-1},\lambda^{-1},1} \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{M^{-1},\lambda,1} \mathbf{Z}$.

(3) $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z} \cong \mathscr{A}_{\rho^{-1}} \rtimes_{M,\lambda,1} \mathbf{Z}$

(4) $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathbf{Z} \cong \mathscr{A}_{\rho^{-1}} \rtimes_{A,\lambda_1,\lambda_2} \mathbf{Z}$.

PROOF. (1) By Lemma 16 $\mathscr{A}_\rho \rtimes_A \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{M,\rho^\kappa,1} \mathbf{Z}$. Now use Lemma 17(1) with $G = \begin{pmatrix} 1 & -\kappa \\ 0 & 1 \end{pmatrix}$.

(2) Use Lemma 17(2) with $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and then Lemma 17(1) with $G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(3) Use Lemma 17(1) with $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(4) By Lemma 16 $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathbf{Z} \cong \mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z}$ and $\mathscr{A}_{\rho^{-1}} \rtimes_{A,\lambda_1,\lambda_2} \mathbf{Z} \cong \mathscr{A}_{\rho^{-1}} \rtimes_{M,\lambda',1} \mathbf{Z}$ where $\lambda = \rho^{2\kappa}\lambda'$. Now by (3) $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z} \cong \mathscr{A}_{\rho^{-1}} \rtimes_{M,\lambda,1} \mathbf{Z}$ then use Lemma 17(1) with $G = \begin{pmatrix} 1 & 2\kappa \\ 0 & 1 \end{pmatrix}$.

So the isomorphism classes of crossed products of affine transformations of $\mathscr{A}_\rho$ with $\mathrm{Trace}(A) = 2, A \neq I_2$, are the isomorphism classes of crossed products of $\mathscr{A}_\rho$ by the automorphisms $\varphi(U) = \lambda U, \varphi(V) = U^m V, \lambda \in \mathbf{T}, m \in \mathbf{N}\backslash\{0\}$. We shall now show these crossed products can be classified by K-theoretical invariants. Note that the tracial state on $\mathscr{A}_\rho$ is invariant under all affine transformations, and therefore induces a tracial state on the associated crossed products.

PROPOSITION 19. *Let* $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $m \in \mathsf{N}\setminus\{0\}$ *and* $\lambda = e^{2\pi i \varepsilon} \in \mathsf{T}$, $0 \leqq \varepsilon < 1$. *Then all tracial states* τ *on* $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z}$ *agree on* $K_0(\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z})$ *and,*

$$\tau_*(K_0(\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z})) \cong \mathsf{Z} + \theta\mathsf{Z} + \varepsilon\mathsf{Z}.$$

PROOF. This is an adaption of the proof in [12] pages 142–143. If $\lambda = \rho = 1$ the result is clear [1]. If $\lambda = e^{2\pi i\varepsilon} \in \mathsf{T}\setminus\{1\}$, define the homomorphism $\tilde{\rho}: C(\mathsf{T}) \to \mathscr{A}_\rho$ by $\tilde{\rho}(g) = g \circ U$, for all $g \in C(\mathsf{T})$. Then it is straightforward to show that $\tilde{\rho}$ is an equivariant homomorphism between $(C(\mathsf{T}), \lambda, \mathsf{Z})$ and $(\mathscr{A}_\rho, \phi_{M,\lambda,1}, \mathsf{Z})$ and that the image of the Rieffel projection in A_λ generates a subgroup of $K_0(\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z})$ isomorphic to $\operatorname{Ker}(1 - (\phi_M)_1)$. It is also easy to prove the result when $\rho \in \mathsf{T}\setminus\{1\}$, by exchanging the roles of λ and ρ.

THEOREM 20. *Let* $\phi_{A,\lambda_1,\lambda_2}$ *and* ϕ_{B,μ_1,μ_2} *be two affine transformations of* $\mathscr{A}_\rho$ *with* $A, B \neq I_2$, $\operatorname{Trace}(A) = \operatorname{Trace}(B) = 2$. *Then* $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathsf{Z} \cong \mathscr{A}_{\rho'} \rtimes_{B,\lambda_1,\lambda_2} \mathsf{Z}$ *if and only if they have the same K-theory, tracial range and twist.*

PROOF. This proof is similar to those in [10] and [13]. We need only show sufficiency since necessity is obvious [1], [13]. By Lemma 16 and Corollary 18 we know that $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathsf{Z} \cong \mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z}$, for some $\lambda \in \mathsf{T}$, and $\mathscr{A}_{\rho'} \rtimes_{B,\mu_1,\mu_2} \mathsf{Z} \cong \mathscr{A}_{\rho'} \rtimes_{M,\lambda',1} \mathsf{Z}$, for some $\lambda' \in \mathsf{T}$ where $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $m > 0$ and $K_1(\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathsf{Z}) \cong K_1(\mathscr{A}_{\rho'} \rtimes_{B,\mu_1,\mu_2} \mathsf{Z}) \cong \mathsf{Z}^3 \oplus \mathsf{Z}_m$. Moreover $\mathscr{A}_\rho \rtimes_{A,\lambda_1,\lambda_2} \mathsf{Z}$ and $\mathscr{A}_{\rho'} \rtimes_{B,\mu_1,\mu_2} \mathsf{Z}$ have the same tracial range, which we shall denote by R.

First assume R has rank one. This implies that both ρ and λ (respectively ρ' and λ') have finite order and therefore there exists a $G \in \mathrm{SL}(2, \mathsf{Z})$ such that $(\rho, \lambda)G = (1, \zeta)$, where $\zeta = e^{\frac{2\pi i}{q}}$ and $q = \operatorname{lcm}\{\operatorname{ord}(\lambda), \operatorname{ord}(\rho)\}$ (respectively a $G' \in \mathrm{SL}(2, \mathsf{Z})$ such that $(\rho', \lambda')G' = (1, \zeta')$, where $\zeta' = e^{\frac{2\pi i}{q'}}$ and $q' = \operatorname{lcm}\{\operatorname{ord}(\lambda'), \operatorname{ord}(\rho')\}$). This implies by Lemma 17, $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z} \cong C(\mathsf{T}^2) \rtimes_{M,\zeta,1} \mathsf{Z}$ and $\mathscr{A}_{\rho'} \rtimes_{M,\lambda',1} \mathsf{Z} \cong C(\mathsf{T}^2) \rtimes_{M,\zeta',1} \mathsf{Z}$. Thus $R = \mathsf{Z} + \dfrac{1}{q}\mathsf{Z} = \mathsf{Z} + \dfrac{1}{q'}\mathsf{Z}$ which implies $q = q'$ and the two algebras are isomorphic.

Now suppose that R has rank two. If ρ has finite order and λ has infinite order by applying Lemma 17 with $G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathsf{Z} \cong \mathscr{A}_\lambda \rtimes_{M,\rho^{-1},1} \mathsf{Z}$. If ρ and λ both have infinite order and $\rho^k = \lambda^l$, for some $k, l \in \mathsf{Z}\setminus\{0\}$ minimal then if we let $q = (k, l) > 0$ so $k = qk'$, $l = ql'$, $(k', l') = 1$ there exist $\alpha, \gamma \in \mathbf{Z}$ such that $G = \begin{pmatrix} \alpha & -k' \\ \gamma & l' \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. Thus by using Lemma 17 we see that $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z} \cong \mathscr{A}_{\hat{\rho}} \rtimes_{M,\zeta,1} \mathbf{Z}$, with $\hat{\rho}$ of infinite order and $\zeta = e^{\frac{2\pi i p}{q}}$, $p, q \in \mathbf{N}$. Therefore we may assume $\rho = e^{2\pi i\theta}$, $0 < \theta < 1$ (respectively $\rho' = e^{2\pi i\theta'}$, $0 < \theta' < 1$) is of infinite order and $\lambda = e^{\frac{2\pi i p}{q}}$, $p, q \in \mathbf{N}$, $p < q$, $(p, q) = 1$ (respectively $\lambda' = e^{\frac{2\pi i p'}{q'}}$, $p', q' \in \mathbf{N}$, $p' < q'$, $(p', q') = 1$). Now the twist of $\mathscr{A}_\rho \rtimes_{M,\lambda,1} \mathbf{Z}$ is p/q or $1 - p/q$ while that of $\mathscr{A}_{\rho'} \rtimes_{M,\lambda',1} \mathbf{Z}$ is p'/q' or $1 - p'/q'$ so $p'/q' = p/q$ or $1 - p/q$ (i.e. $\lambda' = \lambda^{\pm 1}$). This implies $R = \mathbf{Z} + \theta\mathbf{Z} + \frac{p}{q}\mathbf{Z} = \mathbf{Z} + \theta'\mathbf{Z} + \frac{p}{q}\mathbf{Z}$ hence $\theta' = \theta$ or $1 - \theta$ so $\rho' = \rho^{\pm 1}$ and thus the two algebras are isomorphic by Corollary 18(3), and Corollary 18(2) if necessary.

Finally assume that R has rank three. In this case both ρ and λ (respectively ρ' and λ') have to be of infinite order with $\rho^k \neq \lambda^l$ for all $k, l \in \mathbf{Z}\setminus\{0\}$ (respectively $\rho'^k \neq \lambda'^l$ for all $k, l \in \mathbf{Z}\setminus\{0\}$). Hence there exists a matrix $G \in \mathrm{GL}(2, \mathbf{Z})$ such that $(\rho, \lambda)G = (\rho', \lambda')$ so an application of Lemma 17(2), and Corollary 18(2) if necessary, completes the proof in this case.

Although it is a very difficult problem to determine the isomorphism classes of the crossed products, it is relatively easy to classify the fixed point subalgebras of affine transformations of $\mathscr{A}_\rho$.

PROPOSITION 21. *Let* $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. *Then:*

(1) *If* $\mathrm{Trace}(A) \neq 2$, *then* $\mathscr{A}_\rho^{A,\lambda_1,\lambda_2} \cong \mathscr{A}_\rho^{A,e^{\pi i\theta ac},e^{\pi i\theta bd}}$ (*studied in* [2], [5], [6] *and* [7]).

(2) *If* $\mathrm{Trace}(A) = 2$, $A \neq I_2$, *then* $\mathscr{A}_\rho^{A,\lambda_1,\lambda_2} \cong \mathscr{A}_\rho^{M,\lambda,1}$ (cf. *Lemma* 16) *which is isomorphic to* $C(S^1)$ *if* λ *is of finite order and* $\mathbf{C}$ *if* λ *is of infinite order.*

(3) *If* $A = I_2$, *then* $\mathscr{A}_\rho^{A,\lambda_1,\lambda_2}$ *is isomorphic to* $\mathbf{C}$, *if* λ_1, λ_2 *are both of infinite order with* $\lambda_1^k \neq \lambda_2^l$ *for all* $k, l \in \mathbf{Z}\setminus\{0\}$, *is isomorphic to* $\mathscr{A}_{\rho^q}$, *if* λ_1, λ_2 *are both of finite order with* $q = \mathrm{lcm}(\mathrm{ord}(\lambda_1), \mathrm{ord}(\lambda_2))$ *and isomorphic to* $C(S^1)$ *otherwise.*

PROOF. (1) Straightforward using the transformation ϕ defined in the proof of Proposition 3.

(2) Straightforward by applying the techniques in [6] to $\mathscr{A}_\rho^{M,\lambda,1}$.

(3) By using the techniques in [6] it is easy to show the result if $\lambda_1^k \neq \lambda_2^l$ for all $k, l \in \mathbf{Z}\setminus\{0\}$. If $\lambda_1^k = \lambda_2^l$ for some $k, l \in \mathbf{Z}\setminus\{0\}$ and λ_1, λ_2 both have finite order then, as in the proof of Theorem 20, there exists $G \in \mathrm{SL}(2, \mathbf{Z})$ such that $(\lambda_1, \lambda_2)G = (1, \zeta)$, where $\zeta = e^{\frac{2\pi i}{q}}$, $q = \mathrm{lcm}\ (\mathrm{ord}(\lambda_1), \mathrm{ord}(\lambda_2))$. Therefore if we apply the automorphism ϕ_G of $\mathscr{A}_\rho$ we see that $\phi_{I_2,\lambda_1,\lambda_2}$ corresponds to $\phi_{I_2,1,\zeta}$ which obviously has $\mathscr{A}_{\rho^q}$ as its fixed point algebra. Finally if λ_1, λ_2 both have infinite order then, as in

the proof of Theorem 20, there exists $G \in SL(2, \mathbb{Z})$ such that $(\lambda_1, \lambda_2)G = (\hat{\lambda}_1, e^{\frac{2\pi i p}{q}})$, where $q = (k, l)$ and $\hat{\lambda}_1$ is of infinite order. Therefore if we apply the automorphism ϕ_G of $\mathscr{A}_\rho$ we note that $\phi_{I_2,\lambda_1,\lambda_2}$ corresponds to $\phi_{I_2,\hat{\lambda}_1,e^{\frac{2\pi i p}{q}}}$, which has $C(S^1)$ as its fixed point algebra.

REFERENCES

1. B. Blackadar, *K-Theory for Operator Algebras*, MSRI Publications, vol. 5, Springer-Verlag, New York, 1986.
2. O. Bratteli and A. Kishimoto, *Non-commutative spheres* III. Irrational rotations, Comm. Math. Phys. 147 (1992), 605–624.
3. B. Brenken, *Representations and automorphisms of the irrational rotation algebra*, Pacific J. Math. 111 (1984), 257–282.
4. G. Elliott, *On the K-Theory of the C*-algebra generated by a projective representation of a torsion-free discrete abelian group*, Operator algebras and group representations, vol. 1, Pitman, London, 1984.
5. C. Farsi and N. Watling, *Fixed point subalgebras of the rotation algebra*, C.R. Math. Rep. Acad. Sci. Canada XIII (1991), 75–80.
6. C. Farsi and N. Watling, *Trivial fixed point subalgebras of the rotation algebra*, Math. Scand. 72 (1993), 298–302.
7. C. Farsi and N. Watling, *Abstract characterizations of fixed point subalgebras of the rotation algebra*, Canad. J. Math.,to appear.
8. I. Mergory-Cohen, *Finite dimensional representation of classical crossed product algebras*, Pacific J. Math. 135 (1988), 349–359.
9. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and applied Mathematics, vol. XIII, Interscience Publishers, New York, 1966.
10. J. Packer, *C*-algebras generated by projective representations of the discrete Heisenberg group*, J. Operator Theory 18 (1987), 41–66.
11. N. Riedel, *Classification of the C*-algebras associated with minimal rotations*, Pacific J. Math. 101 (1982), 153–161.
12. H. Rouhani, *Quasi rotation C*-algebras*, Pacific J. Math. 148 (1991), 131–151.
13. H-S. Yin, *Two C*-algebras related to the discrete Heisenberg group*, J. Operator Theory 19 (1988), 275–282.
14. Y. Watatani, *Toral automorphisms on irrational rotation algebras*, Math. Japon. 26 (1981), 479–484.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
BOULDER, CO 80309–0395
U.S.A.

DEPARTMENT OF MATHEMATICS
SUNY AT BUFFALO
BUFFALO, NY 14214–3093
U.S.A.

Current Address
DEPARTMENT OF MATHEMATICS
WIDENER UNIVERSITY
CHESTER, PA 19013–5792
U.S.A.