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SUBDIVISION OF SETS IN R^k BY THE HELP OF A REGULAR SIMPLEX

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Abstract.

The complement A^c in R^k of a finite closed convex set A with interior points is divided into disjoint sets $B_r, r = 0, 1, \ldots, k$, by the help of a regular simplex and the sets B_r have the following property: To each r there belongs a direction β_r such that any y belonging to B_r remains in B_r by the translations $y \to y + \beta q_r, \beta > 0$ when the direction q_r makes an angle not larger than arcsin $1/k$ with ρ_r. This is the largest angle for arbitrary convex sets and the given subdivision in R^k.

1. Introduction.

The problem, of which the solution is given in the summary, I presented in [1]. So far as I know, there has been no reaction on this problem. Having recently returned to the problem for an application I found a solution

2. Subdivision by the simplex.

Consider the subdivision of the whole R^k but deal with it as a subspace of R^{k+1} and lay a regular simplex with $k + 1$ vertices in R^{k+1} where we have an orthogonal coordinate system with unit vectors on the axis. We number the axis $0, 1, \ldots, k$. Let the vertices have the coordinates x_j, $0 \leqq j \leqq k$, where

$$x_0 = (1, 0, 0, \ldots, 0, 0), x_1 = (0, 1, 0, \ldots, 0, 0), \ldots, x_k = (0, 0, 0, \ldots, 0, 1).$$

The simplex lies in the hyperplane through the points x_j and thus in R^k. We also talk of x_j as vector from the zero point to x_j. Any two of these vectors are orthogonal to each other. The centroid of the simplex is

$$(1) \quad z = (x_0 + x_1 + x_2 + \cdots + x_k)/(k + 1) = \{1/(k + 1), 1/(k + 1), \ldots, 1/(k + 1)\}.$$

By p_j we denote the unit vector on the direction from the zentroid z to the vertex x_j. The points y in the simplex have the representation

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(2) $$y = \Sigma\, a_j x_j,\ a_0 + a_1 + \cdots + a_k = 1,\ a_j \geqq 0,$$

REMARK. Here and in the following, when writing Σ, we consider summation over all indices $0, 1, \ldots, k$, under consideration if there can be no missunderstanding.

The facets are closed triangles and any three vertices determine a facet. The union of all facets is the surface of the simplex. The surface is a closed set. We form it by inductive construction starting with a triangle, say $\{a_0 \geqq 0, a_1 \geqq 0, a_2 \geqq 0, a_0 + a_1 + a_2 = 1\}$. To this we add successively new vertices and corresponding facets.

LEMMA. *The angle between p_i and p_j for $i \neq j$ is equal to* $\arccos - 1/k$. *Any direction from the centroid is uniquely represented by the vector*

(3) $$p = \Sigma\, a_j p_j,\ a_j \geqq 0,\ |\Sigma\, a_j p_j| = 1.$$

The union of all points ap, $a \geqq 0$, for all p belonging to the different faces by (3) *is the whole* R^k.

PROOF. It follows by the definition of the centroid that the sum of the differences $x_j - z$ is the 0-vector. Since the distances of $x_j - x_i$ are the same for $i \neq j$ and also the distances $x_j - z$ are the same we may change $x_j - z$ to p_j and so get the relation

(4) $$p_0 + p_1 + \cdots + p_k = 0.$$

Forming the scalar produkt of this sum by p_j and remembering that the $p_i p_j$ are equal for all i and j, $i \neq j$ and equal to 1 for $i = j$, we get $p_i p_j = -1/k$. We get the representation (3) from (2) when we change y to $y - z$ and x_j to $x_j - z$ and than normalize $y - z$ and the $x_j - z$ to unit vectors p and p_j. We define the sets C_r of vectors by

(5) $$C_r = \{p\colon p = \Sigma\, a_j p_j,\ a_j > 0 (j < r),\ a_j \geqq 0 (j > r),\ a_r = 0\}.$$

THEOREM 1. *The sets C_r, $r = 0, 1, \ldots, k$ are disjoint and their union determines the whole* R^k. *The vector $-p_r$ belongs to C_r. If any two vectors p and q belong to C_r then $p + q$ belongs to C_r. The smallest angle between $-p_r$ and a vector on the boundary of C_r is not larger than* $\arccos\{(k-1)/2k\}^{1/2}$. *Equality holds for some p on the boudary of C_r.*

REMARK. Note that C_r is determined by any vectors on the rays. We say that such a vector belongs to C_r.

PROOF. The two first statements follow by the definition. Consider a vector p in C_r. It has the representation (5). Only if the corresponding boundary belongs to C_r we may have some other a_j than a_r equal to 0. It follows by (4) and (5) that

$-p_r$ belongs to this set. The angle δ between p and $-p_r$ is obtained from the relation

$$\cos\delta = (-p_r)\cdot p = (1/k)\{\Sigma\, a_j\}\,|p|^{-1} \tag{6}$$

We may normalize the a_j so that $|p| = 1$ and still $a_j > 0 (j < r)$, $a_j \geqq 0 (j > r)$, $a_r = 0$. Then we get

$$\cos\delta = \{(1/\mathrm{k})\Sigma\, \mathrm{a}_j\} \tag{7}$$

subject to the condition

$$|p|^2 = (1 + 1/k)\Sigma(a_j)^2 - (1/k)(\Sigma\, a_j)^2 = 1,$$

where the number of a_j's, different from 0 is at most equal to $k - 1$. The smallest value of δ is obtained for the largest value of $\cos\delta$. It follows by (7) and the condition on p that this largest value is obtained when all a_j, $k - 1$ in number, are equal. So we get

$$\delta = \arccos\{(k - 1)/2k\}^{1/2}.$$

The fact that $p + q$ belongs to C_r if p and q belong to C_r follows by this proof.

Consider now a finite closed convex set A in R^k and assume that A has interior points. Let A^c be the complement of A. We inscribe a regular simplex in A. The sets C_r divide A and A^c into disjoint sets. Any direction p determines a hyperplane $P(p)$ of support of A separating A and the set $\{y: p > x\cdot p)\}$. Consider now the sets

$$B_r = \{y = z + \beta p,\ \beta > 0,\ z \in A \cap P(p),\ p \in C_r\} \tag{8}$$

THEOREM 2. *The sets B_r are disjoint and their union is the set A^c. Any point y in B_r remains in B_r by the translation $y \to y + \beta q$, $\beta > 0$, provided that the angle between $-p_r$ and q is at most equal to* $\arcsin(1/k)$. *This is the largest angle for arbitrary convex sets and the given subdivision in* R^k.

PROOF. Any y in A^c has a shortest distance to A obtained at a point z on the boundary of A and the ray from z to y determines a direction. Hence any y in A^c belongs to some one of the sets B_r. Conversely the direction determines z uniquely. The sets B_r are then disjoint since the sets C_r are disjoint. Consider now B_r and a direction q, and require that

$$B_r + \beta q \in B_r,\ \beta > 0. \tag{9}$$

For any $p_j, j \neq r$ there exists a point x in $P(p_j) \cap A$. Denote the angle between $-p_r$ and p_j by μ. We have

$$\cos\mu = -p_j\cdot p_r = 1/k$$

according to the Lemma. Clearly the angle between q and p_j must not be larger than $\pi/2$ in order that (9) be satisfied. That means that the angle between $-p_r$ and q must not be larger than $\pi/2 - \mu$ in order that (9) holds. But if this condition is satisfied then $q \cdot p_j \geqq 0$ for all $j \neq r$ and

$$q \cdot p = \Sigma\, a_j q \cdot p_j \geqq 0,$$

for p in C_r. This means that y cannot move into A by the translation $y \to y + \beta q$, $\beta > 0$. We observe that $\sin(\pi/2 - \mu) = 1/k$. Nor can y move into any B_s, $s \neq r$. Indeed, by Theorem 1 the vector q belongs to C_r if the angle between q and $-p_r$ isn't larger than arc sin $1/k$ and $p + q$ belongs to C_r if then p belongs to C_r. Let the points y_1 in B_r be trnaslated by q to a point y_2. According to the definition of B_r, the point y_1 determines a vector p in C_r and a hyperplane $P(p)$ such that $y_1 = z_1 + p$, where $z_1 \in P(p) \cap A$.

We may then choos β, $0 < \beta < 1$. The point $z_1 + (1 - \beta)p$, which belongs to C_r, is translated by $\beta p + q$ to y_2. As we just have argued this vector belongs to C_r and then determines a hyperplane $P(\beta p + q)$ and a point z_2 belonging to $P(\beta p + q) \cap A$ such that $y_2 = z_2 + \beta'(\beta p + q)$ with suitable $\beta' > 0$. Hence y_2 belongs to B_r. This fact concludes the proof.

Theorem 2 gives an answer to Problem 2 in [1], i.e. it determines the required subdivision (B_r) which, however, depends not only of the dimension k but also of the convex set A. The other problems in [1] remain open. To given k and given angle μ between $-p_r$ and q_r their certainly belongs a non-empty class of convex sets for which the subdivision (B_r) is the same for all members of the class. It may be of interest to determine this class for given μ or at least to find classes of convex sets (say with some symmetry) which belong to a given subdivision (B_r).

REFERENCE

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