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MEASURES ON LOCALLY COMPACT GROUPS WHOSE FOURIER-STIELTJES TRANSFORMS VANISH AT INFINITY AND GROUP C^*-ALGEBRAS

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Introduction.

Let G be a locally compact group and $M(G)$ the convolution algebra of finite regular Borel measures on G. In this paper we shall be concerned with two subalgebras of $M(G)$. One of these is $R(G)$, the set of all $\mu \in M(G)$ whose Fourier-Stieltjes transforms $\pi \to \pi(\mu)$ vanish at infinity on the unitary dual space $\hat{G}$ of G. Measures in $R(G)$ are sometimes referred to as Rajchman measures. The second one is $M_0(G)$, the subset of $M(G)$ consisting of all measures that are limits of L^1-functions in the group C^*-norm on $M(G)$. $M_0(G)$ can also be defined by $M_0(G) = M(G) \cap C^*(G)$. Subalgebras of $M(G)$, in particular $M_0(G)$, have long been a matter of interest [3, 5, 8, 9, 16, 20, 21].

Clearly, $M_0(G) \subseteq R(G)$, and equality holds (equivalently, $R(G) \subseteq C^*(G)$) if G is either abelian or compact. On the other hand it has been shown in [3] that for G the $ax + b$-group, $R(G)$ is not contained in $C^*(G)$. Our main problem is the extent to which the inclusion $R(G) \subseteq C^*(G)$ remains true for arbitrary locally compact groups. It turns out to hold for so-called SIN-groups (Theorem 2.6), groups with small invariant neighbourhoods of the identity, a very natural class of groups containing all abelian, all compact and all discrete groups. Most likely, $R(G) \subseteq C^*(G)$ fails to hold for any non SIN-group G. In its full generality, however, the problem seems intractable so far. This is, for instance, due to that we know of only a few inheritance properties of the condition $R(G) \subseteq C^*(G)$ (compare Section 2).

In Section 3 we prove for various classes of locally compact groups G, such as simply connected nilpotent Lie groups with one-dimensional center (Theorem 3.3), extensions of compact groups by vector groups (Theorem 3.1) and semi-direct products $G = \mathsf{R} \ltimes N$ with N abelian (Theorem 3.4), that for many $\pi \in \hat{G}$, $\pi(R(G))$ is not contained in $\pi(C^*(G))$. The proofs strongly depend on

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a discontinuity result for translations of certain measures under induced representations (Theorem 1.2).

1. Discontinuity of Translation.

Let G be a locally compact group with left Haar measure and $M(G)$ the convolution algebra of finite regular Borel measures on G. $M(G)$ can be identified with the dual of $C_0(G)$, the space of continuous functions on G that vanish at infinity, with the supremum norm. For $\mu \in M(G)$, $\|\mu\|$ will denote the total variation norm of μ. For each $x \in G$, the left translation operator L_x acts on functions on G by $L_x f(y) = f(x^{-1}y)$ and on $M(G)$ by $L_x\mu(\varphi) = \mu(L_{x^{-1}}\varphi)$, $\varphi \in C_0(G)$. We remind the reader that $\mu \in M(G)$ is absolutely continuous, that is, $\mu \in L^1(G)$, if and only if $x \to L_x\mu$ is continuous from G into $(M(G), \|\cdot\|)$ [8, Corollary 6; 21].

If π is a unitary representation of G on the Hilbert space $\mathscr{H}(\pi)$, then the same letter π stands for the corresponding $*$-representation of $C^*(G)$, the enveloping C^*-algebra of $L^1(G)$, and of $M(G)$. The resulting maximal C^*-norm is denoted $\|\cdot\|_*$. That is, for $\mu \in M(G)$, $\pi(\mu) = \int_G \pi(x)\,d\mu(x)$ and

$$\|\mu\|_* = \sup\{\|\pi(\mu)\| : \pi \text{ a unitary representation of } G\}.$$

$M_0(G)$ is defined to be the closure of $L^1(G)$ in $M(G)$ with respect to $\|\cdot\|_*$. As observed in [3, Theorem 1] and [9, Proposition 3], $\|\cdot\|_*$ can be replaced by the left regular representation norm. Since the representations of $M(G)$, whose restrictions to $L^1(G)$ are non-degenerate, are in one-to-one correspondence with the non-degenerate representations of $L^1(G)$, $M(G)$ embeds canonically into the multiplier algebra $M(C^*(G))$ of $C^*(G)$ and will therefore always be regarded as a subalgebra of $M(C^*(G))$. $M_0(G)$ can then equivalently be defined by $M_0(G) = M(G) \cap C^*(G)$. It has been shown in [8, Theorem 10] that $\mu \in M(G)$ belongs to $M_0(G)$ if and only if $x \to L_x\mu$ is continuous from G into $(M(G), \|\cdot\|_*)$.

Although $M_0(G)$ is not our main concern, we mention for completeness that $L^1(G)$ is strictly contained in $M_0(G)$ unless G is discrete [5, Theorem 5.8]. Moreover, $M_0(G) \subset M_c(G)$, the continuous measures [9, Theorem 1], and this inclusion is strict at least for connected Lie groups [3, Theorem 3]. On the other hand, if G is a compact connected simple Lie group, then every continuous central measure is in $M_0(G)$ [20].

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For our purposes it will be appropriate to consider, for an arbitrary representation π of G, continuity of the mapping $x \to \pi(L_x\mu)$ from G into $\mathscr{L}(\mathscr{H}(\pi))$, the algebra of bounded operators in $\mathscr{H}(\pi)$. The following lemma is nothing but a slight generalization of [8, Theorem 10]. The proof, though being similar, is included for the readers convenience.

LEMMA 1.1. *Let π be a representation of the locally compact group G. For $\mu \in M(G)$ the following conditions are equivalent:*
(i) $\pi(\mu) \in \pi(C^*(G))$.
(ii) *The mapping $x \to \pi(L_x\mu)$ from G into $\mathscr{L}(\mathscr{H}(\pi))$ is continuous.*

PROOF. The proof of (i) $\Rightarrow$ (ii) is straightforward taking into account that $\pi(C^*(G)) = \overline{\pi(L^1(G))}$, $\|\pi(L_y v)\| = \|\pi(v)\|$ for $v \in M(G)$ and $y \in G$, and that $y \to \pi(L_y f)$, $G \to \mathscr{L}(\mathscr{H}(\pi))$ is continuous for every $f \in L^1(G)$.

To show (ii) $\Rightarrow$ (i), let $\varepsilon > 0$ be given. There exists an open neighbourhood U of e in G such that $\|\pi(L_x\mu) - \pi(\mu)\| < \varepsilon$ for all $x \in U$. If $f \in L^1(G)$ is chosen so that $f \geqq 0, f(y) = 0$ for $y \in G \backslash U$ and $\int_G f(y)\,dy = 1$, then $\|\pi(f * \mu) - \pi(\mu)\| \leqq \varepsilon$. Indeed, for any coordinate function

$$\varphi(y) = \langle \pi(y)\xi, \eta \rangle,\ \eta \in \mathscr{H}(\pi),\ \|\xi\| = \|\eta\| = 1,$$

associated to π, we have

$$|\langle \pi(f * \mu - \mu)\xi, \eta \rangle| = \left| \int_G \int_G \varphi(xy) f(x)\,dx\,d\mu(y) - \int_G \varphi(y)\,d\mu(y) \right|$$

$$= \left| \int_G f(x) \int_G [\varphi(x) - \varphi(y)]\,d\mu(y)\,dx \right|$$

$$\leqq \int_U f(x) \left| \int_G \varphi(y)\,dL_x\mu(y) - \int_G \varphi(y)\,d\mu(y) \right| dx$$

$$\leqq \sup_{x \in U} \|\pi(L_x\mu - \mu)\| \leqq \varepsilon.$$

Let H be a closed subgroup of G. For $f \in L^1(H)$, the measure $\mu_f \in M(G)$ is defined by

$$\mu_f(\varphi) = \int_H \varphi(h) f(h)\,dh,\ \varphi \in C_0(G).$$

If σ is a unitary representation of H, we let $\mathrm{ind}_H^G\,\sigma$ denote the representation of G induced by σ, explicitly realized in the form due to Blattner [2]. A readable account of the theory of induced representations can be found in [12, Chapter XI].

THEOREM 1.2. *Let G be a locally compact group, N a closed normal subgroup of G and σ a unitary representation of N. Let $f \in L^1(N)$ be such that $\sigma(f) \neq 0$, and suppose that N is non-open in G. Then the mapping*

$$x \to \mathrm{ind}_N^G \sigma(L_x \mu_f),\ G \to \mathscr{L}(\mathscr{H}(\mathrm{ind}_N^G \sigma))$$

fails to be continuous at e. In particular, $\mathrm{ind}_N^G \sigma(\mu_f) \notin \mathrm{ind}_N^G \sigma(C^*(G))$.

PROOF. We first introduce some more notation. Let $\pi = \mathrm{ind}_N^G \sigma$, and for $\varphi \in C_c(G)$, the space of continuous functions with compact support on G, and $v \in \mathscr{H}(\sigma)$ define $\varepsilon(\varphi, v)\colon G \to \mathscr{H}(\sigma)$ by

$$\varepsilon(\varphi, v)(y) = \int_N \varphi(yn)\sigma(n)v\,dn.$$

$\varepsilon(\varphi, v)$ is continuous, and $\varepsilon(\varphi, v) \in \mathscr{H}(\pi)$ [2]. Moreover, for $S \subseteq G$, χ_S will denote the characteristic function of S.

We claim that f can be assumed to have compact support. For, notice first that, given $\varepsilon > 0$ and any compact subset C of N satisfying $\int_{N \setminus C} |f(n)|\,dn \leqq \varepsilon$, we have

$$|L_x\mu_f(\varphi) - L_x\mu_{f\chi_C}(\varphi)| = \left|\int_N f(n)\varphi(xn)\,dn - \int_C f(n)\varphi(xn)\,dn\right|$$

$$\leqq \|\varphi\|_\infty \int_{N \setminus C} |f(n)|\,dn \leqq \varepsilon\,\|\varphi\|_\infty$$

for all $x \in G$ and $\varphi \in C_c(G)$, so that

$$\|\pi(L_x\mu_f) - \pi(L_x\mu_{f\chi_C})\| \leqq \|L_x\mu_f - L_x\mu_{f\chi_C}\| \leqq \varepsilon.$$

Therefore, for all $x \in G$,

$$\|\pi(L_x\mu_f) - \pi(\mu_f)\| \geqq \|\pi(L_x\mu_{f\chi_C}) - \pi(\mu_{f\chi_C})\| - 2\varepsilon.$$

It follows that $x \to \pi(L_x\mu_f)$ is discontinuous at e once we have seen that $x \to \pi(L_x\mu_{f\chi_C})$ fails to be continuous at e for all sufficiently large compact subsets C of the support of f. On the other hand, $\sigma(f_{\chi_C}) \neq 0$ for all such C.

Thus suppose that f has compact support. Then $\mu_f * \varphi \in C_c(G)$ for every $\varphi \in C_c(G)$, since φ is uniformly continuous. Hence $\varepsilon(\mu_f * \varphi, v)$ is continuous for all $\varphi \in C_c(G)$ and $v \in \mathscr{H}(\sigma)$. Now

$$(1.1) \qquad \pi(L_x\mu_f)\varepsilon(\varphi, v)(y) = \int_N\int_N f(n)\varphi(n^{-1}x^{-1}ym)\sigma(m)v\,dm\,dn$$

$$= \int_N \mu_f * \varphi(x^{-1}ym)\sigma(m)v\,dm = \varepsilon(\mu_f * \varphi, v)(x^{-1}y)$$

for all $x, y \in G$. In particular, denoting by $\varphi | N$ the restriction of φ to N,

$$\pi(\mu_f)\varepsilon(\varphi, v)(e) = \varepsilon(\mu_f * \varphi, v)(e) = \sigma(f * \varphi | N)v.$$

$C_c(G) | N$ contains an approximate identity for $L^1(N)$. In fact, for every neighbourhood V of e in G choose a symmetric function $\psi_V \in C_c^+(G)$ such that $\psi_V(e) > 0$ and $\psi_V | G \backslash V = 0$, and set $\varphi_V = \|\psi_V | N\|_1^{-1}\kappa\psi_V$. Then $g * (\varphi_V | N) \to g$ for each $g \in L^1(N)$. Therefore, since $\sigma(f) \neq 0$, we find $\varphi \in C_c(G)$ and $v \in \mathscr{H}(\sigma)$ such that $\sigma(f * \varphi | N)v \neq 0$. Set

$$c = \tfrac{1}{2}\,\|\sigma(f * \varphi | N)v\| = \tfrac{1}{2}\,\|\varepsilon(\mu_f * \varphi, v)(e)\|.$$

Since $\varepsilon(\mu_f * \varphi, v)$ is continuous and $y \to \|\varepsilon(\mu_f * \varphi, v)(y)\|$ is constant on cosets of N, there exists a relatively compact open neighbourhood V of e in G such that

$$(1.2) \qquad \|\varepsilon(\mu_f * \varphi, v)(y)\| \geqq c$$

for all $y \in VN$. Let

$$(1.3) \qquad d = \sup_{y \in V} \|\varepsilon(\varphi, v)(y)\| = \sup_{y \in VN} \|\varepsilon(\varphi, v)(y)\| < \infty.$$

Now, N being non-open in G, there is a net in $V \backslash N$ converging to e. Therefore the theorem will be proved as soon as, for every $x \in V \backslash N$, we have constructed a $\xi \in \mathscr{H}(\pi)$ (depending on x) satisfying

$$\|\xi\| \leqq d \quad \text{and} \quad \|\pi(L_x\mu_f)\xi - \pi(\mu_f)\xi\| \geqq c.$$

To that end, fix $x \in V \backslash N$ and choose an open neighbourhood W of e in G such that

$$x^{-1}W \cup W \subseteq V \quad \text{and} \quad x^{-1}WN \cap WN = \emptyset.$$

Let Haar measure on G/N be chosen so that Weil's formula holds. Denote by $y \to \dot{y}$ the quotient homomorphism and by $|M|$ the Haar measure of a Borel set M in $\dot{G} = G/N$. We define $\xi \in \mathscr{H}(\pi)$ by

$$\xi(y) = |\dot{W}|^{-1/2}\chi_{WN}(y)\varepsilon(\varphi, v)(y).$$

Using formula (1.1), we obtain for $y \in WN$

$$(1.4)\quad \pi(L_x\mu_f)\xi(y) = |\dot{W}|^{-1/2} \int_N \int_N \chi_{WN}(n^{-1}x^{-1}) f(n)\varphi(n^{-1}x^{-1}ym)\sigma(m)v\, dm\, dn = 0,$$

since $x^{-1}WN \cap WN = \emptyset$ and, by the normality of N, $n^{-1}x^{-1}y \in x^{-1}yN \subseteq x^{-1}WN$ for all $n \in N$. Similarly, again using that N is normal, we get for $y \in WN$

$$(1.5)\qquad \pi(\mu_f)\xi(y) = |\dot{W}|^{-1/2} \int_N f(n)\chi_{WN}(n^{-1}y)\varepsilon(\varphi, v)(n^{-1}y)\, dn$$

$$= |\dot{W}|^{-1/2} \int_N f(n)\varepsilon(\varphi, v)(n^{-1}y)\, dn = |\dot{W}|^{-1/2}\varepsilon(\mu_f * \varphi, v)(y).$$

A combination of (1.2), (1.4) and (1.5) yields

$$(1.6)\qquad \|\pi(L_x\mu_f)\xi(y) - \pi(\mu_f)\xi(y)\| \geqq |\dot{W}|^{-1/2}c$$

for all $y \in WN$. Next recall from [2] that if $\eta \in \mathscr{H}(\pi)$ has compact support modulo N and $\psi \in C_c^+(G)$ satisfies $\int_N \psi(zn)\, dn = 1$ for all z in the support of η, then

$$\|\eta\|^2 = \int_G \psi(z)\, \|\eta(z)\|^2\, dz.$$

Now choose $\psi \in C_c^+(G)$ such that $\int_N \psi(zn)\, dn = 1$ for all $z \in \overline{VN}$. Then by (1.3) and the definition of ξ,

$$\|\xi\|^2 = |\dot{W}|^{-1} \int_{WN} \psi(y)\, \|\varepsilon(\varphi, v)(y)\|^2\, dy$$

$$\leqq |\dot{W}|^{-1} \sup_{y \in WN} \|\varepsilon(\varphi, v)(y)\|^2 \int_{WN} \psi(y)\, dy$$

$$= d^2\, |\dot{W}|^{-1} \int_{\dot{W}} \int_N \psi(yn)\, dn\, d\dot{y} = d^2.$$

Finally, it is easily verified that, for all $y \notin WN \cup x^{-1}WN \subseteq VN$,

$$\pi(L_x\mu_f)\xi(y) = 0 = \pi(\mu_f)\xi(y),$$

so that ψ can also be used to calculate the norm of $\pi(L_x\mu_f)\xi - \pi(\mu_f)\xi$. Using (1.6) we obtain

$$\|\pi(L_x\mu_f)\xi - \pi(\mu_f)\xi\|^2 \geqq \int_{WN} \psi(y)\,\|\pi(L_x\mu_f)\xi(y) - \pi(\mu_f)\xi(y)\|^2\,dy$$

$$\geqq |\dot{W}|^{-1}c^2 \int_{WN} \psi(y)\,dy = c^2.$$

This completes the proof of the theorem.

In Section 3 the preceding theorem will be frequently used to conclude that for some measure $\mu_f, f \in L^1(N)$, and some induced representation $\pi, \pi(\mu_f)$ does not belong to $\pi(C^*(G))$. With a mild abuse of notation Theorem 1.2 can be rephrased as follows.

COROLLARY 1.3. *Let G and N be as in Theorem* 1.2. *Then*

$$L^1(N) \cap C^*(G) = \{0\}.$$

PROOF. Let $0 \neq f \in L^1(N)$ and take for σ the left regular representation λ_N of N. Then $\lambda_N(f) \neq 0$ and by the theorem, $x \to \lambda_G(L_x\mu_f)$ fails to be continuous. Hence so does

$$x \to L_x\mu_f,\ G \to (M(G), \|\cdot\|_*).$$

Theorem 1.2 can also be viewed as a generalization of the well-known fact that the reduced group C^*-algebra $\overline{\lambda_G(L^1(G))}$ does not have a unit unless G is discrete. This follows immediately by taking $N = \{e\}$ and $\sigma = 1$. We do not know whether Theorem 1.2 remains true if we drop the assumption that N be normal in G. This would be interesting from the point of view of dealing with locally compact groups whose irreducible representations are induced from non-normal subgroups, such as semi-simple Lie groups and motion groups.

2. SIN-Groups and Property $R(G) \subseteq C^*(G)$.

In this section we study the fundamental question of when the measure algebras $R(G)$ and $M_0(G) = M(G) \cap C^*(G)$ coincide for a locally compact group G. We remind the reader that $R(G)$ is the subalgebra of $M(G)$ consisting of all those measures whose Fourier-Stieltjes transforms vanish at infinity. More precisely, $\mu \in M(G)$ belongs to $R(G)$ if and only if the function $\pi \to \|\pi(\mu)\|$ on the dual space $\hat{G}$ vanishes at infinity; that is, given $\varepsilon > 0$, there exists a compact subset C of $\hat{G}$ such that $\|\pi(\mu)\| < \varepsilon$ for all $\pi \in \hat{G}\backslash C$. Here, as usual, $\hat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G endowed with the natural topology. Of course, in the non-abelian case, there is a slight inaccuracy in defining the Fourier-Stieltjes transform by

$$\hat{\mu}(\pi) = \pi(\mu) = \int_G \pi(x)\, d\mu(x)$$

on $\hat{G}$, rather than on the set of all irreducible representations, without selecting one representation from each equivalence class.

Next recall that the analogue of the classical Riemann-Lebesgue lemma holds for general locally compact groups (C^*-algebras, as a matter of fact): For every $f \in C^*(G)$ and $\delta > 0$, the set $\{\pi \in \hat{G}\colon \|\pi(f)\| \geqq \delta\}$ is a compact subset of $\hat{G}$ (see [7, Proposition 3.3.7] or [12, Chapter VII]). Thus

$$M_0(G) = M(G) \cap C^*(G) \subseteq R(G)$$

for any locally compact group G, and $R(G) = M_0(G)$ is equivalent to $R(G) \subseteq C^*(G)$. If G is abelian, then $C^*(G) = C_0(\hat{G})$ and hence $R(G) \subseteq C^*(G)$. This also holds for compact groups K. Indeed, if we choose a representative ρ_π from each $\pi \in \hat{K}$ and denote by $\mathscr{C}_0(\hat{K})$ the algebra of all operator fields $\pi \to T_\pi \in \mathscr{L}(\mathscr{H}(\rho_\pi))$ on $\hat{K}$ that vanish at infinity, with the supremum norm, then the canonical embedding from $C^*(K)$ into $\mathscr{C}_0(\hat{K})$ is known to be an isomorphism onto. This shows $R(K) \subseteq C^*(K)$. On the other hand, for G the $ax + b$-group, in [3] there has been constructed a measure in $R(G)\backslash C^*(G)$. It was this example which partially motivated our interest in the problem. We begin by combining the abelian and compact cases.

LEMMA 2.1. *If G is a direct product of a compact group and an abelian group, then $R(G) \subseteq C^*(G)$.*

PROOF. Let $G = A \times K$ where A is abelian and K is compact. Then $\hat{G} = \hat{A} \times \hat{K}$, and it is well known that

$$C^*(G) = C^*(A) \otimes C^*(K) = C_0(\hat{A}) \otimes \mathscr{C}_0(\hat{K}) = C_0(\hat{A}, \mathscr{C}_0(\hat{K})).$$

where $C_0(X, B)$, for any locally compact Hausdorff space X and C^*-algebra B, denotes the C^*-algebra of all continuous mappings from X into B vanishing at infinity. At this point we remind the reader that since $C^*(A)$ is commutative, there is a unique C^*-norm on the algebraic tensor product of $C^*(A)$ and $C^*(K)$.

Let $\mu \in R(G)$, and for $\alpha \in \hat{A}$ and $\pi \in \hat{K}$ put

$$T_\alpha(\pi) = \hat{\mu}(\alpha, \pi) = \int_G \alpha(a)\rho_\pi(k)\, d\mu(a, k) \in \mathscr{L}(\mathscr{H}(\rho_\pi)).$$

Given $\varepsilon > 0$, there exist a compact subset C of $\hat{A}$ and a finite subset F of $\hat{K}$ such that $\|\hat{\mu}(\alpha, \pi)\| \leqq \varepsilon$ for all $(\alpha, \pi) \notin C \times F$. This shows $T_\alpha \in \mathscr{C}_0(\hat{K})$ and that $\alpha \to T_\alpha$ vanishes at infinity on $\hat{A}$. It remains to prove that $\alpha \to T_\alpha$ is continuous at $\alpha_0 \in \hat{A}$.

Since μ is regular, there is a compact subset M of G with $|\mu|(G\backslash M) \leqq \varepsilon$. Denoting by q the projection of G onto A, let

$$U = \{\alpha \in \hat{A}\colon |\alpha(a) - \alpha_0(a)| < \varepsilon \quad \text{for all} \quad a \in q(M)\}.$$

Then, for each $\alpha \in U$,

$$\|T_\alpha - T_{\alpha_0}\| = \sup_{\pi \in \hat{K}} \left\| \int_G [\alpha(a) - \alpha_0(a)]\pi(k)\, d\mu(a,k) \right\|$$

$$\leqq 2\,\|\mu|(G\backslash M) + |\mu|(M) \cdot \sup_{a \in q(K)} |\alpha(a) - \alpha_0(a)| \leqq \varepsilon(2 + \|\mu\|).$$

Our next goal is to show that the property $R(G) \subseteq C^*(G)$ is inherited by certain quotients and by open subgroups. We should remark at this point that if N is a closed normal subgroup of G, then $\widehat{G/N}$ will always be regarded as a closed subset of $\hat{G}$ in the obvious manner. Also, for any open subgroup H of G, $C^*(H)$ is just the closure of $L^1(H) \subseteq L^1(G)$ in $C^*(G)$.

Lemma 2.2. *Let K be a compact normal subgroup of G, and suppose that $R(G) \subseteq C^*(G)$. Then $R(G/K) \subseteq C^*(G/K)$.*

Proof. Let $T_K\colon C_c(G) \to C_c(G/K)$ denote the canonical surjective homomorphism defined by $T_K(\varphi)(xK) = \int_K \varphi(xk)\, dk$, where dk is normalized Haar measure on K. T_K is continuous with respect to C^*-norms and extends to a homomorphism of $C^*(G)$ onto $C^*(G/K)$. Now, let $\mu \in M(G/K)$ and consider $\nu \in M(G)$ given by $\nu(\varphi) = \mu(T_K\varphi)$, $\varphi \in C_c(G)$. Then $\pi(\nu) = \pi(\mu)$ for $\pi \in \widehat{G/K}$ and $\pi(\nu) = 0$ for $\pi \in \hat{G}\backslash\widehat{G/K}$. Indeed, for $\xi, \eta \in \mathscr{H}(\pi)$,

$$\langle \pi(\nu)\xi, \eta \rangle = \int_{G/K} \left(\int_K \langle \pi(k)\xi, \pi(x^{-1})\eta \rangle\, dk \right) d\mu(xK)$$

and, by decomposing $\pi \mid K$ into irreducible representations of K, the above claim follows from the orthogonality relations for elements in $\hat{K}$ [12, Chapter IX, Theorem 4.2]. In particular, if $\mu \in R(G/K)$, then $\nu \in R(G) \subseteq C^*(G)$ so that

$$x \to L_x\nu,\ G \to (M(G), \|\cdot\|_*)$$

is continuous. Easy calculations show that $L_x\nu(\varphi) = L_{xK}\mu(T_K\varphi)$ for all $x \in G$ and $\varphi \in C_c(G)$, and this formula in turn implies that $\pi(L_x\nu) = \pi(L_{xK}\mu)$ for every representation π of G/K. It follows that

$$xK \to L_{xK}\mu,\ G/K \to (M(G/K), \|\cdot\|_*)$$

is continuous. This proves $\mu \in C^*(G/K)$.

As we will see later (Corollary 3.2) it is not true in general that, conversely, $R(G/K) \subseteq C^*(G/K)$ implies $R(G) \subseteq C^*(G)$, for a compact normal subgroup K of G. Although the next lemma will only be needed in the case of group C^*-algebras, we present it for arbitrary C^*-algebras. However, we introduce some more notation before to facilitate further discussion.

For any unitary representation π of the locally compact group G, let ker π denote the kernel of π viewed as a $*$-representation of $C^*(G)$. If S and T are two sets of representations of G, then S is weakly contained in $T(S \prec T)$ if $\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau$, and S and T are weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$. Also recall that the support of a representation π is the closed subset

$$\operatorname{supp} \pi = \{\rho \in \hat{G}: \rho \prec \pi\}$$

of $\hat{G}$, and notice that $\pi \sim \operatorname{supp} \pi$. The analogous notions exist for $*$-representations of general C^*-algebras A and their dual spaces $\hat{A}$ (for all this, compare [7, Chap. 3], [11] and [12]).

LEMMA 2.3. *Let A be a C^*-algebra and B a C^*-subalgebra of A. If C is a compact subset of $\hat{B}$, then*

$$\tilde{C} = \{\pi \in \hat{A}: \operatorname{supp}(\pi | B) \cap C \neq \emptyset\}$$

is contained in some compact subset of $\hat{A}$.

PROOF. For $b \in B$, the norm function $\tau \to \|\tau(b)\|$ is lower semicontinuous on $\hat{B}$ [7, Proposition 3.3.2]. Therefore, for any $\sigma \in C$, there exist $b_\sigma \in B^+$ and $\delta_\sigma > 0$ such that $\|\tau(b_\sigma)\| > \delta_\sigma$ for all τ in some neighbourhood of σ. Since C is compact, we find $b_1, \ldots, b_m \in B^+$ and $\delta_1, \ldots, \delta_m > 0$ such that for each $\tau \in C$, $\|\tau(b_j)\| > \delta_j$ for at least one j, $1 \leqq j \leqq m$. Setting

$$b = \sum_{j=1}^{m} b_j \in B^+ \quad \text{and} \quad \delta = \min(\delta_1, \ldots, \delta_m),$$

it follows that $\|\tau(b)\| > \delta$ for every $\tau \in C$. Now, for $\pi \in \tilde{C}$,

$$\|\pi(b)\| = \|\pi | B(b)\| = \operatorname{supp}\{\|\tau(b)\|: \tau \in \operatorname{supp}(\pi | B)\} > \delta,$$

since $C \cap \operatorname{supp}(\pi | B) \neq \emptyset$. On the other hand, $\{\pi \in \hat{A}: \|\pi(b)\| \geqq \delta\}$ is compact by [7, Proposition 3.3.7].

LEMMA 2.4. *For any open subgroup H of G, $R(G) \subseteq C^*(G)$ implies $R(H) \subseteq C^*(H)$.*

PROOF. Let $\mu \in R(H) \subseteq M(G)$ be given. We claim that $\mu \in R(G)$. If $\varepsilon > 0$, then for some compact subset C of $\hat{H}$, $\|\sigma(\mu)\| < \varepsilon$ for all $\sigma \in \hat{H} \setminus C$. We apply the preceding lemma to $A = C^*(G)$ and $B = C^*(H)$. Thus $\tilde{C}$, as defined above, is contained in some compact subset K of $\hat{G}$. Now, if $\pi \in \hat{G}\backslash K$, then $C \cap \operatorname{supp}(\pi | H) = \emptyset$ and hence

$$\|\pi(\mu)\| = \operatorname{supp}\{\|\sigma(\mu)\| : \sigma \in \operatorname{supp}(\pi | H)\} \leqq \varepsilon.$$

Since $R(G) \subseteq C^*(G)$, the mapping

$$x \to L_x\mu,\ G \to (M(G), \|\cdot\|_*)$$

is continuous. As $\mu \in M(H) \subseteq C^*(G)$, we obtain that

$$h \to L_h\mu,\ H \to (M(H), \|\cdot\|_*)$$

is continuous, and this implies $\mu \in C^*(H)$.

We are now going to establish the equality $R(G) = M_0(G)$, that is, $R(G) \subseteq C^*(G)$, for the most natural class of locally compact groups which on the one hand contains all compact, all abelian and all discrete groups, on the other hand still preserves some of the pleasant features of these three subclasses. This is the class of SIN-groups. Recall that a locally compact group G is called an SIN-group (group with small invariant neighbourhoods) if every neighbourhood of e in G contains a neighbourhood V of e such that $x^{-1}Vx = V$ for all $x \in G$. The announced result will be an immediate consequence of Lemma 2.1, the following lemma and some structure theory and representation theory of SIN-groups.

Before proceeding we need a little more notation. If N is a normal subgroup of G, then G acts on N by inner automorphisms and hence on $\hat{N}$ by $(x, \tau) \to \tau^x$, where $\tau^x(n) = \tau(x^{-1}nx)$, $n \in N$. $G(\tau)$ will denote the orbit of τ under this action.

LEMMA 2.5, *Let N be an open normal subgroup of the locally compact group G and suppose that the following conditions are satisfied:*

(i) *For each $\tau \in \hat{N}$, $\overline{G(\tau)}$ is a minimal closed G-invariant subset of $\hat{N}$.*

(ii) *For every compact subset C of $\hat{N}$, $G(C) = \bigcup_{\tau \in C} G(\tau)$ is relatively compact in $\hat{N}$.*

Then $R(N) \subseteq C^*(N)$ implies $R(G) \subseteq C^*(G)$.

PROOF. In what follows, for a subset M of G and a function ψ on M, $\tilde{\psi}$ and $\psi^{\sim}$ will stand for the extension of ψ to G given by $\tilde{\psi}(x) = 0$ for all $x \in G\backslash M$. Moreover, we fix a representative system A for the cosets of N in G with $e \in A$. Let $\mu \in M(G)$, and for $a \in A$ define $\mu_a \in M(G)$ by $\mu_a(\varphi) = \mu(\widetilde{\varphi | aN})^{\sim}$, $\varphi \in C_c(G)$. Since μ is regular, for every $\varepsilon > 0$ there exists a finite subset F of A such that $\|\mu - \sum_{a \in F} \mu_a\| \leqq \varepsilon$. To prove the lemma it therefore suffices to show that if $\mu \in R(G)$, then $\mu_a \in C^*(G)$ for all a.

To that end notice first that $\mu \in R(G)$ and $\nu \in C^*(G)$ implies $L_x\mu \in R(G)$ and $L_x\nu \in C^*(G)$, respectively, for all $x \in G$. On the other hand, $L_a(L_{a^{-1}}\varphi | N)^{\sim} =$

$(\varphi|aN)^{\sim}$ for any function φ on G, and hence $\mu_a = L_a[(L_{a^{-1}}\mu)_e]$. Thus it remains to verify that $\mu \in R(G)$ implies $\mu_e \in C^*(G)$.

Next, to $\mu \in M(G)$ we associate $\mu' \in M(N)$ defined by $\mu'(\varphi) = \mu(\tilde{\varphi})$, for all $\varphi \in C_c(N)$. Then, for any representation π of G, $\pi(\mu_e) = \int_N \pi(n)\, d\mu(n) = \pi|N(\mu')$. Hence in view of the assumption $R(N) \subseteq C^*(N)$, it is enough to prove that $\mu \in R(G)$ entails $\mu' \in R(N)$.

For that, let σ be any representation of N and $\xi \in \mathscr{H}(\sigma)$ and consider the positive definite function $\varphi_\xi(n) = \langle \sigma(n)\xi, \xi \rangle$, $n \in N$. Then

$$(\mu - \mu_e)^* * \mu_e(\tilde{\varphi}_\xi) = \int_G \int_G \tilde{\varphi}_\xi(xy)\, d(\mu - \mu_e)^*(x)\, d\mu_e(y) = 0,$$

and similarly, $\mu_e^* * (\mu_e - \mu)(\tilde{\varphi}_\xi) = 0$. Hence

$$\mu^* * \mu(\tilde{\varphi}_\xi) = \mu_e^* * \mu_e(\tilde{\varphi}_\xi) + (\mu - \mu_e)^* * (\mu - \mu_e)(\tilde{\varphi}_\xi) \geqq \mu'^* * \mu'(\varphi_\xi).$$

Note that $\tilde{\varphi}_\xi$ is a positive definite function associated to the induced representation $\mathrm{ind}_N^G \sigma$. It follows that

$$\|\sigma(\mu')\|^2 = \sup\{\mu'^* * \mu'(\varphi_\xi) \colon \xi \in \mathscr{H}(\sigma),\ \|\xi\| = 1\}$$
$$\leqq \sup\{\mu^* * \mu(\tilde{\varphi}_\xi) \colon \xi \in \mathscr{H}(\sigma),\ \|\xi\| = 1\} \leqq \|\mathrm{ind}_N^G \sigma(\mu)\|^2.$$

Now, if $\mu \in R(G)$ and $\varepsilon > 0$ is given, then there exists a compact subset K of $\hat{G}$ such that $\|\pi(\mu)\| \leqq \varepsilon$ for all $\pi \in \hat{G} \setminus K$. By [15, Lemma 1.3] there exists a compact subset C of $\hat{N}$ such that $C \cap \mathrm{supp}(\pi|N) \neq \emptyset$ for all $\pi \in K$ (In [15] it is neither required that N be open nor that it be normal. However, we leave it to the reader to check that in the situation at hand, the proof can be simplified). By assumption (ii) the set $\overline{G(C)} \subseteq \hat{N}$ is compact. We claim that $\|\sigma(\mu')\| \leqq \varepsilon$ for every $\sigma \in \hat{N} \setminus \overline{G(C)}$. For that, by the above estimate, it is sufficient to verify that

$$K \cap \mathrm{supp}(\mathrm{ind}_N^G \sigma) = \emptyset.$$

Suppose there exists $\pi \in K \cap \mathrm{supp}(\mathrm{ind}_N^G \sigma)$. Then

$$\pi|N \prec \mathrm{ind}_N^G \sigma|N \sim G(\sigma)$$

and hence $\mathrm{supp}(\pi|N) \subseteq \overline{G(\sigma)}$. On the other hand, $\mathrm{supp}(\pi|N) \cap C \neq \emptyset$. For any $\tau \in C \cap \mathrm{supp}(\pi|N)$ we conclude $\overline{G(\tau)} \subseteq \overline{G(C)} \cap \overline{G(\sigma)}$. Since $\sigma \notin \overline{G(C)}$ this contradicts property (i).

THEOREM 2.6. *For any* SIN-*group* G, $R(G)$ *is contained in* $C^*(G)$.

PROOF. By [14, Theorem 2.13] G contains an open normal subgroup N of the form $N = V \times K$, where V is a vector group and K a compact group and, in addition, the group $I(N, G)$ of inner automorphisms $\alpha_x \colon N \to N$, $n \to x^{-1}nx$,

$x \in G$, has compact closure in Aut (N), the topological group of topological automorphisms of N. In the terminology of [19], $N \in [\mathrm{FIA}]^{-}_{I(N,G)}$. Since $\hat{N}$ is a Hausdorff space, every compact subset of $\hat{N}$ is contained in a G-invariant compact closed subset. Moreover, each orbit closure $\overline{G(\tau)}$, $\tau \in \hat{N}$, is a minimal closed G-invariant subset of $\hat{N}$ [19, Proposition 5.9]. Thus conditions (i) and (ii) in Lemma 2.5 are fulfilled, and the theorem follows since $R(N) \subseteq C^*(N)$ by Lemma 2.1.

REMARK 2.7. It seems reasonable to conjecture that for, say, Lie groups G the condition $R(G) \subseteq C^*(G)$ forces G to be an SIN-group. In fact, such a conjecture is supported by the results presented in Section 3. Likely Lemma 2.4 above could help to reduce to the connected component of G. However, concerning our problem, nothing is known so far for connected semi-simple Lie groups. Also, in the statement of Lemma 2.2 it would be necessary to replace the compact normal subgroup K by an arbitrary closed normal subgroup.

3. Some Special Classes of Locally Compact Groups.

This final section is devoted to proving, for some classes of solvable groups and for extensions of compact groups by vector groups, that for many $\pi \in \hat{G}$, $\pi(R(G))$ is not contained in $\pi(C^*(G))$, provided that G fails to be an SIN-group. The proofs exploit fairly extensive knowledge of the representation theory and structure of the groups involved.

THEOREM 3.1. *Suppose G contains a compact normal subgroup K such that G/K is a vector group, and let $\pi \in \hat{G}$ be infinite dimensional. Then, given any neighbourhood U of π in $\hat{G}$, there exists $\mu \in M(G)$ such that $\pi(\mu) \notin \pi(C^*(G))$ and $\rho(\mu) = 0$ for all $\rho \in \hat{G}\backslash U$.*

PROOF. Choose $\chi \in \hat{K}$ occurring as a subrepresentation in $\pi | K$. χ is G-invariant since $\hat{K}$ is discrete and G/K is connected. We have to employ Mackey's unitary representation theory of group extensions. By [18, Theorems 8.2 and 8.3] there are a multiplier ω on $V = G/K$, an irreducible $\bar{\omega}$-representation σ in $\mathscr{H}(\chi)$ extending χ and an irreducible ω-representation γ of V so that $\pi = \gamma \otimes \sigma$. We do not distinguish in notation between ω (resp. γ) on G/K and the multiplier (resp. multiplier representation)on G obtained by lifting ω (resp. γ) to G, and concerning multiplier representations we refer the reader to [1] and [18].

The classification of the irreducible multiplier representations of the vector group V is well-known and is a consequence of the Stone-von Neumann theorem (see [1, p. 314] or [18, Section 9]). There is a vector subgroup W of V such that, after replacing ω by a similar multiplier if necessary, $\omega | W \times W = 1$, and γ is the ω-representation of V which is ω-induced from the trivial character 1_W of W, $\gamma = \omega - \mathrm{ind}_W^V 1_W$. Let $N = \{x \in G: xK \in W\}$, then

$$\pi = \sigma \otimes \gamma = \sigma \otimes \omega - \mathrm{ind}_N^G 1_N = \mathrm{ind}_N^G (\sigma | N).$$

Since $\dim \pi = \infty$ and $\dim(\sigma | N) = \dim \chi < \infty$, G/N is a non-trivial vector group. Moreover, let

$$\hat{G}_\chi = \{\rho \in \hat{G}: \rho | K \sim \chi) \quad \text{and} \quad \hat{N}_\chi = \{\tau \in \hat{N}: \tau | K \sim \chi\}.$$

Since K is compact, $\hat{G}_\chi$ and $\hat{N}_\chi$ are open in $\hat{G}$ and $\hat{N}$, respectively, and

$$\hat{G}_\chi = \{\pi \otimes \beta: \beta \in \widehat{G/K}\} \quad \text{and} \quad \hat{N}_\chi = \{\sigma | N \otimes \alpha: \alpha \in \widehat{N/K}\}.$$

In addition, the mapping $\tau \to \mathrm{ind}_N^G \tau$ is a homeomorphism between $\hat{N}_\chi$ and $\hat{G}_\chi$. Next we choose an open neighbourhood V of $\sigma | N$ in $\hat{N}$ such that $V \subseteq \hat{N}_\chi$ and $\mathrm{ind}_N^G \tau \in U$ for all $\tau \in V$.

Now, groups with relatively compact commutator subgroups are of polynomial growth and hence their L^1-algebras are $*$-regular [4, Satz 2]. That is, the canonical mapping between the primitive ideal spaces of their C^*- and L^1 -algebras is a homeomorphism. Therefore there exists $f \in L^1(N)$ such that $\sigma | N(f) \neq 0$ and $\tau(f) = 0$ for all $\tau \in \hat{N} \backslash V$. Since $\pi = \mathrm{ind}_N^G(\sigma | N)$, Theorem 1.2 yields $\pi(\mu_f) \notin \pi(C^*(G))$. Finally, we also have $\rho(\mu_f) = 0$ for every $\rho \in \hat{G} \backslash U$. To that end, let $\rho = \mathrm{ind}_N^G \tau$ with $\tau \in \hat{N}_\chi \backslash V$. Then $\tau(f) = 0$, and this implies

$$\rho(\mu_f) = (\rho | N)(f) = 0.$$

Indeed, $\rho | N$ is weakly equivalent to the G-orbit of τ, but τ, being of the form $\sigma | N \otimes \alpha$ for some $\alpha \in \widehat{N/K}$, is G-invariant. If $\rho \in \hat{G} \backslash \hat{G}_\chi$, then $\rho | N \prec \hat{N} \backslash \hat{N}_\chi$ and the same argument as before gives $\mathrm{ind}_N^G(\rho | N)(\mu_f) = 0$. As $\rho \prec \mathrm{ind}_N^G \rho | N$, this shows $\rho(\mu_f) = 0$.

COROLLARY 3.2. *Let G contain a compact normal subgroup K such that G/K is a vector group. If $R(G) \subseteq C^*(G)$, then $G = K \times V$ where V is a vector group.*

PROOF. The previous theorem implies that every irreducible representation of G is finite dimensional. In particular, G can be continuously embedded into a compact group. In addition, G is almost connected. In fact, G/G_0K is totally disconnected and at the same time a quotient of the vector group G/K, so that $G = G_0K$. The Freudenthal-Weil theorem for almost connected groups [7, (16.5.3)] shows $G = C \rtimes V$ where V is a vector group and C is compact. It is obvious that C coincides with the normal subgroup K, and hence $G = K \times V$.

THEOREM 3.3. *Let G be a connected and simply connected nilpotent Lie group with one-dimensional center Z. Then, given $\pi \in \hat{G} \backslash \widehat{G/Z}$ and any neighbourhood U of π in $\hat{G}$, there exists $\mu \in M(G)$ such that $\pi(\mu) \notin \pi(C^*(G))$ and $\rho(\mu) = 0$ for all $\rho \in \hat{G} \backslash U$.*

PROOF. Let $\mathfrak{g}$ denote the Lie algebra of G. Since the center $\mathfrak{z}$ of $\mathfrak{g}$ is one-dimensional, by Kirillov's lemma [6, Lemma 1.1.12] there exist $A, B, C \in \mathfrak{g}$

and an ideal $\mathfrak{n}$ of codimension one with the following properties: $\mathfrak{z} = \mathsf{R}C$, $[A, B] = C$, $\mathfrak{g} = \mathsf{R}A + \mathfrak{n}$, and $\mathfrak{n}$ equals the centralizer of B in $\mathfrak{g}$. Let $N = \exp \mathfrak{n}$ and $M = \exp(\mathsf{R}B + \mathsf{R}C)$, a central subgroup of N.

Then every $\pi \in \hat{G} \backslash \widehat{G/Z}$ is induced from some irreducible representation of N [6, Proposition 2.3.4], necessarily from one in $\hat{N} \backslash \widehat{N/Z}$. Conversely, it follows from Mackey's theory that for every $\tau \in \hat{N} \backslash \widehat{N/Z}$, the induced representation $\operatorname{ind}_N^G \tau$ is irreducible. Indeed, for such τ, the stability group G_τ of τ equals M because otherwise $G_\tau = G$ and hence $(\operatorname{ind}_N^G \tau) | M \sim \tau | M$, a multiple of a character, which is impossible since τ is non-trivial on Z and M is not contained in the center of G.

Fix $\sigma \in \widehat{N/Z}$ so that $\operatorname{ind}_N^G \sigma = \pi$. Inducing being continuous, we find an open neighbourhood V of σ in $\hat{N} \backslash \widehat{N/Z}$ such that $\operatorname{ind}_N^G \tau \in U$ for all $\tau \in V$. Since nilpotent groups are of polynomial growth their L^1-algebras are $*$-regular [4, Satz 2]. Thus there exists $f \in L^1(N)$ such that $\sigma(f) \neq 0$ and $\tau(f) = 0$ for all $\tau \in \hat{\mathrm{N}} \backslash \mathrm{V}$.

Theorem 1.2 now yields $\pi(\mu_f) \notin \pi(C^*(G))$. Finally, consider $\rho \in \hat{G} \backslash U$. Then $\rho(\mu_f) = \rho | N(f) = 0$ provided that $\operatorname{supp}(\rho | N) \cap V = \emptyset$. This latter condition is certainly fulfilled if $\rho \in \widehat{G/Z}$. Let $\rho \in \hat{G} \backslash (\widehat{G/Z} \cup U)$ and pick $\tau \in \hat{N}$ with $\operatorname{ind}_N^G \tau = \rho$. Then $G(\tau) \cap V = \emptyset$, since for $\tau' \in G(\tau) \cap V$, $\operatorname{ind}_N^G \tau' \in U$ and $\operatorname{ind}_N^G \tau' = \operatorname{ind}_N^G \tau = \rho$. It follows that

$$\operatorname{supp}(\rho | N) = \operatorname{supp}(\operatorname{ind}_N^G \tau | N) = \overline{G(\tau)} \subseteq \hat{N} \backslash V.$$

The next theorem considerably extends [3, Theorem 4], where $R(G) \neq M_0(G)$ has been shown for G the $ax + b$-group.

THEOREM 3.4. *Suppose that G is a semi-direct product $G = \mathsf{R} \ltimes N$, where N is abelian and second countable. Then for each infinite dimensional $\pi \in \hat{G}$, there exists $\mu \in M(G)$ such that $\pi(\mu) \notin \pi(C^*(G))$ and $\hat{\mu}$ vanishes outside some compact subset of $\hat{G}$.*

PROOF. We denote the action of R on $\hat{N}$ by $(t, \lambda) \to \lambda^t$. For $\lambda \in \hat{N}$, set $S_\lambda = \{t \in \mathsf{R}: \lambda^t = \lambda\}$, $G_\lambda = S_\lambda \ltimes N$ and $N_\lambda = \{x \in N: \lambda(x) = 1\}$.

The unitary dual of such a semi-direct product of abelian groups is not in general accessible by the Mackey machinery. However, as a consequence of [13], the irreducible representations can be obtained up to weak equivalence as outlined below.

Recall first that if A is a separable C^*-algebra, then the kernel of a homogeneous representation of A is a prime ideal [7, (5.7.6)], and an ideal in A is prime if and only if it is primitive [7, (3.9.1)]. Notice also that for a second countable group H, $C^*(H)$ is separable.

If $\lambda \in \hat{N}$ and $\alpha \in \hat{S}_\lambda$, then $\sigma(t, x) = \alpha(t)\lambda(x)$, $t \in S_\lambda$, $x \in N$, defines a character of G_λ, and $\rho = \operatorname{ind}_{G_\lambda}^G \sigma$ is irreducible by Mackey's theory. Conversely, given $\rho \in \hat{G}$, by Theorem 4.3 of [13] and by the preceding paragraph there exist $\lambda \in \hat{N}$ and $\sigma \in \hat{G}_\lambda$ such that $\sigma | N \sim \lambda$ and $\rho \sim \operatorname{ind}_{G_\lambda}^G \sigma$. We will need that $\rho | N$ is then weakly equivalent to $\mathsf{R}(\lambda)$, the R-orbit of λ. This follows from [10, Theorem 4.5] in case

$S_\lambda = \{0\}$, and from [11, Theorem 5.3] in the general case since N is abelian and G_λ and N are clearly regularly related in the sense of Mackey [17, p. 127]. N_λ is normal in G_λ and $G_\lambda/N_\lambda = S_\lambda \ltimes N/N_\lambda$ is abelian. Hence

$$\sigma(t,x) = \alpha(t)\lambda(x),\ x \in N,\ t \in S_\lambda,$$

for some character α of S_λ. Obviously, ρ is infinite dimensional if and only if $S_\lambda \neq \mathsf{R}$, that is, S_λ is discrete.

Now, let $\pi \in \hat{G}$ be infinite dimensional, and let $\lambda_0 \in \hat{N}$ and $\alpha_0 \in \hat{S}_\lambda$ with

$$\pi \sim \mathrm{ind}_{G_\lambda}^G(\alpha_0\lambda_0).$$

Choose $f \in L^1(N)$ such that $\hat{f}(\lambda_0) \neq 0$ and C, the support of $\hat{f}$, is contained in the open set $\{\lambda \in \hat{N}: S_\lambda \neq \mathsf{R}\}$. Since N is open in G_{λ_0}, $f \in L^1(G_{\lambda_0})$ and $\hat{f}(\alpha_0\lambda_0) = \hat{f}(\lambda_0)$. Therefore, Theorem 1.2 applies and yields

$$\mathrm{ind}_{G_{\lambda_0}}^G(\alpha_0\lambda_0)(\mu_f) \notin \mathrm{ind}_{G_{\lambda_0}}^G(\alpha_0\lambda_0)(C^*(G)).$$

This means that $\pi(\mu_f) \notin \pi(C^*(G))$. Set

$$\tilde{C} = \{\rho \in \hat{G}: \rho \sim \mathrm{ind}_{G_\lambda}^G(\alpha\lambda) \text{ for some } \lambda \in C \text{ and } \alpha \in \hat{S}_\lambda\}.$$

Then, for $\rho \in \hat{G}\backslash\tilde{C}$, $\rho(\mu_f) = (\rho\,|\,N)(f) = 0$. Indeed, if $\rho \sim \mathrm{ind}_{G_\lambda}^G(\alpha\lambda)$ as above, then on the one hand $\rho\,|\,N \sim \mathsf{R}(\lambda)$ and on the other hand $\mathsf{R}(\lambda) \cap C = \emptyset$ since

$$\mathrm{ind}_{G_\lambda}^G(\alpha\lambda^t) = \mathrm{ind}_{G_\lambda}^G(\alpha\lambda)^t = \mathrm{ind}_{G_\lambda}^G(\alpha\lambda)$$

for all $t \in \mathsf{R}$.

Therefore, it remains to prove that $\tilde{C}$ is compact. Since G is second countable, $C^*(G)$ is separable, and hence $\hat{G}$ is second countable [7, (3.3.4)]. Hence $\tilde{C}$ is compact if (and only if) it is sequentially compact. Thus, let $\rho_k \in \tilde{C}$ and $\lambda_k \in C$ with $\rho_k\,|\,N \sim \mathsf{R}(\lambda_k)$, $k \in \mathsf{N}$. Passing to a subsequence if necessary, we can assume that $\lambda_k \to \lambda \in C$.

Suppose first that $S_{\lambda_k} = \{0\}$ for infinitely many $k_j, j \in \mathsf{N}$. Then, by continuity of inducing [11, Section 4],

$$\rho_{k_j} \sim \mathrm{ind}_N^G \lambda_{k_j} \to \mathrm{ind}_N^G \lambda.$$

If, moreover, $S_\lambda = \{0\}$ then $\rho = \mathrm{ind}_N^G \lambda$ is irreducible and $\rho\,|\,N \sim \mathsf{R}(\lambda)$, so that $\rho \in \tilde{C}$. If $S_\lambda \neq \{0\}$, then

$$\rho = \mathrm{ind}_{G_\lambda}^G(1_{S_\lambda} \cdot \lambda) \in \tilde{C} \quad \text{and} \quad \rho_{k_j} \to \rho.$$

Thus we are left with the case $S_{\lambda_k} = \mathsf{Z}t_k$, $t_k > 0$, for each k. Then

$$\rho_k \sim \mathrm{ind}_{G_{\lambda_k}}^G(\alpha_k\lambda_k),\ \alpha_k \in \hat{S}_{\lambda_k}.$$

Again, by passing to a subsequence if necessary, we can assume that

(i) $\alpha_k(t_k) \to z$ for some $z \in \Pi$,

(ii) $S_{\lambda_k} \to S$ *in* $\mathscr{S}(\mathsf{R})$, the set of all closed subgroups of R endowed with Fell's compact-open topology which makes $\mathscr{S}(\mathsf{R})$ a compact space (see [11, p. 427]).

We claim that (ii) and $\lambda_k \to \lambda$ implies $S \subseteq S_\lambda$. To verify this, let $s \in S$ and notice that since $S_{\lambda_k} \to S$ there are $s_k \in S_{\lambda_k}$ such that $s_k \to s$. As the mapping

$$\mathsf{R} \times \hat{N} \to \hat{N}, (t, \chi) \to \chi^t$$

is continuous, we conclude $\lambda_k = \lambda_k^{s_k} \to \lambda^s$. On the other hand, $\lambda_k \to \lambda$. Thus $\lambda^s = \lambda$, so that $s \in S_\lambda$. We now distinguish according to whether the t_k are bounded or not.

In the first case we can assume $t_k \to t$ in R. If $t = 0$, then $\mathsf{Z}t_k \to \mathsf{R}$. Indeed, given $s \in \mathsf{R}$ and $\delta > 0$, for each k choose $n_k \in \mathsf{Z}$ minimal so that $n_k t_k > s - \delta$. Then $n_k t_k \in (s - \delta, s + \delta)$ for all k with $t_k < 2\delta$. But $\mathsf{Z}t_k \to \mathsf{R}$ implies $S_\lambda = \mathsf{R}$, contradicting $\lambda \in C$. Thus $t > 0$ and

$$S_{\lambda_k} = \mathsf{Z}t_k \to \mathsf{Z}t \subseteq S_\lambda.$$

Now, define a character α on $\mathsf{Z}t$ by $\alpha(t) = z$. Then, in Fell's so-called subgroup representation topology [11, Section 3],

$$(S_{\lambda_k} \rtimes N, \alpha_k \lambda_k) \to (\mathsf{Z}t \rtimes N, \alpha\lambda).$$

Employing once more that inducing is continuous even when varying the subgroups continuously, we obtain with $H = \mathsf{Z}t \rtimes N$

$$\rho_k \sim \operatorname{ind}_{G_{\lambda_k}}^G (\alpha_k \lambda_k) \to \operatorname{ind}_H^G (\alpha\lambda).$$

Thus $\rho_k \to \rho$ for every $\rho = \operatorname{ind}_{G_\lambda}^G (\beta\lambda)$, where $\beta \in \hat{S}_\lambda$ is such that $\beta | \mathsf{Z}t = \alpha$. On the other hand, $\rho \in \tilde{C}$ for any such ρ.

Finally, suppose the t_k are unbounded. Then $\mathsf{Z}t_k \to \{0\}$ (compare [11, pp. 427–428]), and as before it follows that $(S_{\lambda_k} \rtimes N, \alpha_k\lambda_k) \to (N, \lambda)$ and therefore $\rho_k \to \operatorname{ind}_N^G \lambda$. Taking $\rho = \operatorname{ind}_{G_\lambda}^G (1_{S_\lambda} \lambda) \in \hat{G}$, we conclude $\rho_k \to \rho$ and $\rho \in \tilde{C}$. This proves that $\tilde{C}$ is compact and finishes the proof of the theorem.

We conclude the paper by determining $M(G) \cap C^*(G)$ for Fell's well-known example of a non-discrete group G whose dual space is nevertheless compact.

EXAMPLE 3.5. G is the semi-direct product $G = \mathsf{Z} \rtimes \mathsf{R}$ where the action of Z on R is given by $(n, x) \to e^n x$. $\hat{G}$ is compact, so that $R(G) = M(G)$. We adopt the notations from the proof of Lemma 2.5 and for $\nu \in M(G)$ denote by ν' its restriction to R. That is, $\nu'(\varphi) = \nu(\tilde{\varphi})$ for $\varphi \in C_0(\mathsf{R})$. It is clear that

$$\|\nu' * \varphi\|_2 \leqq \|\nu * \tilde{\varphi}\|_2$$

for every $\varphi \in L^2(\mathsf{R})$, and this implies

$$\|\nu'\|_* = \|\lambda_{\mathsf{R}}(\nu')\| \leqq \|\lambda_G(\nu)\| = \|\nu\|_*.$$

If $\mu \in C^*(G) \cap M(G)$, then all left translates $L_{(n,0)}\mu$, $n \in \mathsf{Z}$, belong to $C^*(G)$, and the above estimate shows $L_{(n,0)}\mu)' \in C^*(\mathsf{R}) = C_0(\hat{\mathsf{R}})$ for all $n \in \mathsf{Z}$. Conversely, if $\mu \in M(G)$ and $(L_{(n,0)}\mu)' \in C^*(\mathsf{R})$ for all $n \in \mathsf{Z}$, then $\mu \in C^*(G)$ (compare the proof of Lemma 2.5).

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