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COMPLETE SPACES OF VECTOR-VALUED HOLOMORPHIC GERMS

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To the memory of Leopoldo Nachbin.

Abstract.

Let K be a non-empty compact subset of a Fréchet space E and let X be a Banach space. By means of a given representation of the LB-space $H(K,X)$ of germs of holomorphic functions with values in X as a space of linear operators, it is proved that the space $H(K,X)$ is complete if E is quasinormable or if X is complemented in its bidual. If E is a Fréchet-Montel space, X is an $\mathscr{L}_\infty$-space in the sense of Lindenstrauss and Pełczyński and $H(K,X)$ is complete, then $E'_b \hat{\otimes}_\varepsilon X$ must be an LB-space. It is an open problem whether $c_0(E'_b) \simeq E'_b \hat{\otimes}_\varepsilon c_0$ is an LB-space for every Fréchet-Montel space E.

The aim of this note is to study the completeness of the LB-space of germs of holomorphic mappings on a compact subset of a Fréchet space with values in a Banach space by means of a linearization technique of Mujica [30] (see also [25], [31]), The method allows to reduce the problem to the analogous question for LB-spaces of continuous linear mappings from Fréchet spaces to Banach spaces. Since both spaces turn out to be regular LB-spaces, our paper is related to the still open problem of Grothendieck whether every regular LB-space is complete (cf. [2. Prob. 1, p. 78]).

The starting point for the recent research on the spaces $H(K)$ of holomorphic germs on (always non-empty) compact sets K in a Fréchet space E was Mujica's thesis [26]. Mujica proved that $H(K)$ is always a regular LB-space for K and E as above. Dineen [20] (see also [21, Th. 6.1]) proved for the first time that $H(K)$ is even complete and Mujica [27] obtained this result as a consequence of an abstract criterion for the completeness of LB-spaces. Several authors investigated spaces of holomorphic germs on compact subsets of Fréchet spaces. We refer to [1], [6], [7], [20], [21], [26], [27] and [28].

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Spaces of holomorphic germs with values in a Banach space are defined e.g. in [21, 2.4] in the following way: Let E be a Fréchet space with a basis of absolutely convex open 0-neighbourhoods $(V_n)_{n\in\mathbf{N}}$, let K be a compact subset of E and let X be a Banach space. One defines $H(K,X) := \mathrm{ind}_{n\in\mathbf{N}} H^\infty(K + V_n, X)$ with restrictions as linking maps. In [5] the authors have studied the space $H(K,X)$ when K is compact in a Fréchet-Schwartz space E. Chae [14] proved that $H(K,X)$ is a boundedly retractive LB-space, in particular complete, whenever E is a Banach space.

The LB-space of operators $L_i(E,X)$ considered in the paper is defined as follows. Let $E = \mathrm{proj}_{n\in\mathbf{N}} E_n$ be a Fréchet space, where $(E_n, \rho_{n,m}: E_m \to E_n)$ is a reduced projective spectrum of Banach spaces. We define an inductive spectrum $\mathscr{I} = (L(E_n, X), I_{n,m})$, where $I_{n,m}: L_b(E_n, X) \to L_b(E_m, X)$, $I_{n,m}(T) := T \circ \rho_{n,m}$. We define $L_i(E,X)$ to be $L(E,X)$ equipped with the limit topology of the spectrum (the so-called "inductive topology") – algebraically $L(E,X) = \mathrm{ind}\ \mathscr{I}$. By the Banach-Steinhaus Theorem, $L_b(E,X)$ and $L_i(E,X)$ have the same bounded sets and, thus the latter one is a regular LB-space, being the bornological space associated with $L_b(E,X)$. Unfortunately, it can happen that the identity $L_i(E,X) \to L_b(E,X)$ is not open: There are examples of Fréchet-Montel spaces E such that $L_b(E, l_2)$ is not even a DF-space (cf. [37]). Similarly there are Fréchet-Schwartz spaces E and a Banach space X such that $L_b(E,X)$ is also not a DF-space (cf. [32]).

Our notation for locally convex spaces is standard. We denote by t^{b} and t^{bor} the barrelled and bornological topologies associated with t, respectively, i.e, the weakest barrelled and bornological topology finer than the given topology t (see [34, 4.4.10 and 6.2.4]). For other notation and definitions we refer the reader to [23], [24] and [34]. For infinite dimensional holomorphy we refer to [21] and [29].

1. The linearization technique.

The first result gives a "linearization" of the problem. For other linearization theorems see [25], [30, Th. 2.1, 2.4] and [31, Th. 2.1]. The linearization in Prop. 1 was already observed by Bonet and Maestre in [11]. A detailed study of $G(K)$ defined below is contained in [13].

PROPOSITION 1. *For every compact subset K of a Fréchet space E, there is a Fréchet space $G(K)$ such that $H(K,X)$ is canonically isomorphic to $L_i(G(K),X)$ for every Banach space X.*

PROOF. Let (V_n) be a basis of absolutely convex open 0-neighbourhoods in E. We set $W_n := V_n + K$ for every $n \in \mathbf{N}$. Let X be a Banach space. We write $R_{n,m}: H^\infty(W_n, X) \to H^\infty(W_m, X)$ for the restriction map. In case $X = \mathbf{C}$, we denote the restriction by $\rho_{n,m}$. The closed unit ball B_n of $H^\infty(W_n)$ is compact for the compact-open topology τ_0. By the Dixmier-Ng Theorem (cf. [22, p. 211]), $H^\infty(W_n)$ is the strong dual of the Banach space

$$G^\infty(W_n) := \{u \in H^\infty(W_n)' : u|_{B_n} \text{ is } \tau_0\text{-continuous}\}.$$

We have that $p^t_{n,n+1}$ maps $G^\infty(W_{n+1})$ onto a dense subspace of $G^\infty(W_n)$. We set:

$$\sigma_{n,n+1} := \rho^t_{n,n+1}|_{G^\infty_{(W_{n+1})}}.$$

Clearly $\sigma^t_{n,n+1} = \rho_{n,n+1}$ and $(G^\infty(W_n), \sigma^t_{n,n+1})_{n\in\mathrm{N}}$ forms a reduced projective spectrum. We denote by $G(K)$ its limit. By Mujica's theorem [27, Th. 2], if we define $(H(K), \tau_0) := \mathrm{ind}_{n\in\mathrm{N}}(H(W_n), \tau_0)$, then $G(K)$ coincides with the Fréchet space

$$\{u \in H(K)' : u|_{B_n} \text{ is } \tau_0\text{-continuous for every } n \in \mathrm{N}\}$$

and $H(K)$ is the inductive dual of $G(K)$. Actually, $G(K) = (H(K), \tau_0)'$ by [28, Th. 2.1]. We denote by $S_{n,n+1} : L_b(G^\infty(W_n), X) \to L_b(G^\infty(W_{n+1}), X)$ the continuous linear map defined by $S_{n,n+1}(A) := A \circ \sigma_{n,n+1}$ for every $A \in L_b(G^\infty(W_n), X)$. Now, we apply [30, Th. 2.1], which remains true (with the same proof, see the proof of [31, Th. 2.1]) for open subsets of arbitrary locally convex spaces, to conclude that, if δ_x denotes the point evaluation at $x \in W_n$, the linear map

$$T_n : L_b(G^\infty(W_n), X) \to H^\infty(W_n, X),$$

defined by

$$(T_n A)(x) := A(\delta_x), \quad x \in W_n,$$

is an isometric isomorphism for every $n \in \mathrm{N}$. Moreover the following diagram clearly commutes:

$$\begin{array}{ccc} H^\infty(W_n, X) & \xrightarrow{R_{n,n+1}} & H^\infty(W_{n+1}, X) \\ {\scriptstyle T_n}\uparrow & & \uparrow{\scriptstyle T_{n+1}} \\ L_b(G^\infty(W_n), X) & \xrightarrow{S_{n,n+1}} & L_b(G^\infty(W_{n+1}), X). \end{array}$$

This implies that

$$H(K, X) = \operatorname*{ind}_{n\in\mathrm{N}} H^\infty(W_n, X) = \operatorname*{ind}_{n\in\mathrm{N}} L_b(W_n), X) = L_i(G(K), X).$$

By the regularity of $L_i(E, X)$ (see the remarks above) and by the fact that $L_i(E, X)$ is a complemented subspace of $H(K, X)$ for every compact set $K \subseteq E$ [21, Prop. 2.58], we get:

COROLLARY 2. *Let K be a compact subset of a Fréchet space E and let X be a Banach space. Then*

(a) *$H(K,X)$ is a regular* LB-*space;*

(b) *if $H(K,X)$ is complete, then $L_i(E,X)$ is complete.*

COROLLARY 3. *The following conditions are equivalent:*

(a) *$H(K,X)$ is complete for every compact set in every Fréchet space E and for every Banach space X;*

(b) *$L_i(F,X)$ is complete for every Fréchet space F and every Banach space X.*

2. Completeness of spaces of holomorphic germs.

We start with a completeness criterion which is a refined version of [4, 4.9(b)]:

LEMMA 4. *Let (G,t) be a quasicomplete locally convex space with an increasing fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex t-closed bounded sets. Assume that the following condition is satisfied:*

$$(C'_t) \qquad \forall(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+ \,\forall m\, \exists(\gamma_j)_{j\in\mathbb{N}} \subset \mathbb{R}_+ \,\forall n\colon\ \overline{\sum_{j=1}^{n} \gamma_j B_j \cap B_m} \subseteq \bigcup_{i=1}^{\infty} \sum_{i=1}^{k} \lambda_i B_i,$$

where the closure is taken in the topology t. Then $t^{\mathrm{bor}} = t^{\mathrm{b}}$ and $(G,t^{\mathrm{bor}}) := \mathrm{ind}_{n\in\mathbb{N}} G_{B_n}$ is a complete LB-*space.*

PROOF. As (G,t) is quasicomplete, (G,t) is complete [34, Cor. 5.1.8]. Local completeness depends only on the family of bounded sets and (G,t^{bor}) has the same family of bounded sets as (G,t), hence (G,t^{bor}) is locally complete. Since each locally complete quasibarrelled (in particular, bornological) space is barrelled [34, Cor. 5.1.10], (G,t^{bor}) is barrelled. Now, t^{bor} must be finer than t^{b} by the very definition of t^{b}. On the other hand, (G,t^{b}) is a barrelled DF-space, therefore it suffice to show that on bounded sets B_m the topology t^{b} is finer than t^{bor} [34, Cor. 8.3.3].

We fix a 0-neighbourhood $W = \bigcup_{k=1}^{\infty} \sum_{i=1}^{k} \lambda_i B_i$ in (G,t^{bor}) and we fix $m \in \mathbb{N}$. We apply (C'_t) to select (γ_j) and we define: $C_n := \overline{\sum_{j=1}^{n} \gamma_j B_j}$, the closure taken in (G,t). Clearly, $C_n \subseteq C_{n+1}$ and $\bigcup_{n\in\mathbb{N}} C_n$ is absorbing in G. By a result of De Wilde–Houet (cf. [34, 8.2.27]), we have

$$\overline{\bigcup_{n\in\mathbb{N}} C_n}^{(G,t^{\mathrm{b}})} \subseteq 2 \bigcup_{n\in\mathbb{N}} \overline{C_n}^{(G,t^{\mathrm{b}})} \subseteq 2 \bigcup_{n\in\mathbb{N}} C_n.$$

This implies that $V := \bigcup_{n\in\mathbb{N}} C_n$ is a 0-neighbourhood in (G, t^b), and we get from (C'_t):

$$V \cap B_m \subseteq \bigcup_{n\in\mathbb{N}} (C_n \cap B_m) \subseteq W.$$

We present now our "linear" completeness result which implies completeness of some spaces of holomorphic germs.

THEOREM 5. *Let E be a Fréchet space and X a Banach space.*

(a) *If X is complemented in its bidual, then $L_i(E, X)$ is complete.*

(b) *If E is quasinormable, then $L_i(E, X)$ is boundedly retractive and complete.*

(c) *Let, additionally, E be Montel, E or X satisfy the approximation property and let $E'_b \otimes_\varepsilon X$ be bornological (for example, if X is an $\mathscr{L}_\infty$-space or E is a Köthe space $\lambda_p(A)$ for $p = 0$ or $1 \leqq p < \infty$). If $L_i(E, X)$ is complete, then $E'_b \hat{\otimes}_\varepsilon X$ is an LB-space.*

PROOF. (a): Obviously, $L_i(E, X)$ is a complemented subspace of $L_i(E, X'')$. In particular, $L_i(E, X'')$ induces the original topology on bouded subsets of $L_i(E, X)$. Now, it suffices to show that condition (C'_t) from Lemma 4 is satisfied.

Let $C = B^{\circ\circ}$ denote the unit ball of X'' and $\mathscr{C}_n$ the unit ball in $L(E_n, X'')$. Since $\mathscr{C}_n$ is $\sigma(L(E, X''), E \otimes X')$-compact, it follows that

$$\overline{\sum_{i=1}^{m} \gamma_i \mathscr{C}_i} = \sum_{i=1}^{m} \gamma_i \mathscr{C}_i \quad \text{for all} \quad (\gamma_i)_{i \in \mathrm{N}} \subset \mathrm{R}_+, \quad m \in \mathrm{N},$$

where the closure is taken in $L_b(E, X'')$. To check (C'_t) we fix $(\lambda_i) \subseteq \mathrm{R}_+$ and $m \in \mathrm{N}$. Since $L_i(E, X)$ and $L_i(E, X'')$ induce the same topology on bounded sets in $L_i(E, X)$, we may select $(\gamma_j) \subseteq \mathrm{R}_+$ such that

$$\left(\bigcup_{n=1}^{\infty} \sum_{j=1}^{n} \gamma_j \mathscr{C}_j \right) \cap B_m \subseteq \bigcup_{k=1}^{\infty} \sum_{i=1}^{k} \lambda_i B_i.$$

Now, for each $n \in \mathrm{N}$,

$$\overline{\sum_{j=1}^{n} \gamma_j B_j \cap B_m}^{t} \subseteq \sum_{j=1}^{n} \gamma_j \mathscr{C}_j \cap B_m \subseteq \bigcup_{k=1}^{\infty} \sum_{i=1}^{k} \lambda_i B_i.$$

(b): If E is quasinormable, we can apply [18, 5.2 (c) and 5.3] to conclude that for every $n \in \mathrm{N}$ there is $m \geqq n$ such that $L_b(E, X)$, $L_i(E, X)$ and $L_b(E_m, X)$ induce the same topology on the unit ball B_n of $L_b(E_n, X)$. In particular, $L_i(E, X)$ is quasicomplete and, by [34, 8.3.18], complete.

(c): If X is an $\mathscr{L}_\infty$-space, then it has the approximation property and, by the result of Defant and Govearts [16] (see [34, Prop. 11.5.10, Obs. 4.8.3 (c)]) $E'_b \otimes_\varepsilon X$ is bornological. If E is a Köthe space, then it is a T-space in the sense of [12] and therefore, $E'_b \otimes_\varepsilon X$ is bornological [12, Prop. 10]. Of course, it has the approximation property.

Since E is a Fréchet-Montel space and E or X has the approximation property, we have $L_b(E, X) = E'_b \varepsilon X = E'_b \hat{\otimes}_\varepsilon X$. We observe that the injection j: $E'_b \otimes_\varepsilon X \hookrightarrow L_i(E, X)$ is an isomorphism into, since its domain is bornological. The range space is complete and there is a unique continuous extension $\hat{j}: E'_b \hat{\otimes}_\varepsilon X \to L_i(E, X)$ of j. This implies that $\hat{j}$ coincides with the identity.

By Prop. 1 and by the fact [21, Prop. 2.58] that $L_i(E, X)$ is a complemented subspace of $H(K, X)$, we can apply Theorem 5 to the space $H(K, X) = L_i(G(K), X)$. For example, if E is quasinormable, we can apply a result of Mujica [27, Th. 3] to conclude that $H(K) = G(K)'_i$ and $G(K)$ is quasinormable. We have the following corollaries.

COROLLARY 6. *Let K be a compact subset of a Fréchet space E and let X be a Banach space. The space $H(K, X)$ is complete if one of the following conditions is satisfied:*
(a) *X is complemented in its bidual;*
(b) *E is quasinormable.*

COROLLARY 7. *Let K be a compact subset of a Fréchet-Montel space E and let X be a Banach $\mathscr{L}_\infty$-space. If $H(K, X)$ is complete, then $E'_b \hat{\otimes}_\varepsilon X$ is an* LB*-space. In particular, if $H(K, c_0)$ is complete, then $c_0(E'_b)$ is an* LB*-space.*

COROLLARY 8. *Let E be a quasinormable Fréchet space. Let K and U be a compact and an open subset of E, respectively. If X is a Banach space, then $H(K, X)$ is a boundedly retractive* LB*-space and $(H(U, X), \tau_\omega)$ is complete.*

PROOF. It remains to show the second part only. Let (U_n) be a decreasing basis of open 0-neighbourhoods in E. Let $H^K(U, X)$ denote the image of the canonical mapping

$$H(U, X) \to H(K, X)$$

and let $\tilde{H}^K_n(U, X)$ denote the closure of $H^K(U, X) \cap H^\infty(K + U_n, X)$ in $H^\infty(K + U_n, X)$. Since E is quasinormable, we first apply [18, 5.3] to conclude that the LB-space

$$H(K, X) = L_i(G(K), X) = \operatorname*{ind}_{n \in \mathbb{N}} L_b(G^\infty(K + U_n), X) = \operatorname*{ind}_{n \in \mathbb{N}} H^\infty(K + U_n, X)$$

is boundedly retractive, and next apply [6, Lemma 13] to conclude that the LB-space $\tilde{H}^K(U, X) := \operatorname{ind}_{n \in \mathbb{N}} \tilde{H}^K_n(U, X)$ is boundedly retractive as well. Since [26, Lemma 5.6] applies to vector-valued mappings, $(H(U, X), \tau_\omega)$ is isomorphic to the projective limit of the spaces $\tilde{H}^K(U, X)$, with $K \subseteq U$.

In the case $X = \mathbb{C}$, Corollary 8 is due to Avilés and Mujica [1].

In view of Theorem 5 (c) and Prop. 1, it is interesting to explain when $G(K)$ is Montel. Let E be a Fréchet-Montel space, $K \subseteq E$ compact and balanced. By [13], $G(K)$ is Montel iff the completion of the space of projective n-symmetric tensors $\hat{\otimes}_{n,s,\pi} E$ is Fréchet-Montel for every n. This is also equivalent to the coincidence of the topologies τ_0 and τ_ω on $H(E)$. We refer to [17] for the following recent results:

(1) If E is a Fréchet-Montel space with the approximation property such that every bounded subset of $E'_b \hat{\otimes}_\pi X$ is liftable by bounded subsets for every Banach space X, then $G(K)$ is Montel for every compact subset K of E;

(2) If E is a Fréchet-Montel space such that $E \hat{\otimes}_\pi E$ is not Montel [37], then for some (for all) compact subsets K of $E \times E$, the space $G(K)$ is not Montel.

3. Completeness of $L_i(E, X)$.

In view of the previous results, it seems to be interesting to explain connections between various sufficient and necessary conditions for completeness of $L_i(E, X)$. The following "omnibus theorem" covers our full (not completely satisfactory though) knowledge on the completeness of $L_i(E, X)$.

THEOREM 9. *Let E be a Fréchet space and X a Banach space. Let t denote the topology of $L_b(E, X)$ and (B_n) an arbitrary fundamental sequence of bounded sets there. We consider the following conditions:*

(1) *E is quasinormable;*

(2) *X is complemented in its bidual (for example, X is a dual Banach space);*

(3) *$L_i(E, X) = L_b(E, X)$ holds topologically;*

(3′) *$L_i(E, X)$ and $L_b(E, X)$ induce the same topology on bounded sets;*

(4) *$L_i(E, X)$ is a toplogical subspace of $L_i(E, X'')$;*

(4′) *$L_i(E, X)$ and $L_i(E, X'')$ induce the same topology on bounded subsets of the first space;*

(5) *$L_b(E, X)$ satisfies (C_t)*, i.e.,

$$\forall(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+ \; \exists(\gamma_j)_{j\in\mathbb{N}} \subset \mathbb{R}_+ \; \forall n: \overline{\sum_{j=1}^{n} \gamma_j B_j} \subseteq \bigcup_{k=1}^{\infty} \sum_{i=1}^{k} \lambda_i B_i,$$

where the closure is taken in the topology t;

(5′) *$L_b(E, X)$ satisfies (C'_t) – see Lemma 4;*

(6) $t^{\mathrm{bor}} = t^{\mathrm{b}}$;

(7) *$L_i(E, X)$ is complete;*

(8) *$E'_b \hat{\otimes}_\varepsilon X$ is an* LB-*space;*

(9) *$E' \otimes X$ is dense in $L_i(E, X)$.*

We have:

(a) *The following implications always hold:*

$$\begin{array}{ccccccccccc}
(1) & \longrightarrow & (3') & \longrightarrow & (4') & \longrightarrow & (5') & \longrightarrow & (6) & \longrightarrow & (7) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & (3) & \longrightarrow & (4) & \longrightarrow & (5) & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & (2) & & & & & &
\end{array}$$

(b) *If, additionally, E is Montel, E or X satisfies the approximation property and if* $E'_b \otimes_\varepsilon X$ *is bornological (for example, if X is an* $\mathscr{L}_\infty$*-space or E is a Köthe space* $\lambda_p(A)$ *for* $p = 0$ *or* $1 \leq p < \infty$*), then all conditions (3)–(9) are equivalent.*

PROOF. Without loss of generality we may assume that $E = \operatorname{proj}_{n \in \mathbb{N}} E_n$, E_n Banach spaces and B_n is the unit ball of $L_b(E_n, X)$.

(a): Implications (3) ⇒ (3′), (2) ⇒ (4) ⇒ (4′), (5) ⇒ (5′) are trivial and (5′) ⇒ (6) ⇒ (7) are implied immediately by Lemma 4.

The proofs of (3) ⇒ (4) and (3′) ⇒ (4′) are very similar and follow from the following observations: $L_b(E, X)$ is a topological subspace of $L_b(E, X'')$ and the canonical injections

$$L_i(E, X) \hookrightarrow L_i(E, X'') \hookrightarrow L_b(E, X'')$$

are continuous.

Again the proofs of (4) ⇒ (5) and (4′) ⇒ (5′) are very similar and the latter is given in the proof of Theorem 5 (b).

(1) ⇒ (3′): See the proof of Theorem 5 (a).

(b): Since E is a Fréchet-Montel space and E or X has the approximation property, we have $L_b(E, X) = E'_b \varepsilon X = E'_b \hat{\otimes}_\varepsilon X$.

The equivalence of (3) and (8) follows from the fact that $L_i(E, X)$ is the bornological space associated with $L_b(E, X)$.

Since $E' \otimes X$ is dense in $E'_b \hat{\otimes}_\varepsilon X = L_b(E, X)$, (9) clearly follows from (3).

Now, we assume (9). Since $E'_b \otimes_\varepsilon X$ is bornological, it follows that the canonical injection $j: E'_b \otimes_\varepsilon X \hookrightarrow L_i(E, X)$ is a topological isomorphism into. In order to check (3), by [9, 1.2], it is enough to prove that $L_b(E, X)$ and $L_i(E, X)$ induce the same topology on $E' \otimes X$. This is now clear: both spaces induce the injective topology of $E'_b \otimes_\varepsilon X$. Accordingly, (3), (8) and (9) are equivalent.

The rest of the proof is given in the proof of Theorem 5 (c).

It is possible to show by direct arguments that some of the conditions in Theorem 9 imply that $L_i(E, X)$ is complete. We prefer to reduce them to the case covered by Lemma 4 to emphasize that Lemma 4 provides the best criterion.

REMARKS AND EXAMPLES. (a) $X = L^1(0, 1)$ is not a dual space but it is complemented in its bidual.

(b) More examples of Fréchet spaces E such that $L_b(E, X) = L_i(E, X)$ for every Banach space X can be seen in [12] (for example, see [12, Obs. 9a]).

(c) If E is a Köthe echelon space $\lambda_1(A)$ and X is a Banach space, then the proof of [4, 4.8] shows that the condition (5) is satisfied and, accordingly, $L_i(E, X)$ is complete.

(d) We do not know if (7) ⇒ (6) ⇒ (5′) ⇒ (4′), (5′) ⇒ (5) or (5) ⇔ (4′) hold. On the other hand, we know that among other implications between conditions (1)–(7) only those covered by Theorem 9 (a) hold in general.

(3) does not imply (1) or (2): By [3, Cor. 7], if $E = \lambda_1(A)$ is distinguished and X is a Banach space, then $L_b(E, X) = L_i(E, X)$. Take $X = c_0$ and E not quasi-normable.

(2) does not imply (3′): Take $X = \mathbb{K}$ and E any non-distinguished Fréchet space.

(1) does not imply (4): By [33], there is a quojection E and an $\mathscr{L}_\infty$-space X such that $L_b(E, X)$ is not a DF-space. Since X is an $\mathscr{L}_\infty$-space, $L_b(E, X'') = L_i(E, X'')$ follows from a result of [15]. This implies that $L_i(E, X)$ is not a topological subspace of $L_i(E, X'')$.

Non-existence of other implications is obtained by logical operations.

(e) Theorem 9 is also related to the following two questions that remain open [34, 13.8.1 and 13.8.6]:

(1) Is the completion of a bornological DF-space also bornological (or, equivalently, an LB-space)?

(2) Is every regular LB-space complete?

Before we state additional consequences of Theorem 9, we explain first the relation between conditions (4)–(6).

PROPOSITION 10. (a) *The condition (6) above is equivalent to* $(C_{(t^b)})$ *and to* $(C'_{(t^b)})$ *as well.*

(b) *If* $\sigma = \sigma(L(E, X), E' \otimes X)$ *and* X'' *has the bounded approximation property, then* (4) *and* (4′) *are equivalent to* (C_σ) *and* (C'_σ), *resp.*

PROOF. Part (a) is a consequence of Lemma 4. To conclude part (b) it is enough to use that if X'' has the bounded approximation property, then for every Banach space Y the unit ball of $L(Y, X'')$ is contained in a multiple of the closure of the unit ball of $L(Y, X)$ for the topology $\sigma(L(Y, X''), Y \otimes X')$.

By Theorem 9 (b), we obtain immediately:

COROLLARY 11. *If S is a compact Hausdorff space, E is a Fréchet-Montel space, then* $C(S, E'_b)$ *is an* LB-*space iff the conditions (3)–(9) in Theorem 5 hold for* $X = C(S)$. *In particular,* $C(S, E'_b)$ *is bornological iff* $L_i(E, C(S))$ *is complete.*

REMARKS. (a) Bierstedt [35] (comp. also [36], [19]) posed the problem if $C(S, E'_b)$ is an LB-space for every Fréchet-Montel space E. The problem has been solved very recently by S. Dierolf for $c_0(E'_b)$.

(b) An $\mathscr{L}_\infty$-space X is complemented in its bidual iff it is complemented in $C(\beta I)$ for a discrete space I (equivalently, it is injective). In [19, Cor. 6.3] Dierolf and Domański proved that $C(\beta I, E'_b)$ is an LB-space for every Fréchet-Montel space E (which is implied also by Corollary 11 and Theorem 9 above).

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