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GROWTH OF BETTI NUMBERS OVER NOETHERIAN LOCAL RINGS

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Abstract.

Avramov has asked the following question: Is the sequence of Betti numbers of a finitely generated module M over a Noetherian local ring R eventually non-decreasing? In this paper, stronger growth properties for Betti numbers are considered. We give sufficient conditions on the Hilbert function of an Artinian ring R for these stronger growth properties to hold. A positive answer is given to the question of Avramov if one of the Hilbert coefficients is at least as big as the sum of all the others.

1. Introduction.

In this paper, (R, m, k) is a Noetherian local ring and M a finitely generated R-module. We denote by $\mu(M)$ the minimal number of generators of M and by $\lambda(M)$ its length. For $i \geqq 1$, let ∂_i: $R^{b_i} \to R^{b_{i-1}}$ be the ith map in a fixed minimal resolution of M, $K_i = \operatorname{Im} \partial_i$ the ith syzygy of M and $b_i = b_i^R(M) = \dim_k \operatorname{Tor}_i^R(M, k) = \mu(K_i)$, the ith Betti number of M.

In [Av_1, 5.8], L. Avramov proposed the following problem.

PROBLEM 1. *Is the sequence $b_i^R(M)$ eventually non-decreasing for any finitely generated module M over a local ring R?*

Earlier, M. Ramras [*Ra*] considered a weaker problem.

PROBLEM 2. *Is it true that for any finitely generated module M over a local ring R, there are only two possibilities: either the sequence $b_i^R(M)$ is eventually constant, or* $\dim_k b_i^R(M) = \infty$?

Positive results on these problems have been given in papers by L. Avramov, S. Choi, V. N. Gasharov, E. H. Gover, J. Lescot, I. V. Peeva, M. Ramras and L. Sun (Cf. the reference list). We obtain new results in this paper for the Artinian case-Problem 1 has a positive answer if the Hilbert coefficients $e_1, \ldots, e_h$ satisfy certain inequalities, in particular, if one of the e_i's is at least as big as the sum of all the others.

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In many cases these results establish stronger growth properties which we are going to describe.

DEFINITION. (1) M has *strong exponential growth* ([Av_3]) if there is a $\gamma > 1$ such that $b_i^R(M) \geqq \gamma^i$ for $i \gg 0$. M has *alternating exponential growth* if there is a $\gamma > 1$ such that $b_{i+2}^R(M) > \gamma b_i^R(M)$ for $i \gg 0$. M has *termwise exponential growth* if there is a $\gamma > 1$ such that $b_{i+1}^R(M) > \gamma b_i^R(M)$ for $i \gg 0$.

(2) The ring R has *uniform strong* (resp. *alternating, termwise*) *exponential growth* if each finitely generated M has the corresponding property with some γ which depends only on R.

For the rest of the paper, we set $(0:\mathfrak{m}) = J$, $\varepsilon = \dim_k J$ and $e_j = \dim_k(\mathfrak{m}^j/\mathfrak{m}^{j+1})$ for $j \geqq 1$. We denote by $\lambda(M)$ the length of M, and set $h(M) = \sup\{j \in \mathbf{N} \mid \mathfrak{m}^j M \neq 0\}$, $l = \lambda(R)$ and $h = h(R)$.

Gasharov and Peeva ([GP], Proposition 2.2]) proved that for each finitely generated R-module M, $b_{i+1}^R(M) \geqq (2e_1 - l + h - 1)b_i^R(M)$ for $i \gg 0$. So if $2e_1 + h \geqq l + 2$, then $\{b_i^R(M)\}$ is eventually non-decreasing, and if $2e_1 + h > l + 2$, then R has uniform termwise exponential growth. In our main result, this statement is extended to include the other Hilbert coefficients:

MAIN THEOREM. *Let R be an Artinian local ring with $h \geqq 3$. If $2e_j + h \geqq l + j$ for some j such that $2 \leqq j \leqq h$, then $\{b_n^R(M)\}$ is eventually non-decreasing for each finitely generated R-module M. If furthermore $2e_2 + h \neq l + 2$, or $2e_3 + h \neq l + 3$, or $j > 3$, then R has uniform termwise exponential growth.*

Recently, Peeva [P] has shown uniform termwise exponential growth for rings with $2e_1 + h = l + 2$.

2. Preliminary Results.

We begin with some basic observations. We use $\dim_k V$ to denote the dimension of a vector space V over k.

PROPOSITION 2.1. *If M is a finitely generated R-module with* ith *Betti number b_i, and* ith *syzygy K_i, then for $i \geqq 1$, we have*

(a) $JR^{b_i} \subseteq K_{i+1} \subseteq \mathfrak{m}R^{b_i}$;

(b) *the sequence* $0 \to K_{i+1} \to \mathfrak{m}R^{b_i} \to \mathfrak{m}K_i \to 0$ *is exact;*

(c) $b_{i+1} = e_1 b_i - \mu(\mathfrak{m}K_i) + \dim_k \dfrac{K_{i+1} \cap \mathfrak{m}^2 R^{b_i}}{\mathfrak{m}K_{i+1}}$.

PROOF. (a) By minimality of the resolution of M, we have $K_{i+1} = \operatorname{Im} \partial_{i+1} \subseteq \mathfrak{m}R^{b_i}$. Also, $\partial_i(JR^{b_i}) \subseteq J\mathfrak{m}R^{b_{i-1}} = 0$. So $JR^{b_i} \subseteq \operatorname{Ker} \partial_i = \operatorname{Im} \partial_{i+1} = K_{i+1}$.

(b) By (a), one has $\mathfrak{m}K_i = \mathfrak{m}(R^{b_i}/K_{i+1}) \simeq (\mathfrak{m}R^{b_i} + K_{i+1})/K_{i+1} = \mathfrak{m}R^{b_i}/K_{i+1}$.

(c) Any exact sequence $0 \to A \to B \to C \to 0$ of finitely generated modules over a Noetherian local ring $(R, \mathfrak{m}, k)$ gives, after applying $-\otimes_R k$, an exact sequence $A/\mathfrak{m}A \to B/\mathfrak{m}B \to C/\mathfrak{m}C \to 0$ of k-vector spaces. The kernel of the first map is obviously $(A \cap \mathfrak{m}B)/\mathfrak{m}A$. So counting dimensions gives

$$(2.2) \qquad \mu(A) = \dim_k \frac{A}{\mathfrak{m}A} = \dim_k \frac{B}{\mathfrak{m}B} - \dim_k \frac{C}{\mathfrak{m}C} + \dim_k \frac{A \cap \mathfrak{m}B}{\mathfrak{m}A}$$

$$= \mu(B) - \mu(C) + \dim_k \frac{A \cap \mathfrak{m}B}{\mathfrak{m}A}.$$

Applying (2.2) to the exact sequence from (b), we get

$$\text{(c)} \qquad b_{i+1} = \mu(K_{i+1}) = \mu(\mathfrak{m}R^{b_i}) - \mu(\mathfrak{m}K_i) + \dim_k \frac{K_{i+1} \cap \mathfrak{m}^2 R^{b_i}}{\mathfrak{m}K_{i+1}}$$

$$= e_1 b_i - \mu(\mathfrak{m}K_i) + \dim_k \frac{K_{i+1} \cap \mathfrak{m}^2 R^{b_i}}{\mathfrak{m}K_{i+1}}.$$

We include a well-known fact for notation and subsequent references.

REMARK 2.3. *Let M be a finitely generated module over a (not necessarily Noetherian) local ring $(R, \mathfrak{m}, k)$. Then*

(a) *the following statements are eqivalent.*

(1) *$M \simeq M' \oplus k$ for some $M' \subseteq M$*

(2) *$\mathfrak{m}M \not\supseteq \operatorname{Soc}(M)$ where $\operatorname{Soc}(M) = \{x \in M \mid \mathfrak{m}x = 0\}$.*

(b) *$M \simeq M' \oplus k^r$ with $\operatorname{Soc}(M') \subseteq \mathfrak{m}M'$ and $r = \dim_k \dfrac{\operatorname{Soc}(M) + \mathfrak{m}M}{\mathfrak{m}M}$.*

By repeated application of (a), we have $M \simeq M' \oplus k^r$ with $\operatorname{Soc}(M') \subseteq \mathfrak{m}M'$ for some r. Then $\operatorname{Soc}(M) \simeq \operatorname{Soc}(M') \oplus k^r$ and $(\operatorname{Soc}(M) + \mathfrak{m}M)/\mathfrak{m}M \simeq k^r$. Hence $r = \dim_k(\operatorname{Soc}(M) + \mathfrak{m}M)/\mathfrak{m}M$.

PROPOSITION 2.4. *Let $\gamma = \dim_k(J + \mathfrak{m}^2)/\mathfrak{m}^2$. For each finitely generated R-module M, and for $i \geqq 1$, the Betti numbers b_i satisfy*

$$b_{i+1} = \varepsilon b_i + \mu\left(\frac{K_{i+1}}{JR^{b_i}}\right) - \dim_k(JR^{b_i} \cap \mathfrak{m}^2 R^{b_i}) \geqq \gamma b_i$$

PROOF. The equality follows by applying (2.2) to $0 \to JR^{b_i} \to K_{i+1} \to K_{i+1}/JR^{b_i} \to 0$. Further, we have

$$\mu(JR^{b_i}) - \dim_k(JR^{b_i} \cap \mathfrak{m}K_{i+1}) = \dim_k\left(\frac{JR^{b_i}}{JR^{b_i} \cap \mathfrak{m}K_{i+1}}\right)$$
$$\geqq \dim_k\left(\frac{JR^{b_i}}{JR^{b_i} \cap \mathfrak{m}^2R^{b_i}}\right)$$
$$= b_i \dim_k\left(\frac{J + \mathfrak{m}^2}{\mathfrak{m}^2}\right).$$

REMARK 2.5. The proposition shows that the Betti sequence of each R-module is non-decreasing if $(0:\mathfrak{m}) \nsubseteq \mathfrak{m}^2$. This result can also be obtained from [Ch_1, Theorem 1.1]. Indeed, we may assume R is complete and so by Cohen structure theorem, R can be expressed as the homomorphic image S/I_0 of a complete regular local ring $(S, \mathfrak{n})$ with $I_0 \subseteq \mathfrak{n}^2$. Let $J_0 = (I_0 :_S \mathfrak{n})$ and denote the integral closure of ideals by bars. Assuming $\overline{I_0} = \overline{J_0}$, we have $J_0 \subseteq \overline{J_0} = \overline{I_0} \subseteq \overline{\mathfrak{n}^2} = \mathfrak{n}^2$ since any power of the maximal ideal of a regular local ring is integrally closed. This implies $(0:\mathfrak{m}) \subseteq \mathfrak{m}^2$, a contradiction. So $\overline{I_0} \neq \overline{J_0}$ and [Ch_1, Theorem 1.1] applies.

REMARK 2.6. Let R be an Artinian local ring and M be a finitely generated R-module. Then $b_{i+1} \geqq e_1 b_i - (l - e_1 - h + 1)b_{i-1}$. This is contained in the proof of [GP, Proposition 2.2]. It motivates us to study the following inequality for sequences of positive integers $\{b_i\}$ involving 3 consecutive terms:

(2.7) $$b_{i+1} \geqq \lambda b_i - \mu b_{i-1} \text{ for } i \geqq 1 \text{ where } \lambda, \mu > 0.$$

PROPOSITION 2.8. *Suppose $\{b_i\}_{i \geqq 0}$ satisfies* (2.7). *For any integer $i \geqq 1$ and real number $\theta > 0$, if $\dfrac{b_{i+1}}{b_{i-1}} \geqq \theta$, then $\dfrac{b_{i+1}}{b_i} \geqq \dfrac{\lambda\theta}{\mu + \theta}$.*

PROOF. From (2.7) and the assumption, we have $b_{i+1} \geqq \lambda b_i - \mu b_{i-1} \geqq \lambda b_i - \left(\dfrac{\mu}{\theta}\right) b_{i+1}$. Then $\left(1 + \dfrac{\mu}{\theta}\right) b_{i+1} \geqq \lambda b_i$, hence $b_{i+1} \geqq \left(\dfrac{\lambda\theta}{\mu + \theta}\right) b_i$.

An inequality of the form $b_{i+2} \geqq \theta b_i$ is established in some cases by [Ch_2]. Applying it together with Proposition 2.8, we get cases of uniform termwise exponential growth. We first state a definition due to Choi:

An ideal I of a Noetherian local ring $(S, \mathfrak{n})$ satisfies (H_k) if $\overline{(I + \mathfrak{p})/\mathfrak{p}} \neq \overline{((I :_S \mathfrak{n}) + \mathfrak{p})/\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ such that $\operatorname{ht} \mathfrak{p} \leqq k$ or depth $S_{\mathfrak{p}} \leqq 1$.

COROLLARY 2.9. *Let $(S, \mathfrak{n})$ be a Noetherian local ring, let $R = S/I$ be an Artinian local ring and let M be a finitely generated R-module.*

(a) *If $I = \mathfrak{n}J$ with $J \neq R$ and $i > 1$, then $b_{i+1} \geqq \gamma b_i$ with $\gamma = \left(\dfrac{e_1^2}{l - h + 1}\right)$.*

(b) *If I satisfies (H_k) and $i > 1$, then $b_{i+1} \geqq \gamma b_i$ with* $\gamma = \dfrac{(k+1)e_1}{l - e_1 - h + k + 2}$.

(c) *If S is regular with* $\dim S \geqq 2$, *then for $i > 1$, we have $b_{i+1} \geqq \gamma b_i$ where* $\gamma = \dfrac{e_1\theta}{l - e_1 - h + 1 + \theta}$, $\theta = \dbinom{\dim S + o(I) - 2}{o(I) - 1}$ *and $o(I)$ is the order of I.*

PROOF. (a) We may assume R is complete and S is a regular local ring. So $\dim S = e_1$. Then [Ch_2, Proposition 1.3] gives $b_{i+2} \geqq e_1 b_i$ for $i \geqq 1$ and the result follows from Proposition 2.8.

(b) [Ch_2, Proposition 1.4] gives $b_{i+2} \geqq (k+1)b_i$ for $i \geqq 1$ and the result follows from Proposition 2.8.

(c) [Ch_2, Theorem 3.5] gives $b_{i+2} \geqq \dbinom{\dim S + o(I) - 2}{o(I) - 1} b_i$ for $i \geqq 1$ and the result follows from Proposition 2.8.

The corollary can be used to produce numerical examples.

EXAMPLE 2.10. *R has uniform termwise exponential growth in the following cases:*

(a) $h = 3, e_1 = 4$ and $l = 11, 12$ where $\gamma = \frac{16}{9}, \frac{8}{5}$ respectively.

(b) $h = 3, e_1 = 4, l = 11, 12$ and I satisfies $(H_1), (H_2)$ respectively where $\gamma = \frac{8}{7}, \frac{6}{5}$ respectively.

(c) $e_1 \geqq 4$, $h = o(I) + 1$ and $\frac{1}{2}e_1 \leqq o(I) \leqq e_1^2 - \frac{5}{2}e_1$.

We considered another inequality on Betti numbers and have other results on the growth of Betti numbers in [Fa].

3. Main Results.

As pointed out in Section 1, our Main Theorem is an extension of the statement of [GP, Proposition 2.2].

PROPOSITION 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. For any submodule Q of a finitely generated R-module K, there is an inequality*

$$\mu(Q) \leqq \mu(K) + \left(1 - \frac{1}{e_1}\right)\lambda(\mathfrak{m}K).$$

PROOF. We may assume K has finite length and argue by induction on $h(K)$. If $h(K) = 0$, then both Q and K are k-vector spaces, hence $\mu(\mathrm{Q}) \leqq \mu(\mathrm{K})$. Now let $h(K) > 0$ and assume the inequality holds for all R-modues K' with $h(K') < h(K)$. Let $x_1, \ldots, x_s$ be a minimal set of generators of Q. We may assume that $x_1, \ldots, x_p$ $(p \leqq s)$ are part of a minimal set of generators of K and $x_{p+1} \ldots, x_s$ are in $\mathfrak{m}K$. Let $x_1, \ldots, x_p, y_1, \ldots, y_q$ be a minimal set of generators of K. Let L, M and N be the submodules of K generated by $\{x_1,\ldots,x_p\}$, $\{x_{p+1},\ldots,x_s\}$ and $\{y_1,\ldots,y_q\}$, respectively. Then $K = L + N$ and for $p+1 \leqq j \leqq s$, we have $x_j = \sum_{i=1}^{p} r_{ij}x_i + \sum_{i=1}^{q} s_{ij}y_i$, with $r_{ij}, s_{ij} \in \mathfrak{m}$.

Set $x'_j = \sum_{i=1}^{q} s_{ij}y_i$ and $M' = Rx'_{p+1} + \ldots + Rx'_s$. Then $\mu(M') = s - p$ and $M' \subseteq \mathfrak{m}N$. Thus $h(M') < h(K)$ hence the induction hypothesis applies and we obtain:

$$\begin{aligned}
\mu(Q) &= p + \mu(M') \\
&\leqq p + \mu(\mathfrak{m}N) + \left(1 - \frac{1}{e_1}\right)\lambda(\mathfrak{m}^2N) \\
&= p + \frac{1}{e_1}\mu(\mathfrak{m}N) + \left(1 - \frac{1}{e_1}\right)(\mu(\mathfrak{m}N) + \lambda(\mathfrak{m}^2N)) \\
&\leqq p + \mu(N) + \left(1 - \frac{1}{e_1}\right)\lambda(\mathfrak{m}N) \\
&= \mu(K) + \left(1 - \frac{1}{e_1}\right)\lambda(\mathfrak{m}N) \\
&\leqq \mu(K) + \left(1 - \frac{1}{e_1}\right)\lambda(\mathfrak{m}K)
\end{aligned}$$

This finishes the proof of the proposition.

COROLLARY 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and Q, K finitely generated R-modules with $\mathfrak{m}^pK \subseteq Q \subseteq K$ for some $p \geqq 1$. Then*

$$\mu(Q) \leqq \mu(K) + \left(1 - \frac{1}{e_1}\right)[\mu(\mathfrak{m}K) + \ldots + \mu(\mathfrak{m}^{p-1}K)] + \mu(\mathfrak{m}^pK).$$

PROOF. Applying Proposition 3.1 to $\dfrac{Q}{\mathfrak{m}^pK} \subseteq \dfrac{K}{\mathfrak{m}^pK}$, we have

$$\begin{aligned}
\mu\left(\frac{Q}{\mathfrak{m}^pK}\right) &\leqq \mu\left(\frac{K}{\mathfrak{m}^pK}\right) + \left(1 - \frac{1}{e_1}\right)\lambda\left(\frac{K}{\mathfrak{m}^pK}\right) \\
&= \mu(K) + \left(1 - \frac{1}{e_1}\right)[\mu(\mathfrak{m}K) + \ldots + \mu(\mathfrak{m}^{p-1}K)]
\end{aligned}$$

Then the fact $\mu(Q) \leqq \mu\left(\dfrac{Q}{\mathfrak{m}^pK}\right) + \mu(\mathfrak{m}^pK)$ gives the desired inequality.

NOTATION. For an Artinian local ring R and $2 \leqq j \leqq h$, we set

$$u_j = e_{j+1} + \ldots + e_h + j + 1 - h = \lambda(\mathfrak{m}^{j+1}) + j + 1 - h;$$

$$v_j = \left(1 - \frac{1}{e_1}\right)(e_1 + \ldots + e_{j-2}) + e_{j-1};$$

and $$\gamma_j = \frac{e_j - u_j}{1 + v_j}.$$

THEOREM 3.3. *Let R be an Artinian local ring and M be a finitely generated non-free R-module. For $2 \leqq j \leqq h$, if $\gamma_j \geqq 1$, then $b_{i+1} \geqq \gamma_j b_i$ eventually.*

PROOF. For fixed $i \geqq 1$ and $2 \leqq j \leqq h$, consider the exact sequences

(3.4) $$0 \to K_{i+1} \to \mathfrak{m}^j R^{b_i} + K_{i+1} \to \mathfrak{m}^j K_i \to 0$$

$$0 \to \mathfrak{m}^j R^{b_i} \cap K_{i+1} \to \mathfrak{m}^j R^{b_i} \oplus K_{i+1} \to \mathfrak{m}^j R^{b_i} + K_{i+1} \to 0.$$

By [GP, Lemma 2.1], $\mathfrak{m}^j K_i \subseteq \mathfrak{m}^{j+1} R^{b_{i-1}}$ gives

$$\mu(\mathfrak{m}^j K_i) \leqq (\lambda(\mathfrak{m}^{j+1}) + j + 1 - h)b_{i-1} = u_j b_{i-1}.$$

Using the exact sequences (3.4), we have

$$b_{i+1} + u_j b_{i-1} \geqq b_{i+1} + \mu(\mathfrak{m}^j K_i) \geqq \mu(\mathfrak{m}^j R^{b_i} + K_{i+1})$$
$$\geqq \mu(\mathfrak{m}^j R^{b_i}) + \mu(K_{i+1}) - \mu(\mathfrak{m}^j R^{b_i} \cap K_{i+1})$$

Applying Corollary 3.2 with $Q = \mathfrak{m}^j R^{b_i} \cap K_{i+1}, p = j - 1$ and writing $K = K_{i+1}$ yields

$$\mu(\mathfrak{m}^j R^{b_i} \cap K) \leqq \mu(K) + \left(1 - \frac{1}{e_1}\right)[\mu(\mathfrak{m}K) + \ldots + \mu(\mathfrak{m}^{j-2}K)] + \mu(\mathfrak{m}^{j-1}K).$$

So $$b_{i+1} + u_j b_{i-1} \geqq e_j b_i + \mu(K) - v_j\mu(K) - \mu(K) = e_j b_i - v_j\mu(K)$$
$$\geqq e_j b_i - v_j\mu(K).$$

Therefore $(1 + v_j)b_{i+1} \geqq e_j b_i - u_j b_{i-1}$. Hence $b_{i+1} \geqq \lambda b_i - \mu b_{i-1}$ where $\lambda = \frac{e_j}{1 + v_j}$ and $\mu = \frac{u_j}{1 + v_j}$. By assumption, $\lambda - \mu \geqq 1$. Now a finitely generated R-module M necessarily satisfies $b_n \geqq b_{n-1}$ for some n, $1 \leqq n \leqq \mu(M)$. Then by an inductive argument, $b_{i+1} \geqq (\lambda - \mu)b_i \geqq b_i$ for $i \geqq n$.

We now prove the Main Theorem stated in the introduction.

PROOF OF MAIN THEOREM. We first note that for $2 \leqq j \leqq h$, $1 + v_j = 1 + \left(1 - \frac{1}{e_1}\right)e_1 + \left(1 - \frac{1}{e_1}\right)(e_2 + \ldots + e_{j-1}) + e_{j-1} \leqq e_1 + \ldots + e_{j-1}$ with equality exactly when $j = 2, 3$. So

$$\begin{aligned} 0 \leqq 2e_j + h - l - j &= e_j + h - (1 + e_1 + \ldots + e_{j-1} + e_{j+1} + \ldots + e_h + j) \\ &= e_j - (e_1 + \ldots + e_{j-1}) - (e_{j+1} + \ldots + e_h + j + 1 - h) \\ &\leqq e_j - (1 + v_j) - u_j \end{aligned}$$

Hence $\gamma_j = \dfrac{e_j - u_j}{1 + v_j} \geqq 1$ with equality exactly when $2e_j + h = l + j$ with $j = 2, 3$. The result follows from Theorem 3.3.

REMARK 3.5. Some other conditions on the Hilbert coefficients which imply that R has uniform termwise exponential growth are found in [Fa] for the cases $h = 2$ (following the ideas of [L_1]) and for $h = 3$.

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