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SOME DENSE BARRELLED SUBSPACES OF ω AND INCOMPLETE φ-SPACES

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1. Introduction.

We are concerned with Hausdorff locally convex spaces of countably infinite dimension whose bounded sets are finite dimensional. Following [3] we shall call such a space a *φ-space*. Let φ and ω be respectively the direct sum and the product of $\aleph_0$ copies of the scalar field K ($=$ R or C). Then any Hausdorff locally convex space of dimension $\aleph_0$ can be regarded as φ with dual space H, a $\sigma(\omega, \varphi)$-dense subspace of ω, and some topology ξ of the dual pair (φ, H). Clearly $\varphi(\xi)$ is a φ-space if and only if $H(\sigma(H, \varphi))$ is barrelled.

In [5] Köthe identified a large class of φ-spaces for which $H \neq \omega$ yet φ under the Mackey topology $\tau(\varphi, H)$ is complete. He also posed the question of the existence of φ-spaces $\varphi(\tau(\varphi, H))$ which are not complete. An example of such a space has been given by Gutnik ([3], [4]) and other examples are implicit in [11], [12], [13, Theorem 2] and [14]. (See Section 5.) Adapting a technique used by Valdivia in [13] we show that by extending the dual space by one dimension each space in a certain subclass of Köthe's class can be modified to produce a φ-space which is not complete in the associated Mackey topology. This subclass includes some spaces which arise naturally in the study of sequence spaces.

2. Preliminaries.

Let $\mathscr{A}$ be a set of subsets of N having the following properties:

(i) $\bigcup \mathscr{A} = \mathsf{N}$;

(ii) $\mathscr{A}$ is closed under the formation of finite unions and subsets;

(iii) $\mathsf{N} \notin \mathscr{A}$;

(iv) each infinite subset of N contains an infinite element of $\mathscr{A}$.

By (i) and (ii) the set $\{x \in \omega\colon \operatorname{supp} x \in \mathscr{A}\}$ is a $\sigma(\omega, \varphi)$-dense subspace of ω; we shall

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denote it by $\omega_{\mathscr{A}}$. Köthe proved that φ is a complete φ-space under $\tau(\varphi, \omega_{\mathscr{A}})$ ([5, (2) and (9)]).

We shall be concerned with those families $\mathscr{A}$ which satisfy the following additional condition:

(v) given a subset B of $\mathbf{N}$ which is not in $\mathscr{A}$ and a sequence (A_m) of elements of $\mathscr{A}$ then there exist finite sets $F_m \subseteq B \backslash A_m$ $(m \in \mathbf{N})$ such that $\bigcup_{r=1}^{\infty} F_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m).

REMARK. For (v) it is enough to consider increasing sequences (A_m), since by (ii) we can replace an arbitrary sequence (A_m) in $\mathscr{A}$ by the sequence $(\bigcup_{r=1}^{m} A_r)$. Also by (ii) we may assume that each set F_m is non-empty.

We will make use of the following simple consequence of (v) below.

LEMMA. *If $\mathscr{A}$ satisfies property* (v) *then there exist pairwise disjoint non-empty finite subsets G_m of $\mathbf{N}$ $(m \in \mathbf{N})$ such that $\bigcup_{r=1}^{\infty} G_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m).*

PROOF. Apply (v) with $B = \mathbf{N}$ and $A_m = \{1, \ldots, m\}$, choosing the F_m to be non-empty. Put $G_1 = F_1$ and choose $m(1)$ such that $F_1 \subseteq A_{m(1)}$. Letting $G_2 = F_{m(1)}$ we have $G_1 \cap G_2 = \emptyset$ and we can find $m(2) > m(1)$ such that $G_1 \cup G_2 \subseteq A_{m(2)}$. Now put $G_3 = F_{m(2)}$ and continue in this way. The G_m are then pairwise disjoint and non-empty and the result follows since any subsequence of (G_m) is a subsequence of (F_m).

To conclude this section we look at three illustrative examples. The first and second are concerned with general classes $\mathscr{A}$ which have found applications in other situations; our property (v) is satisfied in both.

EXAMPLE 1. Let (ρ_n) be an increasing sequence in $\mathbf{N}$ such that $\rho_1 = 1, \rho_n \to \infty$ and $\rho_n/n \to 0$. Let $\mathscr{A}$ be the set of subsets A of $\mathbf{N}$ such that

$$|\{k \in A : k \leqq n\}| \leqq m_A \rho_n \quad (n \in \mathbf{N})$$

for some constant $m_A \in \mathbf{N}$. It is easily seen that $\mathscr{A}$ satisfies properties (i), (ii) and (iii).

Let M be an infinite subset of $\mathbf{N}$ and define $n(m)$ inductively by

$$n(1) = \min M,$$
$$n(m+1) = \min \{n \in M : n > n(m), \rho_n \geqq m + 1\}.$$

Put $A = \{n(m) : m \in \mathbf{N}\}$. Then if $n < n(1)$ we have $|\{k \in A : k \leqq n\}| = 0 < \rho_n$; otherwise we can find m such that $n(m) \leqq n < n(m+1)$ and then

$$|\{k \in A : k \leqq n\}| = |\{k \in A : k \leqq n(m)\}| = m \leqq \rho_{n(m)} \leqq \rho_n.$$

Thus we have found an infinite element of $\mathscr{A}$ which is contained in M, showing that (iv) holds.

Finally to establish (v) let B be a subset of N which is not in $\mathscr{A}$ and let (A_m) be a sequence of elements of $\mathscr{A}$. By (ii) we have that $B\backslash A_m \notin \mathscr{A}\ (m \in \mathrm{N})$. We can therefore choose a strictly increasing sequence $(n(m))$ in N such that

$$|\{k \in B\backslash A_m : k \leqq n(m)\}| > m\rho_{n(m)}.$$

Then if $F_m = \{k \in B\backslash A_m : k \leqq n(m)\}$ we have

$$|\{k \in \bigcup_{r=1}^{\infty} F_{m(r)} : k \leqq n(m(r))\}| > m(r)\rho_{n(m(r))} \quad (r \in \mathrm{N}).$$

Thus $\bigcup_{r=1}^{\infty} F_{m(r]} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m) and (v) is established.

REMARK. The spaces $\omega_{\mathscr{A}}$ constructed in this way are precisely the *scarce copies* of ω. Given a sequence (ρ_n) as above, the corresponding scarce copy of ω is the subspace of ω spanned by those x such that

$$|(\operatorname{supp} x) \cap \{1, \ldots, n\}| \leqq \rho_n \quad (n \in \mathrm{N})$$

and it is easy to show that y belongs to the linear span of such vectors if and only if $\operatorname{supp} y \in \mathscr{A}$.

Various proofs have been given of the fact that each scarce copy of ω is barrelled. (See for example [1, Theorem 7] or [9, Corollary 3.2].) It is perhaps of interest to note that the above representation of scarce copies shows that this is in fact implicit in Köthe's paper.

EXAMPLE 2. Let g be a real-valued function on N such that $0 < g(n) \leqq n$, $g(n) \leqq g(n+1)\ (n \in \mathrm{N})$ and $g(n) \to \infty$. In [12] Valdivia employs such functions to define certain subspaces of countable products; he is concerned with elements of support $A (\subseteq \mathrm{N})$ satisfying

$$\frac{1}{g(n)} |\{k \in A : k \leqq n\}| \to 0.$$

If $\mathscr{A}$ denotes the set of all such subsets of N, it is easy to see that $\mathscr{A}$ has properties (i), (ii) and (iii).

Now let M be an infinite subset of N and for each $m \in \mathrm{N}$ let $n(m)$ be the least n such that $g(n) > (m+1)^2$. Let y_m equal $\max\{k \in M : k \leqq n(m)\}$ if this set is non-empty and $\min M$ otherwise. Then $A = \{y_m : m \in \mathrm{N}\}$ is an infinite subset of M and if $n(m) \leqq n \leqq n(m+1)$ we have

$$\frac{1}{g(n)} |\{k \in A : k \leqq n\}| \leqq \frac{1}{g(n(m))} |\{k \in A : k \leqq n(m+1)\}| < \frac{m+1}{(m+1)^2} = \frac{1}{m+1}.$$

Since $(n(m))$ is increasing and tends to infinity, it follows that $A \in \mathscr{A}$, thus establishing (iv).

Finally for (v) let B be a subset of N which is not in $\mathscr{A}$ and let (A_m) be any sequence in $\mathscr{A}$. We have

$$0 < \alpha = \limsup_{n \to \infty} \frac{1}{g(n)} |\{k \in B : k \leqq n\}| = \limsup_{n \to \infty} \frac{1}{g(n)} |\{k \in B \backslash A_m : k \leqq n\}| \quad (m \in \mathsf{N}).$$

We can therefore find a strictly increasing subsequence $(n(m))$ of (n) such that

$$\frac{1}{g(n(m))} |\{k \in B \backslash A_m : k \leqq n(m)\}| > \frac{\alpha}{2} \quad (m \in \mathsf{N}).$$

Proceeding as in Example 1 we put $F_m = \{k \in B \backslash A_m : k \leqq n(m)\}$ and see that $\bigcup_{r=1}^{\infty} F_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m) since

$$\limsup_{n \to \infty} \frac{1}{g(n)} |\{k \in \bigcup_{r=1}^{\infty} F_{m(r)} : k \leqq n\}| \geqq \frac{\alpha}{2}$$

for any such $(m(r))$.

Remark. The extreme case $g(n) = n \, (n \in \mathsf{N})$ is of special interest. Here the elements of $\mathscr{A}$ are the subsets of N of *density zero*: $|\{k \in A : k \leqq n\}|/n \to 0$; the corresponding $\omega_{\mathscr{A}}$ is Köthe's space H_0 in [5]. Each of the classes $\mathscr{A}$ considered in Examples 1 and 2 consists of sets of density zero. It is well known and easily shown that H_0 is the union of all the scarce copies of $\omega_{\mathscr{A}}$.

Our final example shows that property (v) is independent of the other properties required of $\mathscr{A}$.

Example 3. Let μ be a 0-1 measure defined on $(\mathsf{N}, \mathscr{P}(\mathsf{N}))$, where $\mathscr{P}(\mathsf{N})$ is the power set of N, such that $\mu(\{n\}) = 0$ for each $n \in \mathsf{N}$. Such a measure is finitely but not countably additive; a simple example is obtained by taking $x_0 \in \beta\mathsf{N} \backslash \mathsf{N}$ and putting $\mu(X) = 1$ if $x_0 \in \mathrm{cl}_{\beta\mathsf{N}} X$ and 0 otherwise. (Note that two disjoint subsets of N have disjoint closures in $\beta\mathsf{N}$.) Now let $\mathscr{A} = \{A \subseteq \mathsf{N} : \mu(A) = 0\}$. Properties (i) and (iii) are obvious and property (ii) is a consequence of the finite additivity of μ. If M is an infinite subset of N and $M \notin \mathscr{A}$, partition M into two disjoint infinite subsets M_1, M_2. We then have $1 = \mu(M) = \mu(\mathrm{M}_1) + \mu(M_2)$, from which it follows that one of M_1, M_2 must be in $\mathscr{A}$, thus establishing (iv).

To see that property (v) does not hold in the present situation assume the contrary and apply the above Lemma. We then have

$$1 = \mu\left(\bigcup_{m=1}^{\infty} G_m\right) = \mu\left(\bigcup_{r=1}^{\infty} G_{2r}\right) = \mu\left(\bigcup_{r=1}^{\infty} G_{2r-1}\right),$$

which is impossible by the finite additivity of μ.

REMARK. In Köthe's approach the family $\mathscr{A}$ arises as the set of complements in N of the elements of a certain type of filter $\mathscr{F}$ on N; in particular we may take for $\mathscr{F}$ any ultrafilter on N with empty intersection ([5, pages 380–381 and (10)]). Example 3 is of this type.

3. The results.

Throughout this section $\mathscr{A}$ will be an arbitrary set of subsets of N satisfying properties (i)–(v). Let $x_0 \in \omega \backslash \omega_{\mathscr{A}}$ and let G be the linear span of $\{x_0\} \cup \omega_{\mathscr{A}}$. It follows from the above discussion that G is a dense barrelled subspace of ω and therefore the $\sigma(\varphi, G)$-bounded sets are finite dimensional. However we have:

PROPOSITION 1. *φ is not complete under $\tau(\varphi, G)$.*

PROOF. By construction $\omega_{\mathscr{A}}$ is a dense hyperplane in G. We show that for each $\sigma(G, \varphi)$-compact absolutely convex set C the intersection $C \cap \omega_{\mathscr{A}}$ is $\sigma(G, \varphi)$-closed. The result will then follow from Grothendieck's completeness criterion. (See for example [7, page 107 Corollary 2]).

Suppose there is a non-empty $\sigma(G, \varphi)$-compact absolutely convex set C such that $C \cap \omega_{\mathscr{A}}$ is not $\sigma(G, \varphi)$-closed. Then we can find $x = (\xi_n) \in C \backslash \omega_{\mathscr{A}}$ and $x_m = (\xi_n^{(m)}) \in C \cap \omega_{\mathscr{A}}$ such that $x_m \to x$. Apply property (v) (and its accompanying Remark) with $B = \operatorname{supp} x$ and $A_m = \bigcup_{r=1}^{m} \operatorname{supp} x_r$ to get non-empty finite sets $F_m \subseteq B \backslash A_m$ such that $\bigcup_{r=1}^{\infty} F_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m).

Now choose a subsequence $(m(r))$ of (m) such that $m(1) = 1$ and

$$|\xi_n^{(m(r+1))} - \xi_n| < \frac{1}{2}|\xi_n| \quad (n \in F_{m(r)}, r \in \mathsf{N}).$$

This is possible since $x_m \to x$ componentwise and $F_{m(r)} \subseteq \operatorname{supp} x$. We then have

$$\frac{1}{2}|\xi_n| < |\xi_n^{(m(r+1))}| < \frac{3}{2}|\xi_n| \quad (n \in F_{m(r)}, r \in \mathsf{N}). \tag{1}$$

Since the set $\{\xi_n^{(m(r+1))} : r \in \mathsf{N}\}$ is bounded for each n, we can choose $\alpha_r \in (0, 1]$ such that

$$\alpha_r \sum_{s=r+1}^{\infty} \frac{1}{2^{3(s-r)}} |\xi_n^{(m(s+1))}| < \frac{1}{4}|\xi_n| \quad (n \in F_{m(r)}, r \in \mathsf{N}). \tag{2}$$

Put $\alpha_0 = 1$ and consider

$$z = (\zeta_n) = \sum_{s=1}^{\infty} \left(\prod_{t=0}^{s-1} \alpha_t \right) \frac{1}{2^{3s-2}} x_{m(s+1)}.$$

The coefficients are positive and

$$0 < \sum_{s=1}^{\infty} \left(\prod_{t=0}^{s-1} \alpha_t \right) \frac{1}{2^{3s-2}} \leqq \sum_{s=1}^{\infty} \frac{1}{2^{3s-2}} < 1.$$

Thus since C is absolutely convex and $\sigma(G, \varphi)$-compact, we deduce that $z \in C$. Further if $n \in F_{m(r)}$ we have

$$\begin{aligned} |\zeta_n| &= \left| \sum_{s=1}^{\infty} \left(\prod_{t=0}^{s-1} \alpha_t \right) \frac{1}{2^{3s-2}} \xi_n^{(m(s+1))} \right| \\ &\leqq \sum_{s=r}^{\infty} \left(\prod_{t=0}^{s-1} \alpha_t \right) \frac{1}{2^{3s-2}} |\xi_n^{(m(s+1))}| \text{ (since } F_{m(r)} \cap \operatorname{supp} x_m = \emptyset \text{ if } m \leqq m(r)) \\ &\leqq \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-2}} \left(|\xi_n^{(m(r+1))}| + \alpha_r \sum_{s=r+1}^{\infty} \frac{1}{2^{3(s-r)}} |\xi_n^{(m(s+1))}| \right) \\ &< \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-2}} \left(\frac{3}{2} |\xi_n| + \frac{1}{4} |\xi_n| \right) \text{ (by (1) and (2))} \\ &< \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-3}} |\xi_n| \end{aligned}$$

and

$$\begin{aligned} |\zeta_n| &\geqq \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-2}} \left(|\xi_n^{(m(r+1))}| - \alpha_r \sum_{s=r+1}^{\infty} \frac{1}{2^{3(s-r)}} |\xi_n^{(m(s+1))}| \right) \\ &> \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-2}} \left(\frac{1}{2} |\xi_n| - \frac{1}{4} |\xi_n| \right) = \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r}} |\xi_n| > 0. \end{aligned}$$

Thus we have

$$\frac{|\zeta_n|}{|\xi_n|} \in \left(\frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r}}, \frac{\prod_{t=0}^{r-1} \alpha_t}{2^{3r-3}} \right) = I_r \quad (n \in F_{m(r)}, r \in \mathbb{N})$$

and $I_r \cap I_s = \emptyset$ if $r \neq s$. Also $z \notin \omega_{\mathscr{A}}$ since $\operatorname{supp} z \supseteq \bigcup_{r=1}^{\infty} F_{m(r)} \notin \mathscr{A}$.

We can now write $x = \lambda_1 x_0 + u$ and $z = \lambda_2 x_0 + v$ for some non-zero scalars λ_1, λ_2 and $u, v \in \omega_{\mathscr{A}}$. Then $z - \lambda_2 \lambda_1^{-1} x \in \omega_{\mathscr{A}}$. However $|\lambda_2 \lambda_1^{-1}|$ can lie in at most one interval I_r and so $\zeta_n - \lambda_2 \lambda_1^{-1} \xi_n = 0$ for only finitely many $n \in \bigcup_{r=1}^{\infty} F_{m(r)}$. Thus $\operatorname{supp}(z - \lambda_2 \lambda_1^{-1} x) \notin \mathscr{A}$, which is a contradiction.

REMARKS. (i) The above proof was inspired by ideas used by Valdivia in [13], especially Part 4 of the proof of Theorem 2.

(ii) Since the dual E' of an ultrabornological space E is necessarily $\tau(E', E)$-complete ([6, Corollary 6.1.25]), it follows from the Proposition that G is not ultrabornological under $\tau(G, \varphi)(=\sigma(G, \varphi))$. However $\omega_{\mathscr{A}}(\tau(\omega_{\mathscr{A}}, \varphi))$ is ultrabornological. This does not depend on property (v) and may be established by showing as in [5, (9)] that any linear functional on $\omega_{\mathscr{A}}$ which is bounded on the $\sigma(\omega_{\mathscr{A}}, \varphi)$-compact absolutely convex sets is continuous ([6, Proposition 6.1.9 and Lemma 6.1.22]).

The topology $\tau(\varphi, G)$ in Proposition 1 is in fact $\sup(\tau(\varphi, \omega_{\mathscr{A}}), \sigma(\varphi, G))$. To see this note that by [10, Lemma and Corollary 1] φ is not complete under the latter topology, η say, and it is topologically isomorphic to a one-codimensional dense subspace of the complete space $\varphi(\tau(\varphi, \omega_{\mathscr{A}})) \times \mathsf{K}$. Thus φ has codimension 1 in its η-completion and the extended topology $\hat{\eta}$ on the completion is a Mackey topology. Now the $\tau(\varphi, G)$-completion of φ must be contained in the η-completion since $\tau(\varphi, G)$ and η are topologies of the same dual pair with η coarser than $\tau(\varphi, G)$ ([7, page 105 Proposition 3]). The two completions must therefore coincide algebraically and topologically and consequently $\eta = \tau(\varphi, G)$ The following result is now immediate.

COROLLARY. *If C is a $\tau(\varphi, G)$-equicontinuous set then there is a $\sigma(\omega_{\mathscr{A}}, \varphi)$-compact absolutely convex set D and $r > 0$ such that $C \subseteq D + \{\lambda x_0 : |\lambda| \leqq r\}$.*

REMARK. The Corollary may also be extracted from [2, Proposition 3.2].

To complete the description of the sets identified in the Corollary we note the following result, which is not unexpected.

PROPOSITION 2. *Let C be a $\sigma(\omega_{\mathscr{A}}, \varphi)$-compact convex set. Then* $\operatorname{supp} C \in \mathscr{A}$.

PROOF. Suppose $\operatorname{supp} C \notin \mathscr{A}$. Since C is separable we can choose $x_m = (\xi_n^{(m)}) \in C$ such that $\bigcup_{m=1}^{\infty} \operatorname{supp} x_m = \operatorname{supp} C$. Now apply property (v) with $B = \operatorname{supp} C$, $A_1 = \emptyset$ and $A_m = \bigcup_{r=1}^{m-1} \operatorname{supp} x_r$ $(m \geqq 2)$ to get non-empty finite sets $F_m \subseteq B \backslash A_m$ such that $\bigcup_{r=1}^{\infty} F_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m).

It is easily seen that if F is any non-empty finite subset of N and $F \subseteq \bigcup_{r=1}^{s} \operatorname{supp} z_r$ where $z_r \in \omega$ $(r = 1, \ldots, s)$ then there is a convex combination of $z_1, \ldots, z_s$ which contains F in its support. Thus we can determine a strictly increasing sequence $(m(r))$ of positive integers and elements $y_r = (\eta_n^{(r)}) \in C$ such that $m(1) = 1$, $F_{m(r)} \subseteq \operatorname{supp} y_r$ and y_r is a convex combination of $x_{m(r)}, \ldots, x_{m(r+1)-1}$ $(r \in \mathsf{N})$. We note that $\eta_n^{(r)} = 0$ if $n \in F_{m(t)}$ and $r < t$.

Now put

$$M_1 = 0,\ M_r = \max\{|\eta_n^{(r)}| : n \in \bigcup_{s=1}^{r} F_{m(s)}\}\ (r \geqq 2),$$

$$\mu_0 = 1,\ \mu_r = \min(1, \min\{|\eta_n^{(r)}| : n \in F_{m(r)}\})\ (r \geqq 1)$$

$$S = \sum_{r=1}^{\infty} \frac{\prod_{s=0}^{r-1} \mu_s}{2^r \prod_{s=1}^{r} (1 + M_s)}$$

and let

$$y = (\eta_n) = \frac{1}{S} \sum_{r=1}^{\infty} \frac{\prod_{s=0}^{r-1} \mu_s}{2^r \prod_{s=1}^{r} (1 + M_s)} y_r.$$

Then $y \in C$ and if $n \in F_{m(t)}$ we have

$$|\eta_n| = \left| \frac{1}{S} \sum_{r=t}^{\infty} \frac{\prod_{s=0}^{r-1} \mu_s}{2^r \prod_{s=1}^{r} (1 + M_s)} \eta_n^{(r)} \right|.$$

$$\geqq \frac{\prod_{s=0}^{t-1} \mu_s}{2^t S \prod_{s=1}^{t} (1 + M_s)} \left(|\eta_n^{(t)}| - \sum_{r=t+1}^{\infty} \frac{\mu_t}{2^{r-t}(1 + M_r)} |\eta_n^{(r)}| \right)$$

$$> \frac{\prod_{s=0}^{t-1} \mu_s}{2^t S \prod_{s=1}^{t} (1 + M_s)} (|\eta_n^{(t)}| - \mu_t) \geqq 0.$$

Thus $\operatorname{supp} y \supseteq \bigcup_{r=1}^{\infty} F_{m(r)}$, which is not in $\mathscr{A}$. This is a contradiction.

COROLLARY. *A subset X of $\omega_{\mathscr{A}}$ is $\tau(\varphi, \omega_{\mathscr{A}})$-equicontinuous if and only if $\operatorname{supp} X \in \mathscr{A}$ and for each $m \in \mathbf{N}$ the set $\{\xi_m : x = (\xi_n) \in X\}$ is a bounded set of scalars.*

PROOF. An equicontinuous set must have its weakly closed convex envelope weakly compact. The Proposition therefore shows that the conditions are necessary.

If X satisfies the given conditions then its $\sigma(\omega, \varphi)$-closed convex envelope C is $\sigma(\omega, \varphi)$-compact and since $\operatorname{supp} C = \operatorname{supp} X$ we have that $C \subseteq \omega_{\mathscr{A}}$. Thus C and therefore also X are $\tau(\varphi, \omega_{\mathscr{A}})$-equicontinuous.

4. Incomplete quotients.

In [5, (15) and top of page 384] Köthe discusses the existence of incomplete quotients and shows in particular that $\varphi(\tau(\varphi, H_0))$ has such a quotient. Köthe's proof applies to all the spaces $\varphi(\tau(\varphi, \omega_{\mathscr{A}}))$ where $\mathscr{A}$ satisfies property (v) or, more generally, the assertion of the Lemma (for example, if $\mathscr{A}$ has no element of density 1 ([5, top of page 384])). Let (G_m) be a sequence of non-empty pairwise disjoint finite subsets of $\mathbf{N}$ such that $\bigcup_{r=1}^{\infty} G_{m(r)} \notin \mathscr{A}$ for every subsequence $(m(r))$ of (m) and put $a_m = (\alpha_n^{(m)})$ where $\alpha_n^{(m)}$ is 1 if $n \in G_m$ and 0 otherwise. Let G be the linear span of the a_m. It is easily seen that the $\sigma(\omega, \varphi)$-closure of G consists of all elements of the form $\sum_{m=1}^{\infty} \lambda_m a_m$, that is, those elements (ξ_n) such that $\xi_n = \lambda_m$ if $n \in G_m$ $(m \in \mathbf{N})$ and $\xi_n = 0$ if $n \notin \bigcup_{m=1}^{\infty} G_m$, where (λ_m) is an arbitrary scalar sequence. The support of such an element is $\bigcup \{G_m : \lambda_m \neq 0\}$ and so the element is in $\omega_{\mathscr{A}}$ if and only if $\lambda_m = 0$ for all but finitely many m. It now follows that G is $\sigma(\omega_{\mathscr{A}}, \varphi)$-closed and that any $\sigma(\omega_{\mathscr{A}}, \varphi)$-compact absolutely convex set which is contained in G must be finite dimensional. (If the latter assertion were false we could adapt the method of Proposition 2 to construct an element in the set with non-zero components on infinitely many G_m.) The quotient topology on φ/G° defined by $\tau(\varphi, \omega_{\mathscr{A}})$ is therefore a weak topology and since the dual space is G, which is infinite dimensional, this quotient cannot be complete.

5. Concluding remarks.

The completion of each of the φ-spaces $\varphi(\tau(\varphi, G))$ considered in Section 3 is again a φ-space but φ-spaces whose Mackey completions are not φ-spaces do exist. An example of such a space was announced by Gutnik in [3].

A simpler example may be deduced from [11] or [14], either of which implies that ω has a dense hyperplane H containing no infinite dimensional Banach disk. Since $\tau(\varphi, H)$ and $\sigma(\varphi, H)$ must then coincide, we have that $\varphi(\tau(\varphi, H))$ is a φ-space whose completion, namely the algebraic dual H^* under $\sigma(H^*, H)$, is not a φ-space. A further example is provided by the space $\varphi(\tau(\varphi, \psi))$, where ψ is the dense barrelled subspace of ω constructed in [12]; the bounded subsets of ψ have dimension at most $\aleph_0$ and therefore any Banach disk in ψ must be finite dimensional. Versions of ψ which do not depend upon the Continuum Hypothesis (applied in [12]) have recently been given by Saxon and Sánchez Ruiz ([8]).

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REFERENCES

1. G. Bennett, *Some inclusion theorems for sequence spaces*, Pacific J. Math. 46(1973), 17–30.
2. M. De Wilde and B. Tsirulnikov, *Barrelledness and the supremum of two locally convex topologies*, Math. Ann. 246(1979/80), 241–248.
3. L. A. Gutnik, *On locally convex spaces that do not have infinite dimensional bounded sets* (Russian), Uspekhi Mat. Nauk 32(1977), 251–252.
4. L. A. Gutnik, *On locally convex spaces that do not have infinite dimensional bounded sets* (Russian), Moscow, 1989.
5. G. Köthe, *Vollständige lokalkonvexe Räume abzählbarer Dimension*, Arch. Math. 13(1962), 377–384.
6. P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, Math. Studies 131, North Holland, 1987.
7. A. P. Robertson and W. J. Robertson, *Topological Vector Spaces* (Second Edition), Cambridge, 1973.
8. S. A. Saxon and L. M. Sánchez Ruiz, *Optimal cardinals for metrizable barrelled spaces*, to appear in J. London Math. Soc.
9. J. Swetits, *A characterization of a class of barrelled sequence spaces*, Glasgow Math. J. 19(1978), 27–31.
10. B. Tsirulnikov, *On the locally convex noncomplete quasi-distinguished spaces*, Bull. Soc. Roy. Sci. Liège 47(1978), 147–152.

11. B. Tsirulnikov, *Remarkable hyperplanes in locally convex spaces of dimension at most c*, Canad. Math. Bull. 24(1981), 369–371.
12. I. Tweddle, *Barrelled spaces whose bounded sets have at most countable dimension*, J. London Math. Soc. (2)29(1984), 276–282.
13. M. Valdivia, *A class of bornological barrelled spaces which are not ultrabornological*, Math. Ann. 194(1971), 43–51.
14. M. Valdivia, *Sur certains hyperplans qui ne sont pas ultra-bornologiques dans les espaces ultra-bornologiques*, C. R. Acad. Sci. Paris Sér. A–B 284 (1977), 935–937.

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