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C^*-ALGEBRAS ASSOCIATED WITH CELLULAR AUTOMATA

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Abstract.

We construct C^*-algebras from linear cellular automata by regarding them as topological dynamical systems. We prove that some of the resulting C^*-algebras become Cuntz's algebra $\mathcal{O}_4$. We show that the limit sets of configurations of cellular automaton evolutions, one of whose examples is the Sierpinski gasket, can be obtained by using the canonical endomorphism Φ_4 of $\mathcal{O}_4$. We also study some automorphisms on these C^*-algebras induced by basic operations on cellular automata.

1. Introduction.

In this paper, we introduce a method to investigate cellular automata from functional analytic point of view. We regard cellular automata as topological dynamical systems on a lattice (cellular space) and construct algebras of operators, called C^*-algebras, on a Hilbert space based on the lattice. Let us consider a d-dimensional k-state cellular automaton. Let φ be its cellular automaton rule. The cellular space $\prod_{\mathbf{Z}^d} \mathbf{Z}_k = \mathfrak{K}_k^d$ is a compact space in the product topology and φ is a continuous map on $\mathfrak{K}_k^d$. We identify the cellular automaton with the topological dynamical system $(\mathfrak{K}_k^d, \varphi)$. Take a φ-invariant probability measure μ on $\mathfrak{K}_k^d$ and consider the Hilbert space $L^2(\mathfrak{K}_k^d, \mu)$ of all square integrable functions on $\mathfrak{K}_k^d$. We represent the commutative C^*-algebra $C(\mathfrak{K}_k^d)$ of all complex valued continuous functions on $\mathfrak{K}_k^d$ on $L^2(\mathfrak{K}_k^d, \mu)$ by multiplication. The rule φ induces a bounded linear operator V_φ on $L^2(\mathfrak{K}_k^d, \mu)$. We define the C^*-algebra associated with the cellular automaton φ, as the C^*-algebra generated by $C(\mathfrak{K}_k^d)$ and V_φ. We denote it by C_φ. We notice that the isomorphism class of the C^*-algebra C_φ of course depends on the choice of the φ-invariant measure μ on $\mathfrak{K}_k^d$. But as long as the Radon -Nikodým derivative with another φ-invariant measure is invertible, the resulting C^*-algebras are isomorphic.

We will treat some 1-dimension 2-state 3-neighborhood linear cellular automata, numbered as 60, 90, 150 by S. Wolfram in [Wo1].

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In Section 3, we first show the C^*-algebra C_{90} associated with the rule 90 becomes a simple C^*-algebra called the Cuntz algebra $\mathcal{O}_4$ of order 4. It is one of a series of simple C^*-algebras which many operator algebraists have been interested in, cf. [Ar], [Cu1], [Cu2], [Cuk], [ETW], [Ev], [Jo], [OP]. We show that the C^*-algebra C_{150} constructed by the rule 150 is also isomorphic to $\mathcal{O}_4$ because the map φ_{150} associated with the rule 150 is topologically conjugate to φ_{90} as continuous map.

In Section 4, we show that the Sierpinski gasket as a limit set of a cellular automaton evolution can be seen in the algebraic structure of the C^*-algebra C_{90}. In fact, in C_{90}, the cellular automaton rule corresponds to the canonical endomorphism Φ_4 of $\mathcal{O}_4$, in the following sense: If U_0 is the self-adjoint unitary $S_2S_1^* + S_1S_2^* + S_4S_3^* + S_3S_4^*$, and $l^{90}(k)$ is the number of cells with value 1 after time k if one starts with a state where only one cell has the value 1, using the cellular automaton rule 90, then we construct a faithful state τ^{90} on $\mathcal{O}_4$ and a number c such that

$$\tau^{90}(\Phi_4^k(U_0)) = c^{l^{90}(k)}.$$

A similar result is established for the cellular automaton rule 150. In [Wi1], [Wi2], S. Willson has showed that the Hausdorff dimension of the limit set is equal to its growth rate dimension (cf. [Ta1]). In our language, Willson's result is thus that

$$\lim_{m\to\infty} \log\left[\sum_{k=0}^{m} \log_{\tau^*(U_0)} \tau^*(\Phi_4^k(U_0))\right]\Big/\log m = \begin{cases}\log_2 3 & (* = 90)\\ \log_2(1+\sqrt{5}) & (* = 150)\end{cases}$$

where τ^*, $* = 90, 150$ are the faithful states on $\mathcal{O}_4$. The values $\log_2 3$ and $\log_2(1+\sqrt{5})$ are the fractal dimensions of the limit sets corresponding to the rule 90 and the rule 150 respectively.

In Section 5, we study automorphisms on cellular automaton C^*-algebras induced by two basic operations on cellular space $\prod_{\mathbf{Z}} \mathbf{Z}_k$. These operations are shift ($\{a_n\} \to \{a_{n+1}\}$) and conjugation ($\{a_n\} \to \{a_n + 1\}$). The C^*-algebras C_{90} and C_{150} are both isomorphic to $\mathcal{O}_4$. However, the automorphism induced by the conjugation on C_{90} is inner while the automorphism induced by the conjugation on C_{150} is outer as an automorphism on $\mathcal{O}_4$. We further show that both automorphisms on $\mathcal{O}_4 (\cong C_{90} \cong C_{150})$ induced by the shift are outer.

In Section 6, we generalize our construction for 2-state cellular automata to general k-state cellular automata. As a consequence, we show that a C^*-algebra associated with a k-state cellular automaton corresponding to φ_{90} is isomorphic to the Cuntz algebra $\mathcal{O}_{k^2}$ of order k^2.

In Section 7, we finally study a C^*-algebra associated with a 2-state cellular automaton rule numbered as 60. We show that the C^*-algebra becomes an inductive limit of a sequence of the Cuntz algebra $\mathcal{O}_2$ of order 2. By a recent theorem of Rørdam [Rø], the algebra itself is then isomorphic to $\mathcal{O}_2$.

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2. Preliminaries on cellular automata.

Let $\mathbf{Z}^d$ be a d-dimensional latice $\mathbf{Z} \times \dots \times \mathbf{Z}$ (d-times product of the integers $\mathbf{Z}$). We fix a natural number k. The state of the cell on each lattice point $i \in \mathbf{Z}^d$ is specified by a number $a_i \in \{0, 1, \dots, k-1\} = \mathbf{Z}_k$. A cellular automaton rule is a map to define the state of the next generation of each cell which depends on only a neighborhood of the lattice point $i \in \mathbf{Z}^d$. Namely, the rule is given by a $\mathbf{Z}_k$-valued map ψ defined on $\mathbf{Z}_k^n$ by specifying that the state at site $i \in \mathbf{Z}^d$ for the next generation should be $\psi(a_{i+r_1}, \dots, a_{i+r_n}) \in \mathbf{Z}_k$ where $\{a_i\}_{i \in \mathbf{Z}^d}$ is the state of the previous generation, and $r_1, \dots r_n$ are fixed distinct elements in $\mathbf{Z}^d$. We call such a map ψ a transition function or simply a rule. Such a system is called a d-dimension k-state n-neighborhood cellular automaton.

For instance, the Pascal's triangle of modulo 2 is realized as a 1-dimension 2-state 3-neighborhood cellular automaton as in the following way. Take the both side $i-1$, $i+1$ and itself i for a point $i \in \mathbf{Z}$ as a neighborhood of i. The transition function ψ is defined by $\psi(a_{i-1}, a_i, a_{i+1}) = a_{i-1} + a_{i+1} \pmod 2$. If we take an initial cell configuration $\{a_i\}$ as $a_i = 1$ $(i = 0)$, $a_i = 0$ $(i \neq 0)$, this cellular automaton evolution is related to the Pascal's triangle of modulo 2.

Following Wolfram [Wo1], we number all 1-dimension 2-state 3-neighborhood cellular automata as in the following way. Let ψ be a transition function, which is defined on $\mathbf{Z}_2^3$. Hence there are eight possibilities of the state of the neighborhood so that we have $2^8 = 256$ possible transition functions. The state of the cell of the next generation of a_i is determined by the map ψ, which is written by

$$\psi(0,0,0) = \psi^0, \quad \psi(0,0,1) = \psi^1, \quad \psi(0,1,0) = \psi^2, \quad \psi(0,1,1) = \psi^3,$$
$$\psi(1,0,0) = \psi^4, \quad \psi(1,0,1) = \psi^5, \quad \psi(1,1,0) = \psi^6, \quad \psi(1,1,1) = \psi^7.$$

We define the number of ψ by $\sum_{i=0}^{i=7} \psi^i 2^i$.

For instance, in the case of the preceding cellular automaton of the Pascal's triangle of modulo 2, one has number 90.

A cellular automaton is said to be symmetric if its transition function ψ is symmetric, that is, $\psi^1 = \psi^4, \psi^3 = \psi^6$. It is natural to restrict cellular automata to ones with $\psi^0 = 0$. If a symmetric cellular automaton rule satisfies the condition $\psi^0 = 0$, it is said to be legal. Thus, in 1-dimension 2-state 3-neighborhood cellular automata, these restrictions leave 32 possible legal cellular automata. In general, the limit sets of linear cellular automata are fractals (cf. [Ta2]). The invariance of the Haar measure under a transition function on the cellular space $\mathfrak{R}_2 = \prod_{\mathbf{Z}} \mathbf{Z}_2$ is an important property. In particular, the Haar measure is invariant under the transition functions 60, 90, 150 which will be considered in the sequel.

3. 1-dimension 2-state cellular automaton C^*-algebras.

We first treat a legal cellular automaton in the class of 1-dimension 2-state 3-neighborhood cellular automata. Hence the cellular space $\mathfrak{R}_2^1 = \prod_{\mathbf{Z}} \{0, 1\}$ is the Cantor set $\mathfrak{R}_2$. Let φ_n be a cellular automaton rule indexed as the number n $(0 \leqq n \leqq 256)$. We denote by C_n the associated C^*-algebra C_{φ_n}.

The cellular automaton associated with the Pascal's triangle of mod 2, that is, φ_{90}, can be considered as the continuous map on $\mathfrak{R}_2$ defined by

$$\varphi_{90}(\{a_i\}) = \{a_{i-1} + a_{i+1}\}, \quad \{a_i\} \in \mathfrak{R}_2.$$

Let us study algebraic structure of the C^*-algebra C_{90}. It is easily seen that if $\{b_i\} \in \mathfrak{R}_2$ is given, one may define $\{a_i\}$ with $\varphi_{90}(\{a_i\}) = \{b_i\}$ by choosing a_0, a_1 arbitrary, and then $a_2, a_3, \ldots$, and $a_{-1}, a_{-2}, \ldots$ by induction. Thus, there are 4 distinct cross sections s_{ij}, $i,j = 0, 1$, of φ_{90} (i.e. $\varphi_{90} \circ s_{ij} = id$) satisfying the conditions

$$P_0(s_{ij}(\{a_n\})) = i, \quad P_1(s_{ij}(\{a_n\})) = j, \quad i,j = 0, 1, \quad \{a_n\} \in \mathfrak{R}_2$$

where each P_k is the function on $\mathfrak{R}_2$ defined by $P_k(\{a_n\}) = a_k$. We conclude:

LEMMA 3.1. *The continuous map φ_{90} is surjective and* 4-*to*-1.

Let $\mu_{1/2}$ be the measure on $\{0, 1\}$ defined by $\mu_{1/2}(\{0\}) = \mu_{1/2}(\{1\}) = 1/2$. The infinite product measure $\prod_{\mathbf{Z}} \mu_{1/2}$ on $\mathfrak{R}_2$ is called the Haar measure and is denoted by μ. The lemma below is easily seen by a direct computation or as a special case of [SR; 2.4. Theorem].

LEMMA 3.2. *The measure μ is φ_{90}-invariant.*

Lemma 3.2 also follows from the next easily proved lemma.

LEMMA 3.3. *The Radon-Nikodým derivative* $(d\mu \circ s_{ij})/d\mu$ *is* $1/4$ $i, j = 0, 1$.

We denote by $\mathfrak{H}$ the Hilbert space $L^2(\mathfrak{H}_2, \mu)$ of all complex valued square integrable functions. We define the bounded linear operator V_{90} coming from the map φ_{90} on $\mathfrak{H}$ by

$$(V_{90}\xi)(\{a_n\}) = \xi(\varphi_{90}(\{a_n\})), \quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{K}_2.$$

Then one can show the following lemma by routine computtion.

LEMMA 3.4.

(i) $(V_{90}^*\xi)(\{a_n\}) = \frac{1}{4} \sum_{i,j=0,1} \xi(s_{ij}(\{a_n\})), \quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{H}_2.$

(ii) $V_{90}^* V_{90} = 1$.

We now discuss the range of the projection $V_{90} V_{90}^*$. Let h_e and h_0 be the homeomorphisms on $\mathfrak{R}_2$ exchanging the state of a cell located in even lattice points and in odd ones, respectively. That is,

$$h_e\colon \begin{cases} a_n \to a_n + 1 & (n\text{: even}) \\ a_n \to a_n & (n\text{: odd}) \end{cases}, \qquad h_o\colon \begin{cases} a_n \to a_n & (n\text{: even}) \\ a_n \to a_n + 1 & (n\text{: odd}) \end{cases}.$$

Let W_e, W_o be the unitaries on $\mathfrak{H}$ defined by

$$(W_*\xi)(\{a_n\}) = \xi(h_*(\{a_n\})), \quad * = e, o, \quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{K}_2.$$

The operator W_* is a self-adjoint unitary. Hence, the decomposition

$$W_* = (1 + W_*)/2 - (1 - W_*)/2$$

is the spectral decomposition of W_*. Put $Q_* = (1 + W_*)/2$, $* = e, o$.

LEMMA 3.5. $V_{90} V_{90}^* = Q_e Q_o$.

PROOF. For a vector $\xi \in \mathfrak{H}$ and an element $\{a_n\} \in \mathfrak{H}_2$, we have

$$(V_{90} V_{90}^* \xi)(\{a_n\}) = \frac{1}{4} \sum_{i,j=0,1} \xi(s_{ij} \circ \varphi_{90}(\{a_n\}))$$

and

$$(Q_e Q_o \xi)(\{a_n\}) = \frac{1}{4}(1 + W_e + W_o + W_e W_o)\xi(\{a_n\}))$$

$$= \frac{1}{4} \sum_{i,j=0,1} \xi(h_e^i \circ h_o^j(\{a_n\})).$$

One easily shows that the set of the four elements $\{s_{ij} \circ \varphi_{90}(\{a_n\})\}_{i,j=0,1}$ coincides with that of the four elements $\{h_e^i \circ h_o^j(\{a_n\})\}_{i,j=0,1}$. Hence we get $V_{90} V_{90}^* = Q_e Q_o$.

For each $n \in \mathbf{Z}$, the continuous function $P_n \in C(\mathfrak{H}_2)$ defined by $P_n(\{a_i\}) = a_n$

satisfies the condition $P_n = P_n^2 = P_n^*$. The sequence of these projections $\{P_n\}$ gives all the information of the configuration of the states of cells. Set $U_n = 1 - 2P_n$. As $\{P_n\}$ generate the C^*-algebra $C(\mathfrak{H}_2)$, these unitaries $\{U_n\}$ generate it.

The proof of the next lemma is left to the reader.

LEMMA 3.6.
(i) *For an even integer n,* $P_n W_e = W_e(1 - P_n)$, $P_n W_o = W_o P_n$.
(ii) *For an odd integer n,* $P_n W_o = W_o(1 - P_n)$, $P_n W_e = W_e P_n$.

COROLLARY 3.7.
(i) *For an even integer n,* $U_n Q_e = (1 - Q_e)U_n$, $U_n Q_o = Q_o U_n$.
(ii) *For an odd integer n,* $U_n Q_o = (1 - Q_o)U_n$, $U_n Q_e = Q_e U_n$.

Now put

$$S_1 = V_{90}, \quad S_2 = U_0 V_{90}, \quad S_3 = U_1 V_{90}, \quad S_4 = U_0 U_1 V_{90}. \tag{3.1}$$

PROPOSITION 3.8. *Keep the above notations. We have the following operator relations of a Cuntz algebra* (cf. [Cu1])

$$S_i^* S_i = 1 \quad (i = 1,2,3,4), \qquad \sum_{i=1}^{4} S_i S_i^* = 1. \tag{3.2}$$

PROOF. Following the direct sum decomposition of the Hilbert space $\mathfrak{H}$:

$$Q_e Q_o + (1 - Q_e)Q_o + Q_e(1 - Q_o) + (1 - Q_e)(1 - Q_o) = 1.$$

one obtains the relation $\sum_{i=1}^{4} S_i S_i^* = 1$ by Corollary 3.7.

Let $C^*(S_i, 1 \leqq i \leqq 4)$ be the C^*-algebra generated by S_i, $i = 1,2,3,4$. Since the generators S_i, $i = 1,2,3,4$ satisfy the relation (3.2), we know, by [Cu1], that $C^*(S_i, 1 \leqq i \leqq 4)$ is uniquely determined up to isomorphism and is simple. It is denoted by $\mathcal{O}_4$.

LEMMA 3.9. *Both unitaries* U_0 *and* U_1 *belong to the* C^*-*algebra* $C^*(S_i, 1 \leqq i \leqq 4)$.

PROOF. By $U_0^2 = 1$, it follows that

$$U_0 S_1 = S_2, \quad U_0 S_2 = S_1, \quad U_0 S_3 = S_4, \quad U_0 S_4 = S_3.$$

Hence from the identity $\sum_{i=1}^{4} S_i S_i^* = 1$, we have

$$U_0 = S_2 S_1^* + S_1 S_2^* + S_4 S_3^* + S_3 S_4^*.$$

Similarly one sees

$$U_1 = S_3S_1^* + S_4S_2^* + S_1S_3^* + S_2S_4^*.$$

Therefore one obtains the following:

PROPOSITION 3.10. *The C^*-algebra $C^*(U_0, U_1, V_{90})$ generated by the three operators U_0, U_1, V_{90} coincides with $C^*(S_i, 1 \leqq i \leqq 4)$, that is, the Cuntz algebra $\mathcal{O}_4$.*

The C^*-algebra $C^*(U_0, U_1, V_{90})$ is a subalgebra of C_{90}, but we will see that they actually coincide by the further discussion.

The rule φ_{90} satisfies the condition

$$\varphi_{90}(\{a_n\}) = \{a_{n-1} + a_{n+1}\} \qquad (\text{mod } 2).$$

As $(a_{n-1} - a_{n+1})^2 = a_{n-1} + a_{n+1}$ (mod 2), the next lemma and the corollary are immediate.

LEMMA 3.11. $V_{90}P_n = (P_{n-1} - P_{n+1})^2 V_{90}, \quad n \in \mathbb{Z}.$

COROLLARY 3.12.

$$V_{90}U_n = U_{n-1}U_{n+1}V_{90}, \quad n \in \mathbb{Z}. \tag{3.3}$$

Hence we obtain

LEMMA 3.13. *For every $n \in \mathbb{Z}$, the unitary U_n belongs to the C^*-algebra $C^*(U_0, U_1, V_{90})$.*

PROOF. By induction, it suffices to show that for an arbitrary but fixed integer k, both operators U_{k+2} and U_{k-1} belong to the *C^*-algebra* $C^*(U_k, U_{k+1}, V_{90})$. By using a similar discussion to the previous one, we can show the identity below from Corollary 3.7:

$$1 = V_{90}V_{90}^* + U_kV_{90}V_{90}^*U_k + U_{k+1}V_{90}V_{90}^*U_{k+1} + U_kU_{k+1}V_{90}V_{90}^*U_{k+1}U_k.$$

Hence, one has, by (3.3):

$$\begin{aligned} U_{k+2} = {} & U_kV_{90}U_{k+1}V_{90}^* + V_{90}U_{k+1}V_{90}^*U_k + U_kU_{k+1}V_{90}U_{k+1}V_{90}^*U_{k+1} \\ & + U_{k+1}V_{90}U_{k+1}V_{90}^*U_{k+1}U_k. \end{aligned} \tag{3.4}$$

This implies U_{k+2} belongs to $C^*(U_k, U_{k+1}, V_{90})$. Similarly, we see that U_{k-1} does to it.

Consequently, we arrive at the theorem below.

THEOREM 3.14. *The C^*-algebra C_{90} $(= C^*(C(\mathfrak{K}_2), V_{90}))$ associated to the cellular automaton φ_{90} is isomorphic to the Cuntz algebra $\mathcal{O}_4$ $(= C^*(S_i, 1 \leqq i \leqq 4))$ under the following correspondence:*

$$S_1 = V_{90}, \quad S_2 = U_0 V_{90}, \quad S_3 = U_1 V_{90}, \quad S_4 = U_0 U_1 V_{90}.$$

PROOF. This is immediate from Proposition 3.10 and Lemma 3.13.

REMARK 3.15. Let S_{ij}, $i,j = 0,1$ be the bounded linear operators on $\mathfrak{H}$ defined by

$$(S_{ij}\xi)(\{a_n\}) = \xi(s_{ij}(\{a_n\})), \quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{K}_2$$

where s_{ij}, $i,j = 0,1$ are the four sections for φ_{90} cited before. We then see the following relations:

$$S_{00} + S_{01} + S_{10} + S_{11} = 4S_1^*, \qquad S_{00} + S_{01} - S_{10} - S_{11} = 4S_2^*,$$
$$S_{00} - S_{01} + S_{10} - S_{11} = 4S_3^*, \qquad S_{00} - S_{01} - S_{10} + S_{11} = 4S_4^*,$$

and we have

$$S_{ij}S_{ij}^* = 4, \qquad \sum_{i,j=0,1} S_{ij}^*S_{ij} = 4.$$

Namely, the four operators $\frac{1}{2}S_{ij}^*$, $i,j = 0,1$ generate the C^*-algebra C_{90}, and they satisfy the Cuntz relations for $\mathcal{O}_4$. Hence we have an another proof of the result that C_{90} is isomorphic to $\mathcal{O}_4$.

There is an another interesting legal cellular automaton rule numbered as 150, which is defined by

$$\varphi_{150}(\{a_n\}) = \{a_{n-1} + a_n + a_{n+1}\} \qquad (\mathrm{mod}\, 2).$$

Corresponding to the relation (3.3), we have

$$V_{150}U_n = U_{n-1}U_nU_{n+1}V_{150}, \qquad n \in \mathbb{Z}. \tag{3.5}$$

By a similar discussion to the previous one or the argument below, one has

PROPOSITION 3.16. *The C^*-algebra C_{150} $(= C^*(C(\mathfrak{K}_2), V_{150}))$ associated with the cellular automaton φ_{150} is isomorphic to the Cuntz algebra $\mathcal{O}_4$ $(= C^*(S_i, 1 \leqq i \leqq 4))$ under the following correspondence:*

$$S_1 = V_{150}, \quad S_2 = U_0 V_{150}, \quad S_3 = U_1 V_{150}, \quad S_4 = U_0 U_1 V_{150}.$$

Once one knows Theorem 3.14, one atomatically obtains Proposition 3.16, because it is known that there is a homeomorphism h on the Cantor set $\mathfrak{K}_2$ satisfying $h \circ \varphi_{90} = \varphi_{150} \circ h$. In fact, take a homeomorphism on $\mathfrak{K}_2$ induced by the following automorphism on the algebra $C(\mathfrak{K}_2)$ defined by the correspondence: $i = 0, 1$

$$\begin{array}{ccc} U_i & \to & U_i \\ U_{i-1}U_{i+1} & \to & U_{i-1}U_iU_{i+1} \\ U_{i-2}U_{i+2} & \to & U_{i-2}U_iU_{i+2} \\ U_{i-3}U_{i-1}U_{i+1}U_{i+3} & \to & U_{i-3}U_{i-2}U_iU_{i+2}U_{i+3} \\ \vdots & & \vdots \end{array}$$

Since $C(\mathfrak{K}_2)$ is the universal C^*-algebra generated by countable infinite mutually commuting self-adjoint unitaries, it is easy to see that the above correspondence gives rise to a well-defined automorphism on it. Hence, by taking a unitary operator W on $\mathfrak{H}$ induced by the homeomorphism h one knows that

$$W^*V_{90}W = V_{150}, \qquad W^*C(\mathfrak{K}_2)W = C(\mathfrak{K}_2)$$

so that both C^*-algebras C_{90} and C_{150} are isomorphic each other.

4. The Sierpinski gasket in $\mathcal{O}_4$ and the growth rate dimension.

In this section, we first show that a cellular automaton evolution may be identified with the canonical endomorphism on the Cuntz algebra $\mathcal{O}_4$. Thus we represent the Sierpinski gasket, as a limit set of a cellular automaton evolution by φ_{90}, in the C^*-algebra C_{90} by using an endomorphism on C_{90}. Hence it is possible to describe the growth rate dimension of the evolution by using a certain state on $\mathcal{O}_4$. We construct a faithful state on $\mathcal{O}_4$ which counts the number of cells with value 1 in the evolution at each stage. Then we describe the growth rate dimension of the limit set. A similar discussion works for the cellular automaton C^*-algebra C_{150}.

We first explain notations following [Cu1]. Let S_i, $1 \leqq i \leqq 4$ be the generators of the algebra C_{90} $(=\mathcal{O}_4)$ defined by (3.1), which satisfy the relation (3.2). Let W_4^k, $k = 1, 2, \ldots$ be the set of all k-tuples $(\mu(1), \ldots, \mu(k))$ with $1 \leqq \mu(i) \leqq 4$. We denote by S_μ the isometry $S_\mu = S_{\mu(1)} \cdots S_{\mu(k)}$ for $\mu \in W_4^k$. Let $\mathcal{F}_{4^k}$ be the C^*-algebra generated by $\{S_\mu S_\nu^*; \mu, \nu \in W_4^k\}$ and $\mathcal{F}_4$ be the C^*-algebra generated by $\bigcup_{k=1}^{\infty} \mathcal{F}_{4^k}$. As in [Cu1], $\mathcal{F}_{4^k}$ is isomorphic to the $4^k \times 4^k$ complex full matrix algebra M_{4^k} because $\{S_\mu S_\nu^*;\ \mu, \nu \in W_4^k\}$ become a system of matrix units of M_{4^k}. The identity $S_\mu S_\nu^* = \sum_{i=1}^{4} S_\mu S_i S_i^* S_\nu^*$ defines an inclusion $\mathcal{F}_{4^k} \subset \mathcal{F}_{4^{(k+1)}} = M_4 \otimes \mathcal{F}_{4^k}$ so that $\mathcal{F}_4$ becomes a UHF-algebra of type 4^∞ ([Cu1; 1.4. Proposition]).

Consider the two sequences of unitaries $\{(\hat{\varphi}_{90}^m\{U_i\})\}_{m \in \mathbf{N}}$, $i = 0, 1$ obtained by iterating the morphism $\hat{\varphi}_{90}$ defined by

$$\hat{\varphi}_{90}(U_{n_1}U_{n_2} \cdots U_{n_k}) = U_{n_1-1}U_{n_1+1} \cdot U_{n_2-1}U_{n_2+1} \cdots U_{n_k-1}U_{n_k+1}$$

We will show in a moment that $\hat{\varphi}_{90}$ is well-defined as a morphism, and extends to the one-sided shift on $\mathscr{F}_4 = \overset{\infty}{\underset{1}{\otimes}} M_4$. We have

$$\begin{aligned} \hat{\varphi}_{90}^0(U_0) &= U_0, & \hat{\varphi}_{90}^0(U_1) &= U_1, \\ \hat{\varphi}_{90}^1(U_0) &= U_{-1}U_1, & \hat{\varphi}_{90}^1(U_1) &= U_0U_2, \\ \hat{\varphi}_{90}^2(U_0) &= U_{-2}U_2, & \hat{\varphi}_{90}^2(U_1) &= U_{-1}U_3, \\ \hat{\varphi}_{90}^3(U_0) &= U_{-3}U_{-1}U_1U_3, & \hat{\varphi}_{90}^3(U_1) &= U_{-2}U_0U_2U_4, \\ \hat{\varphi}_{90}^4(U_0) &= U_{-4}U_4, & \hat{\varphi}_{90}^4(U_1) &= U_{-3}U_5. \\ &\vdots & &\vdots \end{aligned}$$

Namely, each of two sequences $\{\hat{\varphi}_{90}^m(U_i)\}_{m\in\mathbf{N}}$, $i = 0, 1$ shows the cellular automaton evolution stating from a state containing a single cell with value 1.

LEMMA 4.1. *Each unitary U_n, $n \in \mathbf{Z}$ belongs to the algebra $\bigcup_{k=1}^{\infty} \mathscr{F}_{4^k}$. Hence the two sequences $\{\hat{\varphi}_{90}^m(U_i)\}_{m\in\mathbf{N}}$, $i = 0, 1$ belong to the* UHF*-algebra $\mathscr{F}_4$.*

PROOF. We already know that both unitaries U_0, U_1 belong to $\bigcup_{k=1}^{\infty} \mathscr{F}_{4^k}$ as in the proof of Lemma 3.9. Under the assumption that two unitaries U_k, U_{k+1} belong to $\bigcup_{k-1}^{\infty} \mathscr{F}_{4^k}$, we see that U_{k+2}, U_{k-1} also belong to it by the identity (3.4).

We now show that $\hat{\varphi}_{90}$ extends to a morphism of $\mathscr{F}_4$.

PROPOSITION 4.2. *For $i = 0, 1$, $\hat{\varphi}_{90}^m(U_i) = 1 \otimes \cdots \otimes 1 \otimes U_i \in M_4 \otimes \cdots \otimes M_4 \otimes M_4$: $(m + 1)$-times tensor product of M_4.*

To prove Proposition 4.2, we provide notations and a lemma.

Let W_S be a unitary operator on $\mathfrak{H}$ induced by the forward shift (S: $\{a_n\} \to \{a_{n+1}\}$) on $\mathfrak{R}_2$. Put $\sigma^S = \mathrm{Ad}\, W_S$. Since the shift commutes with the rule φ_{90}, we have

$$\sigma^S(V_{90}) = V_{90}, \quad \sigma^S(U_n) = U_{n+1}, \quad n \in \mathbf{Z}.$$

Thus σ^S gives rise to an automorphism on the C^*-algebra C_{90}. Since one has, by the relation (3.4),

$$\begin{aligned} (4.1) \qquad U_2 &= U_0V_{90}U_1V_{90}^* + V_{90}U_1V_{90}^*U_0 + U_0U_1V_{90}U_1V_{90}^*U_1 \\ &\quad + U_1V_{90}U_1V_{90}^*U_1U_0 \\ &= S_2U_1S_1^* + S_1U_1S_2^* + S_4U_1S_3^* + S_3U_1S_4^*, \end{aligned}$$

one obtains, by (3.1),

$$\sigma^S(S_1) = S_1, \quad \sigma^S(S_2) = S_3, \quad \sigma^S(S_3) = S_2U_1, \quad \sigma^S(S_4) = S_4U_1$$

(cf. [MT]).

LEMMA 4.3. $U_{n-1}U_{n+1} = \sum_{i=1}^{4} S_iU_nS_i^*, \quad n \in \mathbb{Z}.$

PROOF. By (4.1), one sees,

$$\begin{aligned} &U_0U_2 \\ &= (S_2S_1^* + S_1S_2^* + S_4S_3^* + S_3S_4^*)(S_2U_1S_1^* + S_1U_1S_2^* + S_4U_1S_3^* + S_3U_1S_4^*) \\ &= \sum_{i=1}^{4} S_iU_1S_i^*. \end{aligned}$$

Since $\sigma^S(\mathrm{U}_n) = \mathrm{U}_{n+1}$, one can show the identity $U_{n-1}U_{n+1} = \sum_{i=1}^{4} S_iU_nS_i^*$ for all $n \in \mathbb{Z}$ by applying the map σ^S $(n-1)$-times to the identity $U_0U_2 = \sum_{i=1}^{4} S_iU_1S_i^*$.

It is clear that Lemma 4.3 implies Proposition 4.2, and, moreover:

COROLLARY 4.4. *$\hat{\varphi}_{90}$ is realized as the canonical endomorphism Φ_4 on $\mathcal{O}_4$ defined by $\Phi_4(X) = \sum_{i=1}^{4} S_iXS_i^*$, $X \in \mathcal{O}_4$ ([Cu1]), that is*

$$\hat{\varphi}_{90}(U_n) = U_{n-1}U_{n+1} = \Phi_4(U_n), \qquad n \in \mathbb{Z}.$$

Note that the commutative C^*-algebra $C(\mathfrak{K}_2)$ coincides with the C^*-algebra $C^*(U_i, 1 \otimes U_i, 1 \otimes 1 \otimes U_i, \ldots, i = 0, 1)$ generated by the following two sequences of unitaries in $\mathcal{F}_4$

$$\begin{array}{lllll} U_0, & 1 \otimes U_0, & 1 \otimes 1 \otimes U_0, & 1 \otimes 1 \otimes 1 \otimes U_0, & 1 \otimes 1 \otimes 1 \otimes 1 \otimes U_0, \ldots \\ U_1, & 1 \otimes U_1, & 1 \otimes 1 \otimes U_1, & 1 \otimes 1 \otimes 1 \otimes U_1, & 1 \otimes 1 \otimes 1 \otimes 1 \otimes U_1, \ldots \end{array}$$

Hence $C(\mathfrak{K}_2)$ is a maximal abelian C^*-subalgebra of $\mathcal{F}_4$. This is because the algebra generated by U_0, U_1 is unitarily equivalent to the algebra consisting of all dagonal elements of M_4, see also [Cu1,CuK].

We will next express the fractal (Hausdorff) dimension (see [Fa]) of the Sierpinski gasket in C^*-algebra language by using the above discussion. In [Wi1], [Wi2], Willson has showed that the frctal dimension of the limit set of the cellular automaton evolution starting from a state containing a single cell with value 1 is equal to its growth rate dimension D_g, defined by

$$D_g = \lim_{t \to \infty} \log N(t)/\log t$$

where $N(t)$ is the number of cells with value 1 until time t.

We express the number $N(t)$ in terms of the C^*-algebra.

Let F_0 be the conditional expectation from $\mathcal{O}_4$ to the UHF-subalgebra $\mathcal{F}_4$ defined by

$$F_0(X) = \int_{\mathbf{T}} \rho_z(X)\,dz, \qquad X \in \mathcal{O}_4$$

where ρ is the action of the circle $\mathbf{T}$ defined by $\rho_z\colon S_i \to zS_i$, $z \in \mathbf{C}$, $|z| = 1$. We next construct a conditional expectation from $\mathcal{F}_4$ to $C(\mathfrak{K}_2)$ in regarding $C(\mathfrak{K}_2)$ as a maximal abelian C^*-subalgebra of $\mathcal{F}_4$. One easily sees that the map e below gives rise to an expectation from M_4 to the subalgebra $C^*(U_0, U_1)$ of M_4:

$$e(A) = \tfrac{1}{4}(A + U_0AU_0 + U_1AU_1 + U_0U_1AU_0U_1), \qquad A \in M_4.$$

The map $\varepsilon^{90} = \prod_1^\infty \otimes e$ yields an expectation from $F_4 (= \prod_1^\infty \otimes M_4)$ to $C(\mathfrak{K}_2)$ $(= \prod_1^\infty \otimes C^*(U_0, U_1))$, under the identification between $C(\mathfrak{K}_2)$ and $\prod_1^\infty \otimes C^*(U_0, U_1)$. Let us consider the faithful tracial state ψ_λ for $0 < \lambda < 1$ on $C(\mathfrak{K}_2)$ defined by the integral induced by the measure $\prod_{-\infty}^{\infty} \otimes \mu_\lambda$, where $\mu_\lambda(\{0\}) = \dfrac{1}{1 + \lambda}, \mu_\lambda(\{1\}) = \dfrac{\lambda}{1 + \lambda}$. By composing these maps, one has a faithful state τ_λ^{90} on C_{90} for each $0 < \lambda < 1$, namely,

$$\tau_\lambda^{90} = \psi_\lambda \circ \varepsilon^{90} \circ F_0\colon C^{90} \xrightarrow{F_0} \mathcal{F}_4 \xrightarrow{\varepsilon^{90}} C(\mathfrak{K}_2) \xrightarrow{\psi_\lambda} \mathbf{C}.$$

One now easily proves:

LEMMA 4.5.

$$\psi_\lambda(U_{i_1}U_{i_2}\dots U_{i_k}) = \left(\frac{1 - \lambda}{1 + \lambda}\right)^k \textit{for distinct numbers } i_1, i_2, \dots, i_k.$$

Let $l^{90}(k)$ be the number of cells with value 1 at time k starting from a state containing a single cell with value 1. The sequence $\{l^{90}(k)\}_{k=0}^\infty$ is inductively determined by the following relations:

$$l^{90}(0) = 1, \quad l^{90}(1) = 2, \quad l^{90}(2^n + k) = 2l^{90}(k) \quad 0 \leqq k \leqq 2^n.$$

Put $c_\lambda = \dfrac{1 - \lambda}{1 + \lambda}$. By Corollary 4.4, one sees

$$\tau_\lambda^{90}(\Phi_4^k(U_n)) = c_\lambda^{l^{90}(k)}, \quad n \in \mathbf{Z}, \quad k = 0, 1, 2, \dots$$

Since $N(m) = \sum_{k=0}^{m} l^{90}(k)$ and the fractal dimension of the associated limit set is $\log_2 3$, we reach

$$\lim_{m\to\infty} \log\left[\sum_{k=0}^{m} \log_{\tau_\lambda^{90}(U_0)} \tau_\lambda^{90}(\Phi_4^k(U_0))\right]\Big/\log m = \log_2 3.$$

We may analogously construct a state τ_λ^{150} for the rle 150 which has similar properties. As a result, we have

$$\lim_{m\to\infty} \log\left[\sum_{k=0}^{m} \log_{\tau_\lambda^{150}(U_0)} \tau_\lambda^{150}(\Phi_4^k(U_0))\right]\Big/\log m = \log_2 (1 + \sqrt{5}).$$

5. Automorphisms induced by operations on cellular automata.

In this section, we study automorphisms on cellular automaton C^*-algebras induced by some basic homeomorphisms on cellular space. Our main purpose to study these automorphisms is to make clear the differences between cellular automaton C^*-algebras associated with different cellular automaton rules. For instance, as we have seen in Section 3, the C^*-algebras C_{90}, C_{150} are mutually isomorphic to $\mathcal{O}_4$ as C^*-algebra. Hence there are no difference between them as C^*-algebras. However, the two rules φ_{90}, φ_{150} are different. We look at some operations on cellular space $\mathfrak{K}_2$ and consider automorphisms on C_{90}, C_{150} induced by them. We see that their behavior on C_{90} are different from those on C_{150}. Namely, we show that the difference between φ_{90} and φ_{150} appears as a difference of a property of certain automorphisms on the two C^*-agebras C_{90} and C_{150}. These automorphisms seem to belong a new class of automorphisms on the Cuntz algebra, which has not been treated in [Ar], [ETW], [Vo], . . . , etc.

The following lemma proved in [MT] is used in the sequel.

LEMMA 5.1. ([MT; Corollary B]). *Let α be an automorphism on the Cuntz algebra $\mathcal{O}_n$ with $\alpha(S_1) = S_1$, where S_1 is a generator of isometries satisfying $\sum_{i=1}^{n} S_i S_i^* = 1$. If α is not trivial, it is outer.*

Let γ be a homeomorphism on $\mathfrak{K}_2$ satisfying the condition

$$\gamma \circ \varphi_* = \varphi_* \circ \gamma, \qquad * = 90 \text{ or } 150.$$

Let W_γ be the unitary on $\mathfrak{H} = L^2(\mathfrak{K}_2, \mu)$ induced by γ. Put $\sigma = \mathrm{Ad}\, W_\gamma$. Lemma 5.1 is used in our situation as:

LEMMA 5.2. *If γ is not trivial, σ gives rise to an outer automorphism on C_{90} and C_{150}.*

PROOF. Under the usual correspondence

$$(5.1)\quad S_1 = V_*,\quad S_2 = U_0 V_*,\quad S_3 = U_1 V_*,\quad S_4 = U_0 U_1 V_*,\quad * = 90, 150,$$

the condition $\gamma \circ \varphi_* = \varphi_* \circ \gamma$ implies $\sigma(S_1) = S_1$.

Let us consider some automorphisms of the C^*-algebras C_* induced by basic homeomorphisms on $\mathfrak{K}_2$. We first deal with automorphisms induced by shift on $\mathfrak{K}_2$. Let w_S be the unitary on $\mathfrak{H}$ induced by the forward shift: S on $\mathfrak{K}_2 = \prod_{\mathbf{Z}} \mathbf{Z}_2$. Put $\sigma^S = \mathrm{Ad}\, W_S$. Since one has $S \circ \varphi_* = \varphi_* \circ S$, $* = 90, 150$, σ^S yields an outer automorphism on C_{90} and on C_{150} by Lemma 5.2. We write these outer automorphisms as σ^S_{90} and σ^S_{150} respectively. Under the correspondence (5.1), one can easily write down automorphisms σ^S_* by using the generators S_i, $1 \leqq i \leqq 4$, in the following way (cf. [MT])

$$\begin{cases} \sigma^S_{90}(S_1) = S_1, & \sigma^S_{90}(S_3) = S_2(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*), \\ \sigma^S_{90}(S_2) = S_3, & \sigma^S_{90}(S_4) = S_4(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*), \end{cases}$$

$$\begin{cases} \sigma^S_{150}(S_1) = S_1, & \sigma^S_{150}(S_3) = S_4(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*), \\ \sigma^S_{150}(S_2) = S_3, & \sigma^S_{150}(S_4) = S_2(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*). \end{cases}$$

Since each automorphism σ^S_*, $* = 90, 150$, shifts U_i to U_{i+1}, $i \in \mathbf{Z}$, $(\sigma^S_*)^n$, $n (\neq 0) \in \mathbf{Z}$ is not trivial and satisfies $(\sigma^S_*)^n(S_1) = S_1$, $n \in \mathbf{Z}$. Therefore we have:

PROPOSITION 5.3. *Both the automorphisms σ^S_{90}, σ^S_{150} on $\mathcal{O}_4$ induced by the shift on $\mathfrak{K}_2$ give rise to outer automorphisms. Moreover each of them yields an outer action of the infinite cyclic group $\mathbf{Z}$ on $\mathcal{O}_4$.*

REMARK 5.4. *Let $\rho_{(24)}$, $\rho_{(34)}$ be the automorphisms on $\mathcal{O}_4$ induced by the permutations (24), (34) on the generators S_1, S_2, S_3, S_4 respectively. Then we have the relations:*

$$\sigma^S_{150} = \sigma^S_{90} \circ \rho_{(34)} = \rho_{(24)} \circ \sigma^S_{90}.$$

Now we refer a compatibility the automorphisms σ^S_* with the states τ^*_λ, $* = 90$, 150.

PROPOSITION 5.5. *τ^{90}_λ (resp. τ^{150}_λ) is invariant under σ^S_{90} (resp. σ^S_{150}). However, it is not invariant under σ^S_{150} (resp. σ^S_{90}).*

PROOF. The invariance of τ^{90}_λ (resp. τ^{150}_λ) under σ^S_{90} (resp. σ^S_{150}) is easy from their definition. We show $\tau^{90}_\lambda \circ \sigma^S_{150} \neq \tau^{90}_\lambda$. As we have $\rho_{(34)}(U_1) = U_0U_1$, we get $\tau^{90}_\lambda \circ \rho_{(34)}(U_1) = \left(\frac{1-\lambda}{1+\lambda}\right)^2$ and hence $\tau^{90}_\lambda \circ \rho_{(34)}(U_1) \neq \tau^{90}_\lambda(U_1)$. Since $\sigma^S_{150} = \sigma^S_{90} \circ \rho_{(34)}$, we conclude $\tau^{90}_\lambda \circ \sigma^S_{150}(U_1) \neq \tau^{90}_\lambda(U_1)$ so that $\tau^{90}_\lambda \circ \sigma^S_{150} \neq \tau^{90}_\lambda$. Similarly, we have $\tau^{150}_\lambda \circ \sigma^S_{90} \neq \tau^{150}_\lambda$.

We do not know whether or not the two automorphisms σ_{90}^S, σ_{150}^S are conjugate on $\mathcal{O}_4$. However the following automorphisms on $\mathcal{O}_4$ make a clear distinction between the two rules φ_{90} and φ_{150}. They are induced by a homeomorphism J on $\mathfrak{K}_2$ called the conjugation, defined by

$$J(\{a_n\}) = \{a_n + 1\} \pmod 2, \quad \{a_n\} \in \mathfrak{K}_2.$$

Let W_J be the unitary on $\mathfrak{H}$ induced by J. Hence we have $W_J = W_e W_o$ where W_e, W_o are unitaries defined in Section 3. Put $\sigma^c = \operatorname{Ad} W_J$ so that one has $\sigma^c(U_n) = -U_n$. We first notice:

$$\varphi_{90} \circ J = \varphi_{90}, \qquad \varphi_{150} \circ J = J \circ \varphi_{150}.$$

By lemma 5.2, we have

LEMMA 5.6. *σ^c gives rise to an automorphism of period 2 on C_{150}, which is outer.*

We denote by σ_{150}^c the above automorphism on C_{150}.
On the other hand, we obtain

LEMMA 5.7. *σ^c gives rise to an automorphism of period 2 on C_{90}, which is inner.*

PROOF. It suffices to show that W_J belongs to C_{90}. We notice that $W_J = W_e W_o$, $W_* = 2Q_* - 1$, $* = e, o$. As in Section 3, we know that

$$Q_e Q_o = V_{90} V_{90}^*, \quad Q_e(1 - Q_0) = U_1 V_{90} V_{90} U_1^*, \quad (1 - Q_e)Q_o = U_0 V_{90} V_{90} U_0^*$$

so that Q_* and hence W_*, $* = e, o$ belong to C_{90}.

We denote by σ_{90}^c the above automorphism on C_{90}.
Thus we conclude the following:

THEOREM 5.8. *The two pairs (C_{90}, σ_{90}^c) and $(C_{150}, \sigma_{150}^c)$ of cellular automaton C^*-algebras with automorphisms induced by the conjugation on $\mathfrak{K}_2$ are not conjugate each other. In fact, σ_{90}^c is inner but σ_{150}^c is outer.*

We can explicitly write the implementing unitary W_J of the inner automorphism σ_{90}^c on $\mathcal{O}_4$ as

$$\begin{aligned} W_J &= (2Q_e - 1)(2Q_o - 1) \\ &= 4Q_e Q_o - 2Q_e - 2Q_o + 1 \\ &= 4S_1S_1^* - 2(S_1S_1^* + S_3S_3^*) - 2(S_1S_1^* + S_2S_2^*) + 1 \\ &= S_1S_1^* - S_2S_2^* - S_3S_3^* + S_4S_4^*. \end{aligned}$$

Hence it follows that

$$\sigma_{90}^c(S_1) = S_1 W_J, \quad \sigma_{90}^c(S_2) = -S_2 W_J, \quad \sigma_{90}^c(S_3) = -S_3 W_J, \quad \sigma_{90}^c(S_4) = S_4 W_J.$$

On the other hand, as we have $\sigma_{150}^c(V_{150}) = V_{150}, \sigma_{150}^c(U_n) = -U_n$, it follows that

$$\sigma_{150}^c(S_1) = S_1, \quad \sigma_{150}^c(S_2) = -S_2, \quad \sigma_{150}^c(S_3) = -S_3, \quad \sigma_{150}^c(S_4) = S_4.$$

6. Generalization to k-state cellular automata.

There is no essential obstruction to generalizing our preceding discussions for 2-state to k-state ($k \geqq 3$). We consider the 3-state version of φ_{90}. It is the Pascal's triangle of modulo 3. Let $\mathfrak{K}_3$ be the infinite product $\prod_{\mathbf{Z}} \mathbf{Z}_3$ of $\mathbf{Z}_3 = \{0, 1, 2\}$. Consider the cellular automaton rule

$$\psi(\{a_n\}) = \{a_{n-1} + a_{n+1}\} \pmod 3 \quad \{a_n\} \in \mathfrak{K}_3.$$

Take a probability measure μ on $\mathfrak{K}_3$ which is the infinite product of the measure $\mu_{1/3}$ on $\mathbf{Z}_3$ defined by

$$\mu_{1/3}(\{0\}) = \mu_{1/3}(\{1\}) = \mu_{1/3}(\{2\}) = \tfrac{1}{3}.$$

It is easy to see that ψ is a 9-to-1 onto map on $\mathfrak{K}_3$ and μ is ψ-invariant. Let V_ψ be the linear operator on the Hilbert space $\mathfrak{H}_3 = L^2(\mathfrak{K}_3, \mu)$ induced by the map ψ. We define two unitaries W_e, W_o on $\mathfrak{H}_3$ induced by similar homeomorphisms h_e, h_o on $\mathfrak{K}_3$ to the previous ones respectively. Let ω be the principal 3-rd root of unity. Put

$$Q_*^0 = \tfrac{1}{3}(1 + W_* + W_*^2), \qquad Q_*^1 = \tfrac{1}{3}(1 + \omega^2 W_* + \omega W_*^2),$$
$$Q_*^2 = \tfrac{1}{3}(1 + \omega W_* + \omega^2 W_*^2) \qquad * = e, o.$$

Hence we have $\quad W_* = Q_*^0 + \omega Q_*^1 + \omega^2 Q_*^2 \qquad * = e, o.$

Corresponding to Lemma 3.5, one has

LEMMA 6.1. $V_\psi^* V_\psi = 1$, $V_\psi V_\psi^* = Q_e^0 Q_o^0$.

Let $E^i \in C(\mathbf{Z}_3) = \mathbf{C}\{0\} \oplus \mathbf{C}\{1\} \oplus \mathbf{C}\{2\}$ $i = 0, 1, 2$ be projections defined by

$$E^i(x) = \begin{cases} 1 & (x = i) \\ 0 & (x \neq i) \quad i, x = 0, 1, 2. \end{cases}$$

Three sequences $\{E_n^i\}_{n \in \mathbf{Z}}$, $i = 0, 1, 2$ of projections in $C(\mathfrak{K}_3)$ are defined by

$$E_n^i(\{a_k\}) = E^i(a_n), \quad i = 0, 1, 2, \quad \{a_n\} \in \mathfrak{K}_3.$$

Put unitary $U_n = E_n^0 + \omega E_n^1 + \omega^2 E_n^2$, $n \in \mathbf{Z}$. Similarly to Lemma 3.6 and Corollary 3.7, one has

LEMMA 6.2. *For* $i = 0, 1, 2 \pmod 3$,
(i) *For an even integer n*, $E_n^i W_e = W_e E_n^{i+1}$, $E_n^i W_o = W_o E_n^i$.
(ii) *For an odd integer n*, $E_n^i W_o = W_o E_n^{i+1}$, $E_n^i W_e = W_e E_n^i$.

COROLLARY 6.3. *For* $j = 0, 1, 2 \pmod 3$,
(i) *For an even integer n*, $U_n Q_e^j = Q_e^{j+1} U_n$, $U_n Q_o^j = Q_o^j U_n$.
(ii) *For an odd integer n*, $U_n Q_o^j = Q_o^{j+1} U_n$, $U_n Q_e^j = Q_e^j U_n$.
Put $S_1 = V_\psi$, $S_2 = U_0 V_\psi$, $S_3 = U_1 V_\psi$, $S_4 = U_0^2 V_\psi$, $S_5 = U_1^2 V_\psi$,

$$S_6 = U_0 U_1 V_\psi,\ S_7 = U_0 U_1^2 V_\psi,\ S_8 = U_0^2 U_1 V_\psi,\ S_9 = U_0^2 U_1^2 V_\psi.$$

It is obvious that $S_i^* S_i = 1$, $1 \leqq i \leqq 9$. By the decomposition of the Hilbert space below

$$1 = (Q_e^0 + Q_e^1 + Q_e^2)(Q_o^0 + Q_o^1 + Q_o^2) = \sum_{\substack{i,j=e,o \\ k,l=0,1,2}} Q_i^k Q_j^l,$$

one has $\sum_{i=1}^{9} S_i S_i^* = 1$. As we see the identity $V_\psi U_n = U_{n-1} U_{n+1} V_\psi$, $n \in \mathbf{Z}$, we consequently have the next theorem by a similar argument to the previous one.

THEOREM 6.4. *The C^*-algebra $C^*(C(\mathfrak{K}_3), V_\psi))$ generated by the commutative C^*-algebra $C(\mathfrak{K}_3)$ and the isometry V_ψ coincides with the Cuntz algebra $\mathcal{O}_9$ $(=C^*(S_i, 1 \leqq i \leqq 9))$ generated by 9 isometries.*

More generally, for a k-state cellular automaton Ψ defined by

$$\Psi(\{a_i\}) = \{a_{i-1} + a_{i+1}\}, \qquad \{a_i\} \in \mathfrak{K}_k = \prod_{\mathbf{Z}} \mathbf{Z}_k,$$

we can summarize our discussion as the following theorem:

THEOREM 6.5. *Let $C^*(C(\mathfrak{K}_k), V_\Psi))$ be the C^*-algebra generated by the commutative C^*-algebra $C(\mathfrak{K}_k)$ and the isometry V_Ψ induced by the cellular automaton rule Ψ. Then $C^*(C(\mathfrak{K}_k), V_\Psi))$ is isomorphic to the Cuntz algebra $\mathcal{O}_{k^2}$ $(=C^*(S_i, 1 \leqq i \leqq k^2))$ generated by k^2 mutually orthogonal isometries $U_0^i U_1^j V_\Psi$, $i,j = 0, 1, \ldots, k-1$ satisfying:*

$$\sum_{i,j=0,1,\ldots,k-1} (U_0^i U_1^j V_\Psi)(U_0^i U_1^j V_\Psi)^* = 1$$

where $U_0 = \sum_{i=0}^{k-1} \omega^i E_0^i$, $U_1 = \sum_{i=0}^{k-1} \omega^i E_1^i$ and ω is the principal k-th root of unity and $\{E_n^i\}$ are projections defined in a similar way to the previous ones.

7. C^*-algebras associated with illegal cellular automata.

Finally, we treat an example of a non-symmetric and hence illegal cellular automaton. It is the 1-dimension 2-state 3-neighborhood cellular automaton numbered as 60 which is defined by

$$\varphi_{60}(\{a_n\}) = \{a_{n-1} + a_n\} \quad (\mathrm{mod}\, 2), \quad \{a_n\} \in \mathfrak{K}_2.$$

It is easy to see that the map φ_{60} is surjective and 2-to-1. As the measure μ cited in Section 3 is also φ_{60}-invariant, our previous discussions basically work for φ_{60}. We denote by V_{60} the operator on the Hilbert space $\mathfrak{H} = L^2(\mathfrak{K}_2, \mu)$ induced by φ_{60} as usual. Let $s_i, i = 0, 1$ be the two cross sections for φ_{60} satisfying $P_0(s_i(\{a_n\})) = i$, $i = 0, 1$, $\{a_n\} \in \mathfrak{K}_2$. Since the Radon-Nikodým derivative $(d\mu \circ s_i)/d\mu = 1/2$, $i = 0, 1$, one has

LEMMA 7.1.

(i) $(V_{60}^*\xi)(\{a_n\}) = \frac{1}{2} \sum_{i=0,1} \xi(s_i(\{a_n\}))$, $\quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{K}_2$.

(ii) $V_{60}^* V_{60} = 1$.

Let h be the homeomorphism on $\mathfrak{K}_2$ defined by $h(\{a_n\}) = \{a_n + 1\}$ and W the unitary on $\mathfrak{H}$ induced by h. Put $Q = (W + 1)/2$.

LEMMA 7.2. $V_{60} V_{60}^* = Q$.

Let U_n, $n \in \mathbf{Z}$ be the self-adjoint unitaries defined in Section 3.

LEMMA 7.3. $U_n Q = (1 - Q)U_n$, $\quad n \in \mathbf{Z}$.

LEMMA 7.4.

$$V_{60} U_n = U_{n-1} U_n V_{60}, \quad n \in \mathbf{Z}. \tag{7.1}$$

We fix an arbitrary integer N henceforth. Put

$$S_1^N = V_{60}, \qquad S_2^N = U_N V_{60}.$$

By Lemma 7.3, we have the following relations

$$S_i^{N*} S_i^N = 1 \quad (i = 1, 2), \qquad \sum_{i=1}^{2} S_i^N S_i^{N*} = 1.$$

As we have the identity

$$U_N = S_2^N S_1^{N*} + S_1^N S_2^{N*},$$

we know the following lemmas.

LEMMA 7.5. *Under fixing an integer N, the C^*-algebra $C^*(U_N, V_{60})$ generated by the operators U_N and V_{60} coincides with the C^*-algebra $C^*(S_1^N, S_2^N)$ generated by S_1^N, S_2^N, which is the Cuntz algebra $\mathcal{O}_2$ of order 2.*

LEMMA 7.6. *The C^*-algebra $C^*(U_k; k \leqq N, V_{60})$ generated by the sequence U_k, $k \leqq N$ and V_{60} coincides with $C^*(U_N, V_{60})$ and hence with $C^*(S_1^N, S_2^N)$ ($\cong \mathcal{O}_2$).*

PROOF OF LEMMA 7.6. It suffices to show that the unitary U_{N-1} belongs to $C^*(U_N, V_{60})$ by induction. As we have

$$1 = V_{60}V_{60}^* + U_N V_{60} V_{60}^* U_N,$$

it follows that, by (7.1),

$$U_{N-1} = U_N V_{60} U_N V_{60}^* + V_{60} U_N V_{60}^* U_N.$$

Hence one sees that U_{N-1} belongs to $C^*(U_N, V_{60})$.

We denote by C_{60}^N the C^*-algebra $C^*(U_k; k \leqq N, V_{60})$. Thus we have a sequence of natural inclusions of C^*-algebras $\{C_{60}^N\}_{N\in \mathbf{Z}}$.

$$\cdots \subset C_{60}^{N-2} \subset C_{60}^{N-1} \subset C_{60}^{N} \subset C_{60}^{N+1} \subset C_{60}^{N+2} \subset \cdots$$

Each of C^*-algebras $\{C_{60}^k\}_{k\in\mathbf{Z}}$ is isomorphic to $\mathcal{O}_2$. We study the inclusion $C_{60}^N \subset C_{60}^{N+1}$ by a C^*-algebra technique.

Put $\alpha_N = \operatorname{Ad} U_{N+1}$. By the relation (7.1), we have

$$\alpha_N(S_1^N) = S_2^N, \qquad \alpha_N(S_2^N) = S_1^N.$$

Namely, α_N yields the "flip-flop" automorphism on $\mathcal{O}_2$ ($\cong C_{60}^N$) studied by R. Archbold. His result in [Ar] says α_N is outer on $\mathcal{O}_2$. Let $C^*(U_{N+1}, C_{60}^N)$ be the C^*-algebra generated by the unitary U_{N+1} and the algebra C_{60}^N. It is nothing but C_{90}^{N+1}. Obviously, there is a canonical surjective homomorphism π_{N+1} from the crossed product $C_{60}^N \times_{\alpha_N} \mathbf{Z}_2$ ($=\mathcal{O}_2 \times_{\alpha_N} \mathbf{Z}_2$) of C_{60}^N by the action α_N of the group $\mathbf{Z}_2$ ($= \{0,1\}$) to the algebra $C^*(U_{N+1}, C_{60}^N)$. By [Ki], $C_{60}^N \times_{\alpha_N} \mathbf{Z}_2$ is simple so that π_{N+1} is injective. Thus we have

LEMMA 7.7. *The C^*-algebra C_{60}^{N+1} is isomorphic to the crossed product $C_{60}^N \times_{\alpha_N} \mathbf{Z}_2$ through the map π_{N+1}. The isomorphism is compatible with two natural inclusions i_N: $C_{60}^N \to C_{60}^{N+1}$ and j_N: $C_{60}^N \to C_{60}^N \times_{\alpha_N} \mathbf{Z}_2$. Namely the following sequence of diagrams is commutative.*

$$\begin{array}{ccccc}
\cdots \to & C_{60}^N & \xrightarrow{i_N} & C_{60}^{N+1} & \to \cdots \\
 & \| & & \| & \\
\cdots \to & C_{60}^N & \xrightarrow{j_N} & C_{60}^N \times_{\alpha_N} \mathbf{Z}_2 & \to \cdots
\end{array}$$

Although the following corollary is a special case of the theorem in [CuE], our approach to the result is completely different from Cuntz-Evans's one.

COROLLARY 7.8 ([CuE; Theorem]). *The crossed product $\mathcal{O}_2 \times_\alpha \mathbf{Z}_2$ of $\mathcal{O}_2$ by the flip-flop automorphism is isomorphic to the original Cuntz algebra $\mathcal{O}_2$.*

We identify C_{60}^{N+1} with $C_{60}^N \times_{\alpha_N} \mathbf{Z}_2$ in the previous way. Therefore we conclude

THEOREM 7.9. *The C^*-algebra C_{60} $(=C^*(C(\mathfrak{K}_2), V_{60}))$ generated by the commutative C^*-algebra $C(\mathfrak{K}_2)$ and the isometry V_{60} is isomorphic to the inductive limit C^*-algebra $\varinjlim_{i_N} C_{60}^N$. Hence C_{60} is also simple.*

PROOF. Since $C(\mathfrak{K}_2)$ is an inductive limit C^*-algebra of the sequence of the C^*-algebras $\{C^*(U_k; k \leqq N)\}_{N \in \mathbf{N}}$, C_{60} is also an inductive limit of the sequence of the C^*-algebras $\{C_{60}^N\}_{N \in \mathbf{N}}$. It is well known that an inductive limit of simple C^*-algebras is also simple.

By a recent result of Rørdam, [Rø], it follows that C_{60} is isomorphic to $\mathcal{O}_2$.

REMARK 7.10. The morphism $\hat{\varphi}_{60}$ given by $\hat{\varphi}_{60}(U_n) = U_{n-1}U_n$ is also represented as the canonical endomorphism Φ_2 on $\mathcal{O}_2$ defined by $\Phi_2(X) = \sum_{i=1}^{2} S_i X S_i^*$, because we have

$$\sum_{i=1}^{2} S_i U_n S_i^* = U_{n-1}U_n \quad (=\hat{\varphi}_{60}(U_n)).$$

We easily see that the sequence $\{\Phi_2^N\}$ of the endomorphism at each C^*-algebra C_{60}^N is compatible with the inclusions $i_N\colon C_{60}^N \to C_{60}^{N+1}$ so that $\{\Phi_2^N\}$ define an endomorphism on $\varinjlim_{i_N} C_{60}^N$. Hence we can continue to discuss on the C^*-algebra C_{60} in a similar fashion to the previous ones C_{90} and C_{150} as in Section 4.

REMARK 7.11. It is easy to generalize our discussions to a general k-state cellular automaton rule corresponding to the rule φ_{60}. Consequently, we have an inductive limit C^*-algebra $\varinjlim_{i_N} \mathcal{O}_k$ of the sequence of the Cuntz algebra $\mathcal{O}_k$ of order k under the inclusion $i_N\colon \mathcal{O}_k \to \mathcal{O}_k \times_{\sigma_k} \mathbf{Z}_k \cong \mathcal{O}_k$ where σ_k is the action induced by the cyclic permutation of generators of isometries $S_1, S_2, \ldots, S_k$.

As a generalization of the above fact $\mathcal{O}_k \times_{\sigma_k} \mathbf{Z}_k \cong \mathcal{O}_k$, M. Izumi privately informed the author about the following fact:

For a finite group G of order n, consider the action α of it on $\mathcal{O}_n$ by $\alpha_g(S_h) = S_{gh}$, $g, h \in G$, where $\{S_g\}_{g \in G}$ are generators of isometries of $\mathcal{O}_n$ with $\sum_{g \in G} S_g S_g^* = 1$. Then the crossed product $\mathcal{O}_n \times_\alpha G$ is isomorphic to $\mathcal{O}_n$.

We notice that this fact may be similarly proved if we start with the cellular space $\prod_{\mathbf{Z}} G$ in place of $\prod_{\mathbf{Z}} \mathbf{Z}_k$ and consider the corresponding map φ_{60}^G on $\prod_{\mathbf{Z}} G$, defined by $\varphi_{60}^{G}(\{g_i\}) = \{g_{i-1}g_i\}, \{g_i\} \in \prod_{Z} G$. In fact, the resulting C^*-algebra is an inductive limit C^*-algebra $\varinjlim_{i_N} \mathcal{O}_n$ of $\mathcal{O}_n$ under the inclusion i_N: $\mathcal{O}_n \to \mathcal{O}_n \times_\alpha G \cong \mathcal{O}_n$.

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