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SATURATED ACTIONS OF FINITE DIMENSIONAL HOPF $*$-ALGEBRAS ON C^*-ALGEBRAS

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1. Introduction.

A result announced by Ocneanu (see [15] for a proof due to Szymański) states that if $\mathbf{N} \subset \mathbf{M}$ is an inclusion of type II_1 factors of depth 2 (see [2]) with $\mathbf{N}' \cap \mathbf{M} = \mathbf{C}I$ and finite index $[\mathbf{M}:\mathbf{N}]$ (see [4]), then $\mathbf{M}$ is isomorphic with the crossed product of $\mathbf{N}$ by an (outer) action of a finite dimensional Kac algebra. Motivated by this result we define below actions of finite dimensional Hopf $*$-algebras on unital C^*-algebras. Actions of Hopf algebras appear in [14]. Actions of finite dimensional Hopf $*$-algebras on von Neumann algebras are considered in [15].

We then define and study in this framework several concepts which are known in the case of group actions, such as; spectral subspaces, spaces of spherical functions inside the crossed product, saturated actions (see [3, 5, 6, 10]). We obtain characterizations of simplicity and primeness of crossed products analogues with those in [10] for group actions.

Finally, using the results we refered to, we prove that if a finite dimensional Hopf $*$-algebra $\mathbf{A}$ acts in a saturated fashion on a C^*-algebra $\mathbf{M}$, then $\mathbf{M} \rtimes \mathbf{A}$ is isomorphic with the basic construction $K(\mathbf{M})(=\mathrm{End}(\mathbf{M}))$ for a pair $\mathbf{M}^{\mathbf{A}} \subset \mathbf{M}$, and the index (in the sense of Watatani, see [16]) of the natural conditional expectation E from $\mathbf{M}$ onto $\mathbf{M}^{\mathbf{A}}$ equals $(\dim \mathbf{A})I$. The latter generalizes a result by Jones for II_1 factors (see [2, 4]).

2. Finite dimensional Hopf $*$-algebras and their actions.

2.1. *Finite dimensional Hopf $*$-algebras.*

For reader's convenience we collect in this section some basic facts about finite dimensional Hopf $*$-algebras. By a finite dimensional Hopf $*$-algebra we understand a compact matrix pseudogroup ([17, Definition 1.1]) corresponding to

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a finite dimensional C^*-algebra, i.e. a finite matrix pseudogroup in Woronowicz's terminology. Throughout this paper **A** denotes a finite dimensional Hopf $*$-algebra. In the following we collect some basic properties of **A**, which are either included in [17, Definition 1.1] or are its immediate consequences [17, Proposition 1.8, Proposition A 2.2].

PROPOSITION 2.1. *Let* **A** *be a finite dimensional Hopf $*$-algebra. Then the following hold.*

1. **A** *is a finite dimensional C^*-algebra.*
2. *There exist linear maps;*
   (a) *comultiplication $\Delta : \mathbf{A} \otimes \mathbf{A}$,*
   (b) *counit $\varepsilon : \mathbf{A} \to \mathbf{C}$ ($\mathbf{C}$ denotes the complex numbers),*
   (c) *antipode $S : \mathbf{A} \to \mathbf{A}$.*
   Comultiplication and counit are C^*-algebra homomorphisms. Antipode is a $*$-preserving antimultiplicative involution. We have $\Delta(I) = I \otimes I$, $\varepsilon(I) = 1$, and $S(I)$, and $S(I) = I$, where I is the identity of **A**.
3. *The following identities hold;*
   (a) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ *is coassociative,*
   (b) $(\varepsilon \otimes \mathrm{id})\,\Delta = (\mathrm{id} \otimes \varepsilon)\,\Delta$,
   (c) $m(S \otimes \mathrm{id})(\Delta(a)) = \varepsilon(a)I = m(\mathrm{id} \otimes S)(\Delta(a))$ *for any $a \in \mathbf{A}$, where $m : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ denotes the multiplication.*

If $a \in \mathbf{A}$, then we denote $\Delta(a) = \sum_i a_i^L \otimes a_i^R$, $\Delta(a_i^L) = \sum_j a_{ij}^{LL} \otimes a_{ij}^{LR}$, and $\Delta(a_i^R) = \sum_j a_{ij}^{RL} \otimes a_{ij}^{RR}$. With this notation we have; $\sum_{i,j} a_{ij}^{LL} \otimes a_{ij}^{LR} \otimes a_i^R = \sum_{i,j} a_i^L \otimes a_{ij}^{RL} \otimes a_{ij}^{RR}$, $\sum_i \varepsilon(a_i^L)a_i^R = a = \sum_i \varepsilon(a_i^R)a_i^L$, and $\sum_i a_i^L\, S(a_i^R) = \varepsilon(a)I = \sum_i S(a_i^L)a_i^R$, for any $a \in \mathbf{A}$.

Consider a map $\sigma : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$, defined as $\sigma : a \otimes b \mapsto b \otimes a$. Since $S^2 = \mathrm{id}$, $\varepsilon S = \varepsilon$ ([17, Theorem 5.6]), and $\Delta S = (S \otimes S)\sigma\Delta$ ([17, Proposition 1.9]), we get $\varepsilon = \varepsilon S = m(\mathrm{id} \otimes S)\Delta S = m(S \otimes \mathrm{id})\sigma\Delta$, and similarly $\varepsilon = m(\mathrm{id} \otimes S)\sigma\Delta$. This means $\sum_i a_i^R S(a_i^L) = \varepsilon(a)I = \sum_i S(a_i^R)a_i^L$ for any $a \in \mathbf{A}$.

The following theorem has been obtained by Woronowicz (see [17]) in much greater generality for compact matrix pseudogroups. For **A** finite dimensional the result takes a simpler form. In particular, the Haar state is a faithful trace, which is not always the case for arbitrary compact matrix pseudogroups.

THEOREM 2.2 (S. L. Woronowicz). *If* **A** *is a finite dimensional Hopf $*$-algebra, then the following hold true.*

1. *There is a unique faithful normalized trace τ on* **A**, *called the Haar trace, such that $(\tau \otimes \mathrm{id})(\Delta(a)) = \tau(a)I = (\mathrm{id} \otimes \tau)(\Delta(a))$ for any $a \in \mathbf{A}$. If p is a minimal central projection in* **A** *such that $p\mathbf{A} \cong M_{d_p}(\mathbf{C})$, then $\tau(f) = d_p(\dim \mathbf{A})^{-1}$ for any f–a minimal projection in $p\mathbf{A}$.*

2. *There exists a basis* $\{\Phi^k_{ij} \mid k = 1, \ldots, N, i,j = 1, \ldots, d_k\}$ *of* **A** *with the following properties;* $\Delta(\Phi^k_{ij}) = \sum_s \Phi^k_{is} \otimes \Phi^k_{sj}$, $\varepsilon(\Phi^k_{ij}) = \delta_{ij}$, $S(\Phi^k_{ij}) = (\Phi^k_{ji})^*$, *and each of the matrices* $[\Phi^k]$, *whose i-j entry equals* Φ^k_{ij}, *is a unitary element of* $M_{d_k}(\mathbf{A})$.

3. *For a basis* $\{\Phi^k_{ij}\}$ *as above, we have*

$$\tau(\Phi^k_{ij}(\Phi^l_{mn})^*) = \frac{1}{d_k}\delta_{kl}\delta_{im}\delta_{jn}.$$

4. *There exists a minimal and central projection e in* **A**, *called the distinguished projection, such that* $ae = \varepsilon(a)e$ *for any* $a \in \mathbf{A}$. *We have* $\varepsilon(e) = 1$, $S(e) = e$, *and* $\tau(e) = (\dim \mathbf{A})^{-1}$. *Moreover,*

$$e = \frac{1}{\dim \mathbf{A}} \sum_{i,k} d_k \Phi^k_{ii}.$$

PROOF. 1. See [17, Theorem 4.2 and Appendix 2].

2. See [17, Theorem 5.7].

3. We have $\tau(\Phi^k_{ij}(\Phi^l_{mn})^*) = M_k^{-1}\delta_{kl}\delta_{im}f_1(\Phi^k_{nj})$ (cf. [17, Formulae 5.9 and 5.25]), where $M_k = f_1(\sum_i \Phi^k_{ii})$ ([17, Theorem 5.6]). f_1 is a linear functional introduced in [17, Theorem 5.6]. However, as remarked by Woronowicz ([17, Appendix 2]), we have $f_1 = \varepsilon$ for **A** finite dimensional. The claim follows.

4. It follows from 3. above that for any l, m, n we have

$$\begin{aligned}\tau\left(\left(\frac{1}{\dim \mathbf{A}}\sum_{i,k} d_k \Phi^k_{ii}\right)(\Phi^l_{mn})^*\right) &= \frac{1}{\dim \mathbf{A}}\delta_{mn} \\ &= \tau(e(\Phi^l_{mn})^*).\end{aligned}$$

Since $\{(\Phi^l_{mn})^*\}$ form a basis of **A**, we have

$$\tau\left(\left(\frac{1}{\dim \mathbf{A}}\sum_{i,k} d_k \Phi^k_{ii}\right)a\right) = \tau(ea),$$

for any $a \in \mathbf{A}$. Faithfulness of τ implies the formula for e.

A simplest example of a Hopf *-algebra is the complex group algebra $\mathbf{C}[G]$ of a finite group G, equipped with $\Delta : g \mapsto g \otimes g$, $\varepsilon : g \mapsto 1$, and $S : g \mapsto g^{-1}$, for $g \in G$. The Haar trace τ is given by $\tau(g) = \delta_{\iota g}$, where ι denotes the neutral element of G. This is in fact a special case of Theorem 2.2.(3). The distinguished projection e equals $|G|^{-1}\sum_{g \in G} g$. This conforms to Theorem 2.2.(4).

PROPOSITION 2.3. $\mathbf{A}^\circ$, *the space of linear functionals on* **A**, *is a Hopf *-algebra with the following structure;*

1. (*multiplication*) $(\phi \otimes \psi)(a) = (\phi \otimes \psi)(\Delta(a))$,
2. (*counit* ε°) $\varepsilon^\circ(\phi) = \phi(I)$,
3. (*comultiplication* Δ°) $\Delta^\circ(\phi)(a \otimes b) = \phi(ab)$,

4. (*antipode* S°) $S^\circ(\phi)(a) = \phi(S(a))$,
5. (*adjoint*) $\phi^*(a) = \overline{\phi(S(a^*))}$.

Here $a, b \in \mathbf{A}$, $\phi, \psi \in \mathbf{A}^\circ$, *and we identify* $\mathbf{A}^\circ \otimes \mathbf{A}^\circ \cong (\mathbf{A} \otimes \mathbf{A})^\circ$. *Moreover,* ε *is the identity and* τ *is the distinguished projection* (i.e. $\phi\tau = \varepsilon^\circ(\phi)$) *in* $\mathbf{A}^\circ$. *The Haar trace* τ° *on* $\mathbf{A}^\circ$ *is given by* $\tau^\circ(\phi) = \phi(e)$.

PROOF. It is clear that Formulae 1 and 5 above introduce a $*$-algebra structure on $\mathbf{A}^\circ$ (see also [17, Formula (1.50)]).

If $\{\Phi_{ij}^k\}$ is a basis of $\mathbf{A}$ as in Theorem 2.2.(2), then the corresponding dual basis $\{\omega_{ij}^k\}$ of $\mathbf{A}^\circ$ (via the duality $\langle a, \phi\rangle = \phi(a)$, $a \in \mathbf{A}, \phi \in \mathbf{A}^\circ$) is easily seen to form a self-adjoint system of matrix units. Thus, $\mathbf{A}^\circ$ is isomorphic to a multimatrix algebra, i.e. a finite dimensional C^*-algebra.

For each minimal central projection p_r in $\mathbf{A}$ let $\{v_{mn}^r\}$ be a self-adjoint system of matrix units in $p_r\mathbf{A} \cong M_{d_r}(\mathbf{C})$. Let $\{\phi_{mn}^r\}$ be the corresponding dual basis of $\mathbf{A}^\circ$ and let $[\phi^r]$ be an element of $M_{d_r}(\mathbf{A}^\circ)$ having ϕ_{mn}^r as its m-n entry. It is clear that the matrix having the matrices $[\phi^r]$ along its diagonal (and zeros elsewhere) satisfies requirements (1) and (2) of [17, Definition 1.1].

It is also clear that the map introduced by Formula 4 above satisfies requirement (3) of [17, Definition 1.1]. The remaining claims of the proposition are easily verified.

One can easily check that $\mathbf{A}$ is canonically isomorphic with $\mathbf{A}^{\circ\circ}$.

If G is a finite group, then it is well known that $\mathbf{C}[G]^\circ$ is an abelian C^*-algebra with minimal projections $\{p_g | g \in G\}$, comultiplication $\Delta(p_g) = \sum_{h \in G} p_h \otimes p_{h^{-1}g}$, counit $\varepsilon(p_g) = \delta_{\iota g}$, and antipode $S(p_g) = p_{g^{-1}}$. We denote $\mathbf{C}[G]^\circ$ by $\mathbf{C}(G)$.

2.2. *Actions and crossed products.*

We denote by $\mathbf{M}$ a C^*-algebra with identity I.

The notions of action and crossed product (in the context of Hopf algebras) were introduced by Sweedler (see [14]). However, he did not consider the $*$-operation.

DEFINITION 2.4. A bilinear map $\cdot : \mathbf{A} \times \mathbf{M} \to \mathbf{M}$ is a (*left*) *action* iff the following hold for any $a, b \in \mathbf{A}$, $x, y \in \mathbf{M}$.

$$I \cdot x = x,$$

$$a \cdot I = \varepsilon(a) I,$$

$$ab \cdot x = a \cdot (b \cdot x),$$

$$a \cdot xy = \sum_i (a_i^L \cdot x)(a_i^R \cdot y),$$

$$(a \cdot x)^* = S(a^*) \cdot x^*.$$

EXAMPLE 2.5. *The following are Hopf $*$-algebra actions.*

1. $a \cdot x = \varepsilon(a)x$ *for any* $a \in \mathbf{A}, x \in \mathbf{M}$. This action is called trivial.

2. $a \cdot x = \sum_i \varrho(a_i^L)x\varrho(S(a_i^R))$ for any $a \in \mathbf{A}$, $x \in \mathbf{M}$, where $\varrho : \mathbf{A} \to \mathbf{M}$ is a C^*-algebra homomorphism with $\varrho(I) = I$.

3. The adjoint action of $\mathbf{A}$ on itself, denoted ad, defined as $\mathrm{ad}(a) \cdot b = \sum_i a_i^L bS(a_i^R)$, for any $a, b \in \mathbf{A}$. This is a special case of the previous example.

4. An action of $\mathbf{A}^\circ$ on $\mathbf{A}$, denoted $\rightharpoonup$, defined as $\phi \rightharpoonup a = (\mathrm{id} \otimes \phi)(\Delta(a))$, for any $a \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$.

5. An action of $\mathbf{A}$ on $\mathbf{A}^\circ$, defined as $(a \cdot \phi)(b) = \phi(S(a)b)$, for any $a, b \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$.

PROOF. It is well known that the maps from Example 2.5 are indeed actions. For illustration, we verify 2.

Let $a, b \in \mathbf{A}$, $x, y \in \mathbf{M}$. We have $I \cdot x = \varrho(I)x\varrho(I) = x$, and $a \cdot I = \sum_i \varrho(a_i^L) I\, \varrho(S(a_i^R)) = \varrho(\sum_i a_i^L S(a_i^R)) = \varepsilon(a)I$.

Since Δ is an algebra homomorphism and S is an antihomomorphism, we get

$$\begin{aligned} ab \cdot x &= \sum_i \varrho((ab)_i^L)x\varrho(S((ab)_i^R)) \\ &= \sum_{i,j} \varrho((a_i^L b_j^L)x\varrho(S((a_i^R b_j^R)) \\ &= \sum_{i,j} \varrho(a_i^L)\varrho(b_j^L)x\varrho(S(b_j^R))\varrho(S(a_i^R)) \\ &= a \cdot (b \cdot x). \end{aligned}$$

Applying twice the identity from Proposition 2.1.(3.a), we get

$$\begin{aligned} \sum_i (a_i^L \cdot x)(a_i^R \cdot y) &= \sum_{i,j,k} \varrho(a_{ij}^{LL})x\varrho(S(a_{ij}^{LR})a_{ik}^{RL})y\varrho(S(a_{ik}^{RR})) \\ &= \sum_{i,j,k} \varrho(a_{ijk}^{LLL})x\varrho(S(a_{ijk}^{LLR})a_{ij}^{LR})y\varrho(S(a_i^R)) \\ &= \sum_{i,j,k} \varrho(a_{ij}^{LL})x\varrho(S(a_{ijk}^{LRL})a_{ijk}^{LRR})y\varrho(S(a_i^R)) \\ &= \sum_i \varrho\left(\sum_j \varepsilon(a_{ij}^{LR})a_{ij}^{LL})xy\varrho(S(a_i^R)\right) \\ &= \sum_i \rho(a_i^L)xyS(a_i^R) \\ &= a \cdot xy. \end{aligned}$$

Finally, since $\Delta S = (S \otimes S)\sigma\Delta$ and both S and Δ preserve $*$, we get

$$\begin{aligned}(a\cdot x)^* &= \left(\sum_i \varrho(a_i^L)x\varrho(S(a_i^R))\right)^* \\ &= \sum_i \varrho(S((a_i^R)^*))x^*\varrho((a_i^L)^*) \\ &= S(a^*)\cdot x^*.\end{aligned}$$

If $\mathbf{A}$ acts on $\mathbf{M}$, then we define $\alpha:\mathbf{A}\to \mathrm{End}_{\mathbf{C}}(\mathbf{M})$ as $\alpha(a)(x) = a\cdot x$ for $a\in\mathbf{A}$, $x\in\mathbf{M}$. We call an action *faithful* iff α has trivial kernel.

DEFINITION 2.6. *If* $\mathbf{A}$ *acts on* $\mathbf{M}$, *then we define a new* $*$*-algebra* $\mathbf{M}\rtimes\mathbf{A}$, *called the crossed product. in the following way.*

$\mathbf{M}\rtimes\mathbf{A}$ *is just* $\mathbf{M}\otimes\mathbf{A}$ *(the algebraic tensor product) as a vector space but multiplication and* $*$*-operation are defined as follows.*

$$(x\otimes a)(y\otimes b)\overset{\text{def}}{=}\sum_j x(a_i^L\cdot y)\otimes a_i^R b,$$

$$(x\otimes a)^* \overset{\text{def}}{=}\sum_i (a_i^L)^*\cdot x^*\otimes (a_i^R)^*$$

for $a,b\in\mathbf{A}$, $x,y\in\mathbf{M}$.

It is well known (see [14]) that $\mathbf{M}\rtimes\mathbf{A}$ is an associative algebra with identity $I\otimes I$. One can easily check that the maps $\mathbf{A}\to\mathbf{M}\rtimes\mathbf{A}$, $a\mapsto I\otimes a$, and $\mathbf{M}\to\mathbf{M}\rtimes\mathbf{A}, x\mapsto x\otimes I$, are injective $*$-homomorphisms. Identifying a with $I\otimes a$ and x with $x\otimes I$ we can write xa (instead of $x\otimes a = (x\otimes I)(I\otimes a)$) for elements of the crossed product.

For any $a\in\mathbf{A}$, $x\in\mathbf{M}$ the following useful formula holds;

$$xa = \sum_i a_i^R(S(a_i^L)\cdot x). \tag{1}$$

Indeed,

$$\begin{aligned}\sum_i a_i^R(S(a_i^L)\cdot x) &= \sum_{i,j}(a_{ij}^{RL}\cdot(S(a_i^L)\cdot x))a_{ij}^{RR} \\ &= \sum_{i,j}(a_{ij}^{LR}\cdot(S(a_{ij}^{LL})\cdot x))a_i^R \\ &= \sum_i\left(\left(\sum_j a_{ij}^{LR}S(a_{ij}^{LL})\right)\cdot x\right)a_i^R \\ &= x\left(\sum_i \varepsilon(a_i^L)a_i^R\right) \\ &= xa.\end{aligned}$$

We can now check that $\mathbf{M}\rtimes\mathbf{A}$ is in fact a $*$-algebra. Indeed, for any $a\in\mathbf{A}$, $x\in\mathbf{M}$

$$
\begin{aligned}
(xa)^{**} &= \left(\sum_i ((a_i^L)^* \cdot x^*\right)(a_i^R)^*)^* \\
&= \sum_{i,j} (a_{ij}^{RL} \cdot (S(a_i^L) \cdot x))\, a_{ij}^{RR} \\
&= \sum_i a_i^R (S(a_i^L) \cdot x) \\
&= xa.
\end{aligned}
$$

Thus, $*$ in $\mathbf{M} \bowtie \mathbf{A}$ is an involution.

It follows immediately from our definitions that $(xa)^* = a^* x^*$ for $a \in \mathbf{A}$, $x \in \mathbf{M}$. Hence, for any $a, b \in \mathbf{A}$, $x, y \in \mathbf{M}$

$$
\begin{aligned}
((xa)(yb))^* &= \left(\sum_i x(a_i^L \cdot y)\, a_i^R b\right)^* \\
&= \sum_i b^* (a_i^R)^* (a_i^L \cdot y)^* x^* \\
&= b^* \left(\sum_i (a_i^R)^* (S(a_i^L)^* \cdot y^*)\right) x^* \\
&= b^* y^* a^* x^* \\
&= (yb)^* (xa)^*.
\end{aligned}
$$

Thus, $*$ in $\mathbf{M} \bowtie \mathbf{A}$ is antimultiplicative, and we see that $\mathbf{M} \bowtie \mathbf{A}$ is a unital $*$-algebra.

DEFINITION 2.7. If $\mathbf{A}$ acts on $\mathbf{M}$, then there is a *dual action* of $\mathbf{A}^\circ$ on $\mathbf{M} \bowtie \mathbf{A}$ defined as

$$
\phi \cdot xa \stackrel{\text{def}}{=} x(\phi \rightharpoonup a),
$$

for $a \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$, $x \in \mathbf{M}$. Here $\rightharpoonup$ denotes the action from Example 2.5.(4).

One can easily verify that this is in fact an action.

If G is a finite group acting on $\mathbf{M}$ and $g \in G$ is identified with an element of the crossed product $\mathbf{M} \bowtie G$, then $g \cdot x = gxg^{-1} = gxS(g)$ for any $x \in \mathbf{M}$. This formula readily extends to an arbitrary Hopf $*$-algebra as follows. If $\mathbf{A}$ ats on $\mathbf{M}$, then for any $a \in \mathbf{A}$, $x \in \mathbf{M}$

$$
a \cdot x = \sum_i a_i^L x S(a_i^R). \tag{2}
$$

Here, as usual, we identify $\mathbf{A}$ and $\mathbf{M}$ with appropriate subalgebras of the crossed product. Indeed, we have

$$\begin{aligned}\sum_i a_i^L x S(a_i^R) &= \sum_{i,j} (a_{ij}^{LL} \cdot x) a_{ij}^{LR} S(a_i^R) \\ &= \sum_{i,j} (a_i^L \cdot x) a_{ij}^{RL} S(a_{ij}^{RR}) \\ &= \left(\sum_i \varepsilon(a_i^R) a_i^L\right) \cdot x \\ &= a \cdot x.\end{aligned}$$

We define a linear map $F: \mathbf{M} \rtimes \mathbf{A} \to \mathbf{M}$ as

$$F: xa \mapsto \tau(a)x, \tag{3}$$

for any $a \in \mathbf{A}$, $x \in \mathbf{M}$.

PROPOSITION 2.8. *F is a faithful* (i.e. $F(ff^*) = 0$ *implies* $f = 0$, *for* $f \in \mathbf{M} \rtimes \mathbf{A}$) *conditional expectation from* $\mathbf{M} \rtimes \mathbf{A}$ *onto* $\mathbf{M}$.

PROOF. It is clear that $F(x) = x$ *and* $F(xya) = xF(ya)$ for any $a \in \mathbf{A}$, $x, y \in \mathbf{M}$. Moreover,

$$\begin{aligned}F((xa)^*) &= \sum_i F(((a_i^L)^* \cdot x^*)(a_i^R)^*) \\ &= \left(\sum_i \tau(a_i^R) a_i^L\right)^* . x^* \\ &= \overline{\tau(a)} x^* \\ &= F(xa)^*\end{aligned}$$

for any $a \in \mathbf{A}$, $x \in \mathbf{M}$. Thus, F is a conditional expectation.

Let $\{a_i\}$ be a basis of $\mathbf{A}$ such that $\tau(a_i a_j^*) = \delta_{ij}$. Suppose that we have $F((\sum_i x_i a_i)(\sum_i x_i a_i)^*) = 0$ for some $\{x_i\}$ in $\mathbf{M}$. Hence, $0 = \sum_{i,j} F(x_i a_i a_j^* x_j^*) = \sum_{i,j} \tau(a_i a_j^*) x_i x_i^* = \sum_i x_i x_i^*$, and we have $\sum_i x_i a_i = 0$. Thus, F is faithful.

2.3. *An action of* $\mathbf{A}$ *on* $\mathbf{A}^\circ$.

Before moving forward we want to take a closer look at the action of $\mathbf{A}$ on $\mathbf{A}^\circ$ from Example 2.5.(5). The main purpose of this is to prove Proposition 2.10 below.

We consider a Hilbert space $l^2(\mathbf{A}, \tau)$, whose vectors are elements of $\mathbf{A}$, with inner product $\langle a, b \rangle = \tau(b^* a)$. An $a \in \mathbf{A}$, considered an element of $l^2(\mathbf{A}, \tau)$, will be denoted by the same symbol. $\mathscr{B}(l^2(\mathbf{A}, \tau))$ denotes the C^*-algebra of linear endomorphisms of $l^2(\mathbf{A}, \tau)$. Recall that the Haar trace τ, viewed as an element of $\mathbf{A}^\circ$, is the distinguished projection in $\mathbf{A}^\circ$.

The following proposition is essentially a very special case of duality for Hopf algebra actions.

PROPOSITION 2.9. *If* $\mathbf{A}$ *acts on* $\mathbf{A}^\circ$ *as in Example* 2.5 (5), *then the crossed product* $\mathbf{A}^\circ \bowtie \mathbf{A}$ *is isomorphic to* $\mathscr{B}(l^2(\mathbf{A}, \tau))$. *Moreover, the unique normalized trace* T *on* $\mathbf{A}^\circ \bowtie \mathbf{A}$ *is given by* $T(\phi a) = \tau^\circ(\phi)\tau(a)$, *for* $a \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$.

PROOF. We define a linear map $\theta\colon \mathbf{A}^\circ \bowtie \mathbf{A} \to \mathscr{B}(l^2(\mathbf{A}, \tau))$ by setting $\theta(\phi a)(b) = (\phi \otimes \mathrm{id})(\Delta(ab))$, for $a \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$, $b \in l^2(\mathbf{A}, \tau)$. We claim that θ is a C^*-algebra isomorphism.

At first we observe that the following four identities hold; $\theta(ab) = \theta(a)\theta(b)$, $\theta(\phi\psi) = \theta(\phi)\theta(\psi)$, $\theta(\phi a) = \theta(\phi)\theta(a)$, and $\theta(a\phi) = \theta(a)\theta(\phi)$ for any $a, b \in \mathbf{A}$, $\phi, \psi \in \mathbf{A}^\circ$. Since their proofs are similar, we check only the fourth one. Indeed, for any $a \in \mathbf{A}$, $\phi \in \mathbf{A}^\circ$, $b \in l^2(\mathbf{A}, \tau)$

$$\begin{aligned}
\theta(a\phi)(b) &= \theta\left(\sum_i (a_i^L \cdot \phi)a_i^R\right)(b) \\
&= \sum_i ((a_i^L \cdot \phi) \otimes \mathrm{id})(\Delta(a_i^R b)) \\
&= \sum_{i,j,k} \phi(S(a_i^L)a_{ij}^{RL}, b_k^L)a_{ij}^{RR}b_k^R \\
&= \sum_{i,j,k} \phi(S(a_{ij}^{LL})a_{ij}^{LR}b_k^K)a_i^R b_k^R \\
&= \left(\sum_i \varepsilon(a_i^L)a_i^R\right)\left(\sum_k \phi(b_k^L)b_k^R\right) \\
&= a(\phi \otimes \mathrm{id})(\Delta(b)) \\
&= (\theta(a)\theta(\phi))(b).
\end{aligned}$$

For any $a, b \in \mathbf{A}$, $\phi, \psi \in \mathbf{A}^\circ$ we have

$$\begin{aligned}
\theta((\phi a)(\psi b)) &= \sum_i \theta(\phi(a_i^L \cdot \psi)a_i^R b) \\
&= \theta(\phi)\theta\left(\sum_i (a_i^L \cdot \psi)a_i^R\right)\theta(b) \\
&= \theta(\phi)\theta(a\psi)\theta(b) \\
&= \theta(\phi a)\theta(\psi b).
\end{aligned}$$

Thus, θ is an algebra homomorphism. We leave it to the reader to verify that θ preserves $*$.

Take an arbitrary $a \in \mathbf{A}$ with $\tau(aa^*) = 1$. For any $b \in l^2(\mathbf{A}, \tau)$ we have $\theta(a\tau a^*)(b) = \theta(a)((\tau \otimes \mathrm{id})(\Delta(a^*b))) = \tau(a^*b)\theta(a)(I) = \tau(a^*b)a$. Thus $\theta(a\tau a^*)$ is an orthogonal projection onto the one-dimensional subspace of $l^2(\mathbf{A}, \tau)$ spanned by a. It follows that $\theta(\mathbf{A}^\circ \bowtie \mathbf{A}) = \mathscr{B}(l^2(\mathbf{A}, \tau))$ and θ is a C^*-algebra isomorphism.

For T, defined in the proposition, we have $T(I) = 1$ and it suffices to show that $T(\phi ab) = T(b\phi a)$ and $T(\phi a\psi) = T(\psi\phi a)$ for any $a, b \in \mathbf{A}$, $\phi, \psi \in \mathbf{A}^\circ$. Indeed,

$$\begin{aligned} T(b\phi a) &= \sum_i T((b_i^L \cdot \phi)b_i^R a) \\ &= \sum_i \tau^\circ(b_i^L \cdot \phi)\tau(b_i^R a) \\ &= \sum_i \phi(S(b_i^L)e)\tau(b_i^R a) \\ &= \phi(e)\tau\left(\left(\sum_i \varepsilon(b_i^L)b_i^R\right)a\right) \\ &= \tau^\circ(\phi)\tau(ba) \\ &= T(\phi ab). \end{aligned}$$

The other identity is established in a similar fashion.

For any p, a minimal central projection in $\mathbf{A}$, we denote by $\{v_{ij}^p \mid i,j = 1,\dots,d_p\}$ a system of matrix units in $p\mathbf{A}$. That is $v_{ij}^p v_{mn}^q = \delta_{pq}\delta_{jm}v_{in}^p$, $(v_{ij}^p)^* = v_{ji}^p$, and $\sum_i v_{ii}^p = p$.

Proposition 2.10. *For $\{v_{ij}^p\}$, a basis of $\mathbf{A}$ as above, we have*

$$\Delta(e) = \sum_{i,j,p} \frac{1}{d_p} v_{ij}^p \otimes S(b_{ji}^p).$$

Proof. Since $T(\tau) = (\dim \mathbf{A})^{-1}$, we see that τ is a minimal projection in $\mathbf{A}^\circ \bowtie \mathbf{A}$. Hence, there is a inear functional λ on $\mathbf{A}$ such that $\tau a\tau = \lambda(a)\tau$ for any $a \in \mathbf{A}$. Taking T of both sides we get $\lambda(a) = \tau(a)$.

We define a linear map γ: $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}^\circ \bowtie \mathbf{A}$ as γ: $a \otimes b \mapsto a\tau S(b)$. This map is a vector space isomorphism. To see this one can choose a basis $\{a_i\}$ of $\mathbf{A}$ such that $\tau(a_i a_j^*) = \delta_{ij}$, and check that $T(\gamma(a_i \otimes a_j)^*\gamma(a_m \otimes a_n)) = (\dim \mathbf{A})^{-1}\delta_{im}\delta_{jn}$. This shows that $\{\gamma(a_i \otimes a_j)\}$ are linearly independent.

It follows from Formula (2) that $\gamma(\Delta(a)) = a \cdot \tau$ for any $a \in \mathbf{A}$. Since clearly $e \cdot \tau = (\dim \mathbf{A})^{-1}\varepsilon$ and ε is the identity of $\mathbf{A}^\circ$, we have $\gamma(\Delta(e)) = (\dim \mathbf{A})^{-1}I$.

Since $\tau v_{ij}^p \tau = d_p(\dim \mathbf{A})^{-1}\delta_{ij}\tau$, one can check that $\{(\dim \mathbf{A})d_p^{-1}v_{ij}^p\tau v_{ji}^p\}$ are pairwise orthogonal projections in $\mathbf{A}^\circ \bowtie \mathbf{A}$. Since each of these projections has trace(T) equal to $(\dim \mathbf{A})^{-1}$, they sum up to I. Thus, we have

$$\begin{aligned} \gamma\left(\sum_{i,j,p} \frac{1}{d_p} v_{ij}^p \otimes S(v_{ji}^p)\right) &= \sum_{i,j,p} \frac{1}{d_p} v_{ij}^p \tau v_{ji}^p \\ &= \frac{1}{\dim \mathbf{A}} I. \end{aligned}$$

Since γ is a $1 - 1$ map, our claim follows.

By virtue of Theorem 2.2.(2 and 4) we have

$$\Delta(e) = \frac{1}{\dim \mathbf{A}} \sum_{i,j,k} d_k \Phi^k_{ij} \otimes \Phi^k_{ji}. \tag{4}$$

Thus, $\sigma(\Delta(e)) = \Delta(e)$, where $\sigma\colon \mathbf{A} \otimes \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$, $\sigma\colon a \otimes b \mapsto b \otimes a$. Consequently, the formula from Proposition 2.10 may be rewritten as

$$\Delta(e) = \sum_{i,j,p} \frac{1}{d_p} S(v^p_{ij}) \otimes v^p_{ji}. \tag{5}$$

A finite dimensional Hopf *-algebra $\mathbf{A}$ has two kinds of particularly useful bases. In the first class there are bases composed of systems of matrix units $\{v^p_{ij}\}$. Such bases are related to the structure of irreducible modules of $\mathbf{A}$. In the second class there are bases of the form $\{\Phi^k_{ij}\}$, related to the structure of irreducible comodules of $\mathbf{A}$. Proposition 2.10 and Formulae (4) and (5) show how to express $\Delta(e)$ in bases of both kinds.

2.4. *Fixed point algebra.*
The notion of fixed point algebra for an action of a Hopf algebra appears in [14].

DEFINITION 2.11. *If* $\mathbf{A}$ *acts on* $\mathbf{M}$, *then the set of fixed points, denoted by* $\mathbf{M}^{\mathbf{A}}$, *is defined as*

$$\mathbf{M}^{\mathbf{A}} \stackrel{\text{def}}{=} (x \in \mathbf{M} \mid (\forall a \in \mathbf{A}) a \cdot x = \varepsilon(a)x\}.$$

It is easily seen that the fixed points from a unital *-subalgebra of $\mathbf{M}$.

PROPOSITION 2.12. *If* $\mathbf{A}$ *acts on* $\mathbf{M}$, *then* $E\colon \mathbf{M} \to \mathbf{M}$, *defined as* $E\colon x \mapsto e \cdot x$, *is a faithful conditional expectation from* $\mathbf{M}$ *onto* $\mathbf{M}^{\mathbf{A}}$ *such that*

$$E((a \cdot x)y) = E(x(S(a) \cdot y)) \tag{6}$$

for any $a \in \mathbf{A}$, $x, y \in \mathbf{M}$.

PROOF. A straightforward calculation shows that E is a conditional expectation onto $\mathbf{M}^{\mathbf{A}}$. We verify that E is faithful. Indeed, suppose that $E(x^*x) = 0$ for some $x \in \mathbf{M}$. By virtue of Theorem 2.2.(2 and 4) we have

$$\begin{aligned} 0 &= e\cdot x^*x \\ &= \frac{1}{\dim \mathbf{A}} \sum_{i,k} \Phi_{ii}^k \cdot x^*x \\ &= \frac{1}{\dim \mathbf{A}} \sum_{i,j,k} (\Phi_{ij}^k \cdot x^*)(\Phi_{ji}^k \cdot x) \\ &= \frac{1}{\dim \mathbf{A}} \sum_{i,j,k} (\Phi_{ji}^k \cdot x)^*(\Phi_{ji}^k \cdot x). \end{aligned}$$

Thus, $\Phi_{ji}^k \cdot x = 0$ for any i, j, k. Since $\{\Phi_{ji}^k\}$ form a basis of $\mathbf{A}$, we conclude that $x = I \cdot x = 0$.

In order to establish Formula (6), it sufficies to verify that in $\mathbf{M} \rtimes \mathbf{A}$

$$E((a\cdot x)y)e = E(x(S(a)\cdot y))e.$$

At first we observe that

$$(a\cdot x)e = axe \tag{7}$$

for any $a \in \mathbf{A}$, $x \in \mathbf{M}$. Indeed, $axe = \sum_i (a_i^L \cdot x)a_i^R e = ((\sum_i \varepsilon(a_i^R)a_i^L)\cdot x)e = (a\cdot x)e$. By virtue of Formula (2) we have

$$\begin{aligned} E((a\cdot x)y)e &= (e\cdot((a\cdot x)y))e \\ &= e(a\cdot x)ye \\ &= \sum_i e a_i^L x S(a_i^R) y e \\ &= exS\left(\sum_i \varepsilon(a_i^L)a_i^R\right) ye \\ &= ex(S(a)\cdot y)e \\ &= E(x(S(a)\cdot y))e. \end{aligned}$$

This completes the proof.

EXAMPLE 2.13. *The following hold true:*

1. *The fixed point algebra for the adjoint action from Example* 2.5.(3) *is equal to the center of* $\mathbf{A}$.

2. *The fixed algebra for the dual action of* $\mathbf{A}^\circ$ *on* $\mathbf{M} \rtimes \mathbf{A}$ *equals* $\mathbf{M}$.

PROOF. 1. By virtue of Proposition 2.12 the fixed point algebra is spanned by $\{e\cdot v_{mn}^q\}$, where $\{v_{mn}^q\}$ is a system of matrix units described in the previous section.

By virtue of Proposition 2.10 we have $e \cdot v^q_{mn} = \sum_{i,j,p} d_p^{-1} v^p_{ij} v^q_{mn} v^p_{ji} = \delta_{mn} d_q^{-1} \sum_i v^q_{ii} = \delta_{mn} d_q^{-1} q$. The claim follows.

2. The distinguished projection in $\mathbf{A}^\circ$ is τ. For any $a_i \in \mathbf{A}$, $x_i \in \mathbf{M}$ we have $\tau \cdot (\sum_i x_i a_i) = \sum_i x_i (\tau \rightharpoonup a_i) = \sum_i \tau(a_i) x_i$. The claim follows.

2.5. *Covariant modules.*

We assume that $\mathbf{A}$ acts on $\mathbf{M}$. Let $\mathscr{H}$ be a Hilbert space that is both hermitian $\mathbf{A}$- and $\mathbf{M}$-module. By a hermitian module we understand a Hilbert space of a $*$-representation of a C^*-algebra. We say that the two modules are *covariant* iff

$$a(x\xi) = \sum_i (a^L_i \cdot x)(a^R_i \xi) \tag{8}$$

for any $a \in \mathbf{A}$, $x \in \mathbf{N}$, $\xi \in \mathscr{H}$. In such a case we also say that the corresponding representations are covariant. If ${}_A\mathscr{H}$ and ${}_M\mathscr{H}$ are covariant, then $\mathscr{H}$ becomes a hermitian $(\mathbf{M} \rtimes \mathbf{A})$-module via

$$(xa)\xi \stackrel{\text{def}}{=} x(a\xi).$$

We omit a tedious but not complicated verification of this fact.

PROPOSITION 2.14. *If $\mathscr{H}$ is a hermitian $\mathbf{M}$-module, then there are covariant $\mathbf{A}$- and $\mathbf{M}$-module structures on $\mathscr{K} = \mathscr{H} \otimes l^2(\mathbf{A}, \tau)$, related to the following representations;*

$$\varrho(a)\colon \xi \otimes b \mapsto \xi \otimes ab,$$

$$\pi(x)\colon \xi \otimes b \mapsto \sum_i (S(b^L_i) \cdot x)\xi \otimes b^R_i,$$

for $a \in \mathbf{A}$, $x \in \mathbf{M}$, $b \in l^2(\mathbf{A}, \tau)$, $\xi \in \mathscr{H}$. Therefore, there is a representation θ of $\mathbf{M} \rtimes \mathbf{A}$ on $\mathscr{K}$, and this representation is faithful if the representation of $\mathbf{M}$ on $\mathscr{H}$ is faithful. If $\mathbf{A}$ and $\mathbf{M}$ are identified with subalgebras of $\mathbf{M} \rtimes \mathbf{A}$ as in Section 2.2, then θ extends both ϱ and π.

PROOF. Again, a tedious but straightforward calculation shows that ϱ and π are covariant representations. Assuming that $\mathscr{H}$ is a faithful $\mathbf{M}$-module we check that $\mathscr{K}$ is a faithful $(\mathbf{M} \rtimes \mathbf{A})$-module.

Suppose that $\sum_{ijk} x_{ijk} \Phi^k_{ij}$ acts trivially on $\mathscr{K}$, for some $x_{ijk} \in \mathbf{M}$. For any $\xi \in \mathscr{H}$ we have

$$\begin{aligned} 0 &= \left(\sum_{i,j,k} x_{ijk} \Phi^k_{ij} \right) (\xi \otimes I) \\ &= \sum_{i,j,k,m} ((\Phi^k_{mi})^* \cdot x_{ijk}) \xi \otimes \Phi^k_{mj}. \end{aligned}$$

Hence, for any j, k, m we have

$$\sum_i (\Phi^k_{mi})^* \cdot x_{ijk} = 0.$$

Since matrix $[\Phi^k]$ is unitary, for any n we get

$$\begin{aligned} 0 &= \sum_m \Phi^k_{mn} \cdot \left(\sum_i (\Phi^k_{mi})^* \cdot x_{ijk} \right) \\ &= \sum_i \sum_m \Phi^k_{mn} (\Phi^k_{mi})^* \cdot x_{ijk} \\ &= x_{njk}. \end{aligned}$$

This completes the proof.

In what follows we keep the notation of the previous proposition. We denote by $\mathscr{B}(\mathscr{H})$ the C^*-algebra of bounded operators on $\mathscr{H}$. The following proposition allows us to introduce a norm in $\mathbf{M} \bowtie \mathbf{A}$, with which $\mathbf{M} \bowtie \mathbf{A}$ is a C^*-algebra.

PROPOSITION 2.15. *The image of* $\mathbf{M} \bowtie \mathbf{A}$ *in* $\mathscr{B}(\mathscr{H})$ *under the representation* θ *constructed in Proposition* 2.14 *is norm closed.*

PROOF. Let $f_\alpha = \sum_{i,j,k} x^{(\alpha)}_{ijk} \Phi^k_{ij}$, with $x^{(\alpha)}_{ijk} \in \mathbf{M}$, be a net of elements of $\mathbf{M} \bowtie \mathbf{A}$ such that $\theta(f_\alpha) \xrightarrow[\alpha]{\|\cdot\|} T$, for some $T \in \mathscr{B}(\mathscr{H})$.

Let F be the conditional expectation from $\mathbf{M} \bowtie \mathbf{A}$ onto $\mathbf{M}$ as in Formula (3). Clearly $\|\pi(F(g))\| \leq \|\theta(g)\|$ doe any $g \in \mathbf{M} \bowtie \mathbf{A}$. Theorem 2.2.(3) implies that $x^{(\alpha)}_{mnl} = d_l F(f_\alpha (\Phi^l_{mn})^*)$ for any m, n, l. Therefore $\pi(x^{(\alpha)}_{mnl}) = d_l \pi(F(f_\alpha(\Phi^l_{mn})^*))$ converges in norm to some $\pi(y_{mnl})$, $y_{mnl} \in \mathbf{M}$. Consequently,

$$\theta(f_\alpha) \xrightarrow[\alpha]{\|\cdot\|} \theta\left(\sum_{i,j,k} y_{ijk} \Phi^k_{ij} \right).$$

This completes the proof.

For any k we have a unitary matrix $[\varrho(\Phi^k)] \in M_{d_k}(\mathscr{B}(\mathscr{H}))$, whose $i-j$ entry equals $\varrho(\Phi^k_{ij})$. We consider $\beta(\Phi^k)$ – an automorphism of $M_{d_k}(\mathscr{B}(\mathscr{H}))$ given by

$$\beta(\Phi^k)(T) \stackrel{\text{def}}{=} [\varrho(\Phi^k)]\, T [\varrho(\Phi^k)]^*,$$

for any $T \in M_{d_k}(\mathscr{B}(\mathscr{H}))$. For any $x \in \mathbf{M}$ we denote by $D_k(x)$ a diagonal matrix in $M_{d_k}(\mathscr{B}(\mathscr{H}))$ with $\pi(x)$ along its diagonal. It follows from Formula (2) and Theorem 2.2.(2) that

$$\pi(\Phi^k_{ij} \cdot x) = i - j \text{ entry of } \beta(\Phi^k)(D_k(x))$$

for any $x \in \mathbf{M}$, and any i, j, k. As an immediate consequence we have the following.

PROPOSITION 2.16. *If* $\mathbf{A}$ *acts on* $\mathbf{M}$, *then* $\alpha(a)\colon \mathbf{M} \to \mathbf{M}$ *is norm continuous for any* $a \in \mathbf{A}$, *where* $\alpha(a)\colon x \mapsto a \cdot x$.

3. Spherical Functions.

Throughout this section we assume that **A** acts on **M**. For p, a projection in **A**, the *spectral subspace* (of the first kind) of **M** corresponding to p is

$$\mathbf{M}_1(p) \stackrel{\text{def}}{=} \{p \cdot x \mid x \in \mathbf{M}\}. \tag{9}$$

PROPOSITION 3.1. *With the above definition we have the following.*

1. *If* $I = \sum_i p_i$, then $\mathbf{M} = \oplus_i \mathbf{M}_1(p_i)$.
2. $\mathbf{M}_1(p)^* = \mathbf{M}_1(S(p))$.
3. *Each* $\mathbf{M}_1(p)$ *is an* $\mathbf{M}^{\mathbf{A}}$*-bimodule.*
4. *If p is central, then* $\mathbf{M}_1(p)$ *is* **A***-invariant.*

We leave the straightforward proof of this proposition to the reader.

Let p be a minimal central projection in **A**. There is a left $\mathbf{A} \otimes p\mathbf{A}$-module structure on $\mathbf{M} \otimes p\mathbf{A}$, namely

$$(a \otimes b)(x \otimes c) \stackrel{\text{def}}{=} a \cdot x \otimes bc. \tag{10}$$

Let $\{v_{ij}^p\}$ be a system of matrix units in $p\mathbf{A}$. We define

$$v_L^p \stackrel{\text{def}}{=} \sum_{i,j} v_{ij}^p \otimes v_{ji}^p, \tag{11}$$

$$v_R^p \stackrel{\text{def}}{=} d_p^{-1} \sum_{i,j} S(v_{ij}^p) \otimes v_{ji}^p. \tag{12}$$

Note that v_L^p is the unique unitary element in $p\mathbf{A} \otimes p\mathbf{A}$ with trace $(\tau \otimes \tau)$ equal to $d_p(\dim \mathbf{A})^{-1}$ and such that $(u \otimes w)v_L^p = v_L^p(w \otimes u)$ for any u, $w \in p\mathbf{A}$. Thus, the definition of v_L^p is independent on the choice of $\{v_{ij}^p\}$. Since $v_R^p = d_p^{-1}(S \otimes \mathrm{id})(v_L^p)$, it is well defined too.

Following [3, 10] (see also [6, 7]) we give the following definitions.

DEFINITION 3.2. *The spectral subspaces (of the second kind) of* **M** *corresponding to p are*

$$\mathbf{M}_2^L(p) \stackrel{\text{def}}{=} \{v_L^p(x \otimes p) \mid x \in \mathbf{M}\},$$

$$\mathbf{M}_2^R(p) \stackrel{\text{def}}{=} \{v_R^p(x \otimes p) \mid x \in \mathbf{M}\}.$$

It is clear that $M_2^R(p) = (M_2^L(p))^*$.

DEFINITION 3.3. *For p, q, minimal central projections in* **A**, *we define the subspace of spherical functions inside the crossed product* $\mathbf{M} \rtimes \mathbf{A}$ *as*

$$S_{p,q} \stackrel{\text{def}}{=} p(\mathbf{M} \rtimes \mathbf{A})q.$$

If $p = q$, then we put $S_p = S_{p,q}$.

Each S_p is a hereditary C^*-subalgebra of $\mathbf{M} \rtimes \mathbf{A}$.

An easy proof of the following proposition is omitted.

PROPOSITION 3.4. *For any p, q, r, s, minimal central projections in* **A**, *with* $q \neq r$, *we have the following.*

1. $S_{p,q}S_{r,s} = \{0\}$.
2. $(S_{p,q})^* = S_{q,p}$.
3. $S_{p,q}S_{q,s} \subseteq S_{p,s}$.
4. $S_{p,q}S_{q,p}$ *is a two-sided ideal of* S_p.
5. $S_{p,e} = \mathbf{M}_1(p)e$.

We denote $\mathbf{A}' \cap (\mathbf{M} \bowtie \mathbf{A})$ by $\mathbf{I}$, and set $\mathbf{I}_p = \mathbf{I} \cap S_p$.

It is clear from Example 2.5.(2) and Formula (2) that an action of **A** on **M** extends to an action of **A** on $\mathbf{M} \bowtie \mathbf{A}$ given by

$$a \cdot f \overset{\text{def}}{=} \sum_i a_i^L f S(a_i^R),$$

for $a \in \mathbf{A}$, $f \in \mathbf{M} \bowtie \mathbf{A}$. The fixed point algebra for this action is denoted by $(\mathbf{M} \bowtie \mathbf{A})^{\mathbf{A}}$.

PROPOSITION 3.5. *For any p, a minimal central projection in* **A**, *we have the following.*

1. $\mathbf{I} = (\mathbf{M} \bowtie \mathbf{A})^{\mathbf{A}}$.
2. *The map* $p\mathbf{A} \otimes \mathbf{I}_p \to \mathbf{M} \bowtie \mathbf{A}$, *given by* $pa \otimes f \mapsto paf$, *establishes a* $*$-*isomorphism between* $p\mathbf{A} \otimes \mathbf{I}_p$ *and* S_p.
3. *A map* $p\mathbf{A} \otimes \mathbf{I}_p \to \mathbf{M} \bowtie \mathbf{A}$, *given by* $pa \otimes f \mapsto paf$, *establishes a* $*$-*isomorphism between* $p\mathbf{A} \otimes \mathbf{I}_p$ *ad* S_p.

PROOF. 1. If f belongs to $\mathbf{I}$, then $a \cdot f = \sum_i a_i^L f S(a_i^R) = (\sum_i a_i^L S(a_i^R))f = \varepsilon(a)f$, for any $a \in \mathbf{A}$. Hence, $f \in (\mathbf{M} \bowtie \mathbf{A})^{\mathbf{A}}$. Conversely, let $f \in (\mathbf{M} \bowtie \mathbf{A})^{\mathbf{A}}$. Then, by virtue of Proposition 2.10, $f = e \cdot f = \sum_{i,j,p} v_{ij}^p f v_{ji}^p$. Thus, for any v_{mn}^q we have $v_{mn}^q f = \sum_j v_{mj}^q f v_{jn}^q = f v_{mn}^q$. Hence, $f \in \mathbf{I}$.

2. and 3. are clear.

We define a left **A**-module structure on $\mathbf{M} \otimes \mathbf{A}$ as

$$a(x \otimes {}^{\beta}) \overset{\text{def}}{=} \sum_{i,j} a_i^L \cdot x \otimes a_{ij}^{RR} b S(a_{ij}^{RL}), \tag{13}$$

for $a, b \in \mathbf{A}$, $x \in \mathbf{M}$. We leave it to the reader to verify that this definition in fact produces a module structure. Similarly to [3, 5, 6, 10] we have the following two propositions.

PROPOSITION 3.6. *Let* $\psi: \mathbf{M} \bowtie \mathbf{A} \to \mathbf{M} \otimes \mathbf{A}$ *be defined as* $\psi: xa \mapsto x \otimes S(a)$, *for* $a \in \mathbf{A}$, $x \in \mathbf{M}$. *For any p, a minimal central projection in* **A**, ψ *establishes a* $*$-*isomorphism between* $\mathbf{I}_p$ *and* $e(\mathbf{M} \otimes S(p)\mathbf{A})$.

PROOF. Clearly ψ is a linear isomorphism from $\mathbf{M} \rtimes \mathbf{A}$ onto $\mathbf{M} \otimes \mathbf{A}$. Moreover, for any $a, b \in \mathbf{A}$, $x \in \mathbf{M}$ we have

$$\begin{aligned}\psi(a \cdot xb) &= \psi\left(\sum_i a_i^L x b S(a_i^R)\right)\\ &= \psi\left(\sum_{i,j} (a_{ij}^{LL} \cdot x) a_{ij}^{LR} b S(a_i^R)\right)\\ &= \sum_{i,j} a_{ij}^{LL} \cdot x \otimes a_i^R S(b) S(a_{ij}^{LR})\\ &= \sum_{i,j} a_i^L \cdot x \otimes a_{ij}^{RR} S(b) S(a_{ij}^{RL})\\ &= a\psi(xb).\end{aligned}$$

Hence, by virtue of Proposition 2.12, ψ is a linear isomorphism from $(\mathbf{M} \rtimes \mathbf{A})^{\mathbf{A}}$ onto $e(\mathbf{M} \otimes \mathbf{A})$. Proposition 3.5 implies that ψ is a linear isomorphism from $\mathbf{I}_p$ onto $e(\mathbf{M} \otimes S(p)\mathbf{A})$.

Moreover, ψ is a *-homomorphism. Indeed, let $\sum_i x_i a_i$ and $\sum_j y_j b_j$ be in $\mathbf{I}_p$. Since $\mathbf{I}_p \subseteq \mathbf{A}'$, we have

$$\begin{aligned}\psi\left(\left(\sum_i x_i a_i\right)\left(\sum_j y_j b_j\right)\right) &= \sum_i \psi\left(x_i\left(\sum_j y_j b_j\right) a_i\right)\\ &= \psi\left(\sum_{i,j} x_i y_j b_j a_i\right)\\ &= \sum_{i,j} x_i y_j \otimes S(a_i) S(b_j)\\ &= \psi\left(\sum_i x_i a_i\right)\psi\left(\sum_j y_j b_j\right).\end{aligned}$$

Hence, ψ preseerves multiplication.

Let $f = \sum_{\substack{i \\ k}} x_{ij}^k \Phi_{ij}^k$ be in $\mathbf{I}$, for some $x_{ijk} \in \mathbf{M}$. Since f commutes with $(\Phi_{mn}^l)^*$ for any m, n, l, we have

$$\begin{aligned}\sum_{i,j,k} x_{ijk} \Phi_{ij}^k (\Phi_{mn}^l)^* &= \sum_{i,j,k} (\Phi_{mn}^l)^* x_{ijk} \Phi_{ij}^k\\ &= \sum_{i,j,k,s} ((\Phi_{ms}^l)^* \cdot x_{ijk})(\Phi_{sn}^l)^* \Phi_{ij}^k\\ &= \sum_{i,j,k,s} (S(\Phi_{sm}^l) \cdot x_{ijk} (\Phi_{sn}^l)^* \Phi_{ij}^k.\end{aligned}$$

Applying conditional expectation F (as in Formula (3)) to both sides of the above equality and taking into account Theorem 2.2.(3) we get

$$x_{mnl} = \sum_s S(\Phi^l_{sm}) \cdot x_{snl}, \tag{14}$$

for any m, n, l. Thus, we have

$$\begin{aligned}
\psi(f^*) &= \psi\left(\sum_{i,j,k,s} ((\Phi^k_{is})^* \cdot x^*_{ijk})(\Phi^k_{sj})^*\right) \\
&= \sum_{i,j,k,s} (\Phi^k_{is})^* \cdot x^*_{ijk} \otimes (S(\Phi^k_{sj}))^* \\
&= \sum_{s,j,k} \left(\sum_i S(\Phi^k_{is}) \cdot x_{ijk})^* \otimes (S(\Phi^k_{sj})\right)^* \\
&= \left(\sum_{s,j,k} x_{sjk} \otimes S(\Phi^k_{sj})\right)^* \\
&= \psi(f)^*.
\end{aligned}$$

Hence, ψ preserves $*$.

PROPOSITION 3.7. *For any p, a minimal central projection in $\mathbf{A}$, we have*

$$\psi(e \cdot S_{p,e}S_{e,p}) = \mathbf{M}^R_2(S(p))\mathbf{M}^L_2(S(p)).$$

PROOF. For $X, Y \subset \mathbf{M} \rtimes \mathbf{A}$ we denote $XY = \operatorname{span}\{xy \mid x \in X,\ y \in Y\}$. We have $\mathbf{MA} = \mathbf{M} \rtimes \mathbf{A} = \mathbf{AM}$. Therefore, $S_{p,e}S_{e,p} = p(\mathbf{M} \rtimes \mathbf{A})e(\mathbf{M} \rtimes \mathbf{A})p = p\mathbf{MA}e\mathbf{AM}p = \mathbf{pM}e\mathbf{M}p$. This together with Formula (7) implies that

$$S_{p,e}S_{e,p} = \operatorname{span}\{(p \cdot x)eyp \mid x, y \in \mathbf{M}\}.$$

Taking adjoints of both sides of the equality in Formula (7) we get

$$e(a \cdot x) = exS(a), \tag{15}$$

for any $a \in \mathbf{A}$, $x \in \mathbf{M}$. Hence, taking into account Formulae (7) and (15), we get

$$\begin{aligned}
e \cdot ((p \cdot x)eyp) &= \sum_i e^L_i(p \cdot x)eypS(e^R_i) \\
&= \sum_i (e_i \cdot (p \cdot x))eyS(e^R_i)p \\
&= \sum_i (pe_i \cdot x)e(e^R_i \cdot y)p,
\end{aligned}$$

for any $x, y \in \mathbf{M}$. By virtue of Proposition 2.10 and Formula (5) we get

$$\begin{aligned}\psi(e\cdot((p\cdot x)eyp)) &= \psi\left(\sum_{i,j,q}(pS(v^q_{ij})\cdot x)e(v^q_{ji}\cdot y)p\right)\\ &= \psi\left(\sum_{i,j}(S(v^{S(p)}_{ij})\cdot x\right)e(v^{S(p)}_{ji}\cdot y)p)\\ &= \psi\left(\sum_{i,j,m,n,q}(S(v^{S(p)}_{ij})\cdot x\right)(v^q_{mn}\cdot(v^{S(p)}_{ji}\cdot y))S(v^q_{nm})p)\\ &= \psi\left(\sum_{i,j,m}(S(v^{S(p)}_{ij})\cdot x\right)(v^{S(p)}_{mi}\cdot y)S(v^{S(p)}_{jm}))\\ &= \sum_{i,j,m}(S(v^{S(p)}_{ij})\cdot x)(v^{S(p)}_{mi}\cdot y)\otimes v^{S(p)}_{jm}.\end{aligned}$$

However, it is an immediate consequence of Formulae (11) and (12) that

$$(v^{S(p)}_R(x\otimes p))(v^{S(p)}_L(y\otimes p)) = \sum_{i,j,m}(S(v^{S(p)}_{ij})\cdot x)(v^{S(p)}_{mi}\cdot y)\otimes v^{S(p)}_{jm}$$

for any $x, y \in \mathbf{M}$. This completes the proof.

REMARK 3.8. *It follows from the proof of the above proposition that we have* $(e\cdot S_{p,e}S_{e,p}) \subseteq S_{p,e}S_{e,p}$, *and consequently*

$$e\cdot S_{p,e}S_{e,p} = S_{p,e}S_{e,p}\cap \mathbf{I}.$$

4. Saturated Actions.

Let $\mathbf{A}$ act on $\mathbf{M}$. Following [1, 5] we consider below a natural $[\mathbf{M} \rtimes \mathbf{A}) - \mathbf{M}^{\mathbf{A}}$ bimodule, which under some conditions becomes an imprimitivity bimodule (see [12]). We also look at the algebras of $\mathbf{M}^{\mathbf{A}}$-module endomorphisms of $\mathbf{M}_{M^A}$ (see [8, 9, 12, 13]).

$\mathbf{M}e \subset \mathbf{M} \rtimes \mathbf{A}$ becomes an $\mathbf{M}^{\mathbf{A}}$-valued inner product bimodule (see [8, 9, 12, 13]) if we set

$$\langle xe, ye\rangle_{M^A} \stackrel{\text{def}}{=} E(y^*x).$$

As a consequence of Proposition 3.1 we have the following.

PROPOSITION 4.1. *If* $I = \sum_i p_i$, *then* $\mathbf{M}e = \oplus_i \mathbf{M}_1(p_i)e$ *is a direct sum of pairwise orthogonal* $\mathbf{M}^{\mathbf{A}}$-*subbimodules.*

We consider a left $(\mathbf{M} \rtimes \mathbf{A})$-module structure on $\mathbf{M}e$ given by

$$(xa)ye \stackrel{\text{def}}{=} x(a\cdot y)e.$$

We also consider an $(\mathbf{M} \rtimes \mathbf{A})$-valued inner product on $\mathbf{M}$ defined by

$$\langle xe, ye\rangle_{M\rtimes A} \stackrel{\text{def}}{=} xey^*.$$

The following definition is due to Rieffel (see [11]).

DEFINITION 4.2. *We say that an action of* $\mathbf{A}$ *on* $\mathbf{M}$ *issaturated iff* $\mathbf{M}e$ is an $(\mathbf{M} \rtimes \mathbf{A}) - \mathbf{M}^{\mathbf{A}}$ *imprimitivity bimodule* (*in the sense of* [12, *Definition* 6.10]).

The following two theorems are Hopf $*$-algebra analogues of the results in [10] for compact group actions.

THEOREM 4.3. *The following are equivalent.*

1. *The action is saturated.*
2. *The two-sided ideal generated by* S_e *equals* $\mathbf{M} \rtimes \mathbf{A}$.
3. *For any* p, *a minimal central projection in* $\mathbf{A}$, $S_{p,e}S_{e,p}$ *equals* S_p.
4. *For any* p, *a minimal central projection in* $\mathbf{A}$, $e(\mathbf{M} \otimes p\mathbf{A})$ *is equal to* $\mathbf{M}_2^R(p)\mathbf{M}_2^L(p)$.

PROOF. It is clear that $\mathbf{M}e$ may fail to be the imprimitivity bimodule only if the range of the $(\mathbf{M} \rtimes \mathbf{A})$-valued inner product is not dense. This, however, is equivalent to 2. (since $\mathbf{M} \rtimes \mathbf{A}$ contains identity, every dense ideal equals the whole algebra). Thus, 1. and 2. are equivalent.

It is clear that 2. implies 3. Equivalence of 3. and 4. follows from Propositions 3.6 and 3.7.

Suppose that 3. holds, and for each p, a minimal central projection in $\mathbf{A}$, let x_i^p, $y_i^p \in \mathbf{M}_1(p)$ be such that $\sum_i x_i^p e y_i^p = p$. We have $\sum_{i,p} x_i^p e y_i^p = I$, and consequently S_e generates $\mathbf{M} \rtimes \mathbf{A}$. Thus 2. holds.

One can verify without difficulty that the actions from Example 2.5.(4 and 5) are saturated. More interesting examples of saturated actions of finite dimensional Hopf $*$-algebra on C^*- and von Neumann algebras will be provided in a subsequent paper.

The proof of the following theorem is almost identical with those of [10, Theorems 3.4 and 3.11, Corollaries 3.7 and 3.12]. Therefore, we verify the characterizations of simplicity only.

THEOREM 4.4. *The following are equivalent.*

1. $\mathbf{M} \rtimes \mathbf{A}$ *is simple* (*prime*).
2. *For any* p, *a minimal central projection in* $\mathbf{A}$
   (a) $S_{p,e} \neq \{0\}$,
   (b) S_p *is simple* (*prime*).
3. *For any* p, *a minimal central projection in* $\mathbf{A}$
   (a) $\mathbf{M}_1(p) \neq \{0\}$,
   (b) $e(\mathbf{M} \otimes p\mathbf{A})$ *is simple* (*prime*).

PROOF. $(2 \Leftrightarrow 3)$ Since $S_{p,e} = \mathbf{M}_1(p)e$ by Proposition 3.4.(5), and $S_p \cong$

$M_{d_p}(\mathbf{C}) \otimes \mathbf{I}_p \cong M_{d_p}(\mathbf{C}) \otimes e(\mathbf{M} \otimes S(p)\mathbf{A})$ by Propositions 3.5. (3) and 3.6, it is clear that 2. and 3. are equivalent.

(1 ⇔ 2) Let $\mathbf{M} \rtimes \mathbf{A}$ be simple, and suppose for a moment that $S_{p,e} = \{0\}$ for some p, a minimal central projection in $\mathbf{A}$. But then $p(\mathbf{M} \rtimes \mathbf{A}) S_e(\mathbf{M} \rtimes \mathbf{A}) = S_{p,e}(\mathbf{M} \rtimes \mathbf{A}) = \{0\}$, and the ideal generated by S_e is proper, a contradiction. Thus, $S_{p,e} \neq \{0\}$ for any p. It is well known that simplicity of $\mathbf{M} \rtimes \mathbf{A}$ implies that each hereditary subalgebra of $\mathbf{M} \rtimes \mathbf{A}$, in particular S_p, is simple.

Conversely, if $S_{p,e} \neq \{0\}$ for any p, and each S_p is simple, then $S_{p,e}S_{e,p}$ equals S_p (by virtue of Proposition 3.4.(4)). Since by Theorem 4.3 $\mathbf{M} \rtimes \mathbf{A}$ and $S_e = \mathbf{M}^{\mathbf{A}}e$ are Morita equivalent, it follows that $\mathbf{M} \rtimes \mathbf{A}$ is simple.

$\mathbf{M}$ is an $\mathbf{M}^{\mathbf{A}}$-valued inner product module if we set

$$\langle x, y \rangle_{M^A} = E(y^* x).$$

If $T\colon \mathbf{M} \to \mathbf{M}$ is a norm bounded right $\mathbf{M}^{\mathbf{A}}$-module endomorphism, then a norm bounded operator $T^*\colon \mathbf{M} \to \mathbf{M}$ is called the adjoint of T if $\langle Tx, y \rangle_{M^A} = \langle x, T^* y \rangle_{M^A}$ for any $x, y \in \mathbf{M}$ (see [8, 12]). We denote by $\mathrm{End}(\mathbf{M})$ the *-algebra of those norm bounded right $\mathbf{M}^{\mathbf{A}}$-module endomorphism of $\mathbf{M}$ that possess adjoints.

We have a *-algebra embedding $L\colon \mathbf{M} \hookrightarrow \mathrm{End}(\mathbf{M})$, given by $L_x(y) = xy$. It easily follows from Propositions 2.12 and 2.16 that $\alpha\colon \mathbf{A} \to \mathrm{End}(\mathbf{M})$ is a *-algebra homomorphism. We also consider $e_M \in \mathrm{End}(\mathbf{M})$, defined as $e_M(x) = E(x)$. Clearly e_M is a projection commuting with $L(\mathbf{M}^{\mathbf{A}})$, and such that for any $x \in \mathbf{M}$

$$e_M L_x e_M = L_{E(x)} e_M. \tag{16}$$

The algebra of compact operators is defined as

$$\mathrm{K}(\mathbf{M}) \stackrel{\mathrm{def}}{=} \mathrm{span}\,\{L_x e_M L_y \mid x, y \in \mathbf{M}\}.$$

Clearly $\mathrm{K}(\mathbf{M})$ is a two-sided self-adjoint ideal of $\mathrm{End}(\mathbf{M})$.

The notion of the index of a conditional expectation in the context of C^*-algebras was introduced by Watatani in [16]. If $I \in \mathbf{N} \subset \mathbf{M}$ are C^*-algebras and $E\colon \mathbf{M} \to \mathbf{N}$ is a conditional expectation, then E is said to be of index finite type if there exists a finite family $\{(u_1, w_1), \ldots, (u_n, w_n)\}$ of elements of $\mathbf{M}$ such that

$$\sum_i E(xu_i)w_i = x = \sum_i u_i E(w_i x)$$

for any $x \in \mathbf{M}$. Such a family is called a quasi-basis. If E is of index finite type then $\mathrm{Index}(E) = \sum_i u_i w_i$, and this definition does not depend upon the choice of a quasi-basis. One can always choose a quasi-basis (if there is one) such that $w_i = u_i^*$.

Proposition 4.5. *If an action of* $\mathbf{A}$ *on* $\mathbf{M}$ *is saturated, then* $\mathbf{M} \rtimes \mathbf{A} \cong \mathrm{K}(\mathbf{M}) = \mathrm{End}(\mathbf{M})$ *and* $\mathrm{Index}(E) = (\dim \mathbf{A})I$.

Proof. By virtue of Theorem 4.3 we have

$$\mathbf{M} \rtimes \mathbf{A} = \mathrm{span}\{xey \mid x, y \in \mathbf{M}\}. \tag{17}$$

We define a map ϕ: $\mathbf{M} \rtimes \mathbf{A} \to \mathrm{K}(\mathbf{M})$ as

$$\phi\colon \sum_i x_i e y_i \mapsto \sum_i L_{x_i} e_M L_{y_i},$$

for $x_i, y_i \in \mathbf{M}$. Saturatedness implies $\sum_i x_i e y_i = 0$ iff $(\forall t \in \mathbf{M})(\sum_i x_i e y_i) te = 0$ iff $(\forall t \in \mathbf{M})(\sum_i L_{x_i} e_M L_{y_i})(t) = 0$. Thus ϕ is well defined and injective. Clearly ϕ is surjective too. With help of Formula (16) one can easily check that ϕ is a $*$-homomorphism. Since $\mathrm{K}(\mathbf{M})$ contains the identity, it follows that $\mathrm{K}(\mathbf{M}) = \mathrm{End}(\mathbf{M})$.

By (17), one can choose a finite family $\{u_1, \ldots, u_n\}$ in $\mathbf{M} \rtimes \mathbf{A}$ such that $I = \sum_i u_i e u_i^*$. For any $x \in \mathbf{M}$ we have $xe = (\sum_i u_i e u_i^*) xe = \sum_i E(u_i^* x)e$ (by virtue of Formula (7)). Hence $x = \sum_i u_i E(u_i^* x)$. It follows that $\{(u_i, u_i^*)\}$ is a quasi basis. With F as in Formula (3) we have $I = F(I) = F(\sum_i u_i e u_i^*) = \sum_i u_i F(e) u_i^* = \dim \mathbf{A})^{-1} \sum_i u_i u_i^*$. Hence, $\mathrm{Index}(E) = \sum_i u_i u_i^* = (\dim \mathbf{A})I$.

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