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KOROVKIN THEORY IN LINDENSTRAUSS SPACES

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Abstract.

In this paper the author investigates the Korovkin closure in simplex spaces with respect to positive linear contractions and also the weak sequential Korovkin closure with respect to contractions in general real and complex Lindenstrauss spaces. These Korovkin closures are characterized via their uniqueness spaces in the separable case. We also give some results in the non-separable case.

1. Introduction.

Let K be a compact convex set in a locally convex Hausdorff topological vector space, $H \subseteq A(K)$ a subspace of the space $A(K)$ of all real-valued continuous affine functions and let $T: A(K) \to A(K)$ be a positive linear operator. One main task of Korovkin theory is to characterize the Korovkin closure $\mathrm{Kor}^+(H, T)$ of H with respect to T, i.e. the space of all functions $f \in A(K)$ with the following property: If $(L_i)_{i \in I}$ is a net of positive linear operators $L_i: A(K) \to A(K)$ such that $\lim_{i \in I} \|L_i h - Th\| = 0 \forall h \in H$, then it follows that $\lim_{i \in I} \|L_i f - Tf\| = 0$. In [Alt] F. Altomare studied this problem in the case of general compact convex sets for $T =$ identity operator. He proved the so called standard characterization of $\mathrm{Kor}^+(H, \mathrm{id})$ for test function spaces H lying in the center of $A(K)$ (cf. [Alt, Th. 1.5]). Using selection theorems, Leha-Papadopoulou ([L-P]) and G. M. Ustinov ([U1], [U3]) obtained results in the case where K is a metrizable Choquet simplex (cf. 2.8, too). If K is a Choquet simplex, then $A(K)$ is a Lindenstrauss space. For Lindenstrauss spaces more general selection theorems are available ([L-L], [O]), therefore it is natural to study Korovkin theory in Lindenstrauss spaces.

In the second part, we study the Korovkin closure with respect to positive contractions in simplex spaces, i.e. ordered Lindenstrauss spaces E (cf. 2.1). For separable simplex spaces we characterize the positive contractive Korovkin closure with respect to a general positive contraction $T: E \to E$ via its uniqueness space $U_{+,1}(H, T)$. As an application, we obtain the result $\mathrm{Kor}^+(H, T) =$

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$U_+(H,T)$ for $A(K)$ spaces, K a metrizable Choquet simplex, $T: A(K) \to A(K)$ a positive linear operator. We also prove a result in the non-separable case.

In the third part, we study the weak sequential Korovkin closure $\sigma\omega\text{-Kor}^1(H,T)$ with respect to contractions $T: E \to E$ in general real and complex Lindenstrauss spaces (without order). We characterize $\sigma\omega\text{-Kor}^1(H,T)$ for separable E and obtain a partial result for more special test function spaces H in the non-separable case.

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2. Positive contractive approximation in simplex spaces.

2.1. Let E be a real or complex Lindenstrauss space, i.e. a Banach space with a dual that is linearly and isometrically isomorphic to an abstrct L space (cf. [L-L], [Lac]). Special cases are the (real) simplex spaces introduced by Effros in [Eff]. A simplex space is a real Banach space with an order such that

(i) E^+ is a closed cone.

(ii) E^+ is an AL-space with order induced by the positive linear (bounded) functionals on E.

It is shown in [Eff] that a simplex space E is linearly order and isometrically isomorphic to $A_0(K)$ where

$$A_0(K) := \{f: K \to \mathrm{R}: f \text{ affine continuous}, f(0) = 0\}$$

and K is the w*-compact convex set

$$K := B_1^+(E') := \{p \in E': p \geqq 0, \|p\| \leqq 1\}.$$

We now give the basic definitions of Korovkin theory in these spaces. If $H \subseteq E$ is a subspace and $T: E \to E$ a positive linear contraction (i.e. $\|T\| \leqq 1$), then the positive contractive Korovkin closure of H with respect to T is

$$\mathrm{Kor}^{+,1}(H,T) = \{y \in E: \text{ If } (L_i)_{i\in I} \text{ is a net of positive linear contractions } L_1: E \to E, \text{ then } \lim_{i\in I} \|L_i h - Th\| = 0 \,\forall h \in H \text{ implies } \lim_{i\in I} \|L_i y - Ty\| = 0\}.$$

It is the aim of this section to show that for separable simplex spaces $\mathrm{Kor}^{+,1}(H,T)$ coincides with the so called positive contractive uniqueness closure of H with respect to T:

$$U_{+,1}(H,T) = \{y \in E: \text{ If } p \in B_1^+(E') \text{ and } q \in \overline{\partial_e B_1^+(E')} \text{ such that } p = q \circ T \text{ on } H, \text{ then } p(y) = q(Ty)\}.$$

Here $\partial_e B_1^+(E')$ denotes the extreme points of $B_1^+(E')$. The above mentioned results of Effros ([Eff, Th. 2.2]) and the Krein-Milman theorem entail the equation:

$$\|x\| = \sup_{p \in B_1^+(E')} |p(x)| = \sup_{q \in \partial_e B_1^+(E')} |q(x)|$$

for every $x \in E$. Using standard techniques of Korovkin theory (cf. [M]) and the above equation we get:

2.2. PROPOSITION. *Let E be a simplex space and $H \subseteq E$ a subspace. Then for a positive linear contraction $T: E \to E$, $\|T\| \leqq 1$, the inclusion*

$$U_{+,1}(H, T) \subseteq \mathrm{Kor}^{+,1}(H, T)$$

always holds true.

To prove the reverse inclusion is more difficult, because the standard techniques of Korovkin theory using multiplication operators and Urysohn functions are not applicable for general test function spaces H (but cf. [Alt] for test function spaces lying in the center of an $A(K)$ space). Leha-Papadopoulou ([L], [L-P]) and G. M. Ustinov ([U1], [U2], [U3], cf. 2.8 too) used selection theorems to prove an analogous characterization of the Korovkin closure in more special cases. We need the following selection theorem by Lazar:

2.3. THEOREM ([Laz]). *Let E be a Fréchet space and $\Phi: K \to 2^E$ a convex lower semicontinuous correspondence such that $\Phi(k)$ is closed for any $k \in K$. Let K be a Choquet simplex, then there exists an affine continuous selection mapping $f: K \to E$ with $f(k) \in \Phi(k) \forall k \in K$. If F is a closed face of K and φ is an affine continuous selection map of $\Phi_{|F}$, then f can be chosen such that $f_{|F} = \varphi$ holds.*

A convex correspondence $\Phi: K \to 2^E$ is a map such that $\Phi(k) \neq \emptyset$, $\Phi(k)$ is convex for any $k \in K$ and

$$\lambda\Phi(k_1) + (1 - \lambda)\Phi(k_2) \subseteq \Phi(\lambda k_1 + (1 - \lambda)k_2),$$

$k_1, k_2 \in K$, $0 < \lambda < 1$. Φ is lower semicontinuous, if for any U open in E the set

$$\{k \in K: \Phi(k) \cap U \neq \emptyset\}$$

is open in K.

Let E be a separable Banach space and $K \subseteq E'$ a w*-compact convex set. A well known folk theorem says, that K with it's w*-topology can be embedded in a Fréchet space. Using a modification of Leha-Papadopoulou's ideas, we obtain the following:

2.4. LEMMA. *Let E be a separable simplex space, $H \subseteq E$ a subspace and $L: E \to E$ a positive linear continuous operator with $\|L\| \leqq 1$. Further, let $p \in B_1^+(E')$, $q \in \partial_e B_1^+(E')$ and $(q_n)_{n \in \mathbb{N}}$ be a sequence in $\partial_e B_1^+(E')$ with $\lim_{n \to \infty} q_n = q$. Finally, let $p = q \circ L$ on H. Then there exists a sequence $(L_n)_{n \in \mathbb{N}}$ of positive linear contractions $L_n: E \to E$ such that*

$$\lim_{n\to\infty} \|L_n h - Lh\| = 0 \forall h \in H \quad and \quad q_n \circ L_n = p \forall n \in \mathbb{N}$$

holds.

PROOF. As a subspace of E the space H is also separable. Let $(h_n)_{n\in\mathbb{N}}$ be a dense sequence of H. Define the convex correspondence Ψ_n: $B_1^+(E') \to 2^F$,

$$\Psi_n(v) := \left\{ w \in B_1^+(E') : |\omega(h_i) - L'v(h_i)| < \frac{1}{n} \forall i \in \mathbb{N}_n \right\} \neq \emptyset,$$

$\mathbb{N}_n := \{1, \ldots, n\}$, $v \in B_1^+(E')$. F is the Fréchet space in which $B_1^+(E')$ is embedded. The lower semicontinuity is a direct consequence of the following:

Claim: Let $v \in B_1^+(E')$ and $(v_k)_{k\in\mathbb{N}}$ be a sequence in $B_1^+(E')$ with $v_k \to v$ and suppose $w \in \Psi_n(v)$. Then $w \in \Psi_n(v_k)$ for almost every $k \in \mathbb{N}$.

This is easy: We have $|w(h_i) - L'v(h_i)| < \frac{1}{n}$ for $i \in \mathbb{N}_n$. Now put

$$\varepsilon := \max_{i\in\mathbb{N}_n} |w(h_i) - L'v(h_i)| < \frac{1}{n}.$$

Because of $v_k \to v$ there exists $k_0 \in \mathbb{N}$, such that for all $k \geqq k_0$

$$|L'v_k(h_i) - L'v(h_i)| < \frac{1}{n} - \varepsilon.$$

This implies for $k \geqq k_0$:

$$|w(h_i) - L'v_k(h_i)| \leq |w(h_i) - L'v(h_i)| + |L'v(h_i) - L'v_k(h_i)| < \varepsilon + \frac{1}{n} - \varepsilon = \frac{1}{n}.$$

Now consider $\Phi_n(v) := \overline{\Phi_n(v)} \subseteq B_1^+(E')$, Φ_n: $B_1^+(E') \to \bar{c}(B_1^+(E'))$. Here $\bar{c}(B_1^+(E'))$ is the set of all closed convex subsets of $B_1^+(E')$. Then Φ_n is a convex lower semicontinuous correspondence because of the equation:

$$\{v \in B_1^+(E') : \Psi_n(v) \cap U \neq \emptyset\} = \{v \in B_1^+(E') : \overline{\Psi_n(v)} \cap U \neq \emptyset\}$$

for open subsets U in F.

Let $(q_k)_{k\in\mathbb{N}}$ as in the Lemma 2.4 be given. The relation $p \in \Phi_n(q) \forall n \in \mathbb{N}$ and the above claim yield after passing to a suitable subsequence of $(q_k)_{k\in\mathbb{N}}$ (also denoted $(q_k)_{k\in\mathbb{N}}$):

$$p \in \Phi_n(q_n) \forall n \in \mathbb{N}. \tag{1}$$

The map f_n: $[0, q_n] \to B_1^+(E')$, $f_n(\lambda 0 + (1 - \lambda)q_n) := (1 - \lambda)p$, $\lambda \in [0, 1]$, is an affine continuous selection of $\Phi_{n|[0,q_n]}$ and $[0, q_n]$ is a closed face of $B_1^+(E')$. By Lazar's selection theorem there exists an affine continuous selection φ_n: $B_1^+(E') \to B_1^+(E')$ with the properties:

(2) $$\varphi_n(v) \in \Phi_n(v) \forall v \in B_1^+(E'),\ \varphi_n(q_n) = p,\ \varphi_n(0) = 0 \forall n \in \mathsf{N}.$$

Now the operators $L_n\colon E \to E$ can be constructed as follows. Let $x \in E$ be given. Define by $\varphi_{n,x}\colon B_1^+(E') \to \mathsf{R}$, $\varphi_{n,x}(v) := \varphi_n(v)(x)$, $v \in B_1^+(E')$, an affine w*-continuous function. The unique extension of $\varphi_{n,x}$ to a linear functional on E' is by the Krein-Šmulian theorem w*-continuous. Therefore this extension is of the form $z_x \in E$, and we define: $L_n\colon E \to E$, $L_n(x) := z_x$. Then L_n is a positive linear contraction:

$$\|L_n(x)\| = \sup_{v \in B_1^+(E')} |\varphi_{n,x}(v)| = \sup_{v \in B_1^+(E')} |\varphi_n(v)(x)| \leqq \|x\|,$$

and for $x \geqq 0$ it follows:

$$v(L_n(x)) = \varphi_n(v)(x) \geqq 0 \forall v \in B_1^+(E'),$$

because $\varphi_n(v) \in \overline{\Psi_n(v)} \subseteq B_1^+(E')$. Finally, we get for $x \in E$:

$$L_n' q_n(x) = q_n(L_n x) = \varphi_n(q_n)(x) = p(x).$$

To prove $\lim_{n \to \infty} \|L_n h - Lh\| = 0 \forall h \in H$, choose $\varepsilon > 0$ and $h \in H$ and a positive integer $n_0 \in \mathsf{N}$ such that

$$\|h_{n_0} - h\| < \frac{\varepsilon}{3} \quad \text{and} \quad \frac{1}{n_0} < \frac{\varepsilon}{3}.$$

For $n \geqq n_0$ we obtain:

$$\begin{aligned}
&\|L_n h - Lh\| \\
&\leqq \|L_n h - L_n h_{n_0}\| + \|L_n h_{n_0} - L h_{n_0}\| + \|L h_{n_0} - Lh\| \\
&\leqq \frac{\varepsilon}{3} + \|L_n h_{n_0} - L h_{n_0}\| + \frac{\varepsilon}{3} \\
&\leqq \frac{2\varepsilon}{3} + \sup_{v \in B_1^+(E')} |v(L_n h_{n_0} - L h_{n_0})| \\
&\leqq \frac{2\varepsilon}{3} + \sup_{v \in B_1^+(E')} |(\varphi_n(v))(h_{n_0}) - L'v(h_{n_0})| \\
&\leqq \frac{2\varepsilon}{3} + \frac{1}{n_0} < \varepsilon,
\end{aligned}$$

because $\varphi_n(v) \in \Phi_n(v) = \overline{\Psi_n(v)}$ for all $v \in B_1^+(E')$. This completes the proof.

2.5. THEOREM. *Let E be a separable simplex space, $H \subseteq E$ a linear subspace and $T\colon E \to E$ a positive linear contraction. Then*

$$\operatorname{Kor}^{+,1}(H, T) = U_{+,1}(H, T).$$

PROOF. Only the inclusion $\mathrm{Kor}^{+,1}(H, T) \subseteq U_{+,1}(H, T)$ remains to be proved. Let $y \in \mathrm{Kor}^{+,1}(H, T)$, $p \in B_1^+(E')$, $q \in \overline{\partial_e B_1^+(E')}$ such that $p(h) = T'q(h)$ for every $h \in H$. Choose a sequence $(q_n)_{n \in \mathbb{N}}$ in $\partial_e B_1^+(E')$ with $\lim_{n \to \infty} q_n = q$. Then Lemma 2.4 guarantees the existence of a sequence $(L_n)_{n \in \mathbb{N}}$ of positive linear contractions $L_n: E \to E$ with the properties:

$$\lim_{n \to \infty} \|L_n h - Th\| = 0 \forall h \in H \text{ and } q_n(L_n y) = p(y) \forall n \in \mathbb{N}.$$

The relation $y \in \mathrm{Kor}^{+,1}(H, T)$ implies $\lim_{n \to \infty} \|L_n y - Ty\| = 0$, and in particular it follows that $\lim_{n \to \infty} q_n(L_n y) = q(Ty)$. Because of $q_n(L_n y) = p(y) \forall n \in \mathbb{N}$ we finally get: $p(y) = q(Ty)$.

2.6. Let K be a compact convex set in a locally convex Hausdorff topological vector space and denote by $A(K)$ the space of all real-valued affine continuous functions on K with the supremum norm and the natural order. For a positive linear operator $T: A(K) \to A(K)$ and a subspace $H \subseteq A(K)$ we define the positive Korovkin closure of H with respect to T as in the introduction. The respective uniqueness closure is:

$$U_+(H, T) := \{f \in A(K): \text{ If } \mu \in A'_+(K) \text{ is a positive linear functional and } x \in \overline{\partial_e K} \text{ such that } \mu(h) = \delta_x \circ T(h) \forall h \in H, \text{ then } \mu(f) = \delta_x \circ T(f)\}.$$

The Korovkin theory of these spaces was studied in [Alt] for $T = \mathrm{id}$. For cofinal subspaces $H \subseteq A(K)$ (i.e. H contains a strictly positive function $h_0 \in H$) the inclusion $U_+(H, T) \subseteq \mathrm{Kor}^+(H, T)$ always holds. (Use a modification of the proof of Th. 1.1 in [M].) As an application of the above lemma we obtain the following result:

2.7. THEOREM. *Let K be a metrizable Choquet simplex, $H \subseteq A(K)$ a cofinal subspace and $T: A(K) \to A(K)$ be a positive linear operator. Then*

$$\mathrm{Kor}^+(H, T) = U_+(H, T).$$

PROOF. If suffices to prove the inclusion "$\subseteq$". Let f be in $\mathrm{Kor}^+(H, T)$, $x \in \overline{\partial_e K}$ and $\mu \in A'_+(K)$ a positive linear functional such that $\mu(h) = \delta_x \circ T(h) \forall h \in H$ holds and choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $\partial_e K$ with $\lim_{n \to \infty} x_n = x$.

Claim: If $h_0 \geqq 1$, $h_0 \in H$, then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of positive linear operators $T_n: A(K) \to A(K)$ with $\|T_n\| \leqq \|Th_0\| \forall n \in \mathbb{N}$,

$$\lim_{n \to \infty} \|T_n h - Th\| = 0 \forall h \in H \text{ and } \delta_{x_n} \circ T_n = \mu \forall n \in \mathbb{N}.$$

To see this, consider in the case $Th_0 \neq 0$ the operator $\|Th_0\|^{-1} T$ and the positive linear functional $\|Th_0\|^{-1} \mu$. The positivity of T and μ imply

$$\|Th_0\|^{-1}T(1) \leqq \|Th_0\|^{-1}T(h_0) \leqq 1$$

and

$$\|Th_0\|^{-1}\mu(1) \leqq \|Th_0\|^{-1}\mu(h_0) = \|Th_0\|^{-1}\delta_x \circ T(h_0)$$
$$= \|Th_0\|^{-1}Th_0(x) \leqq 1.$$

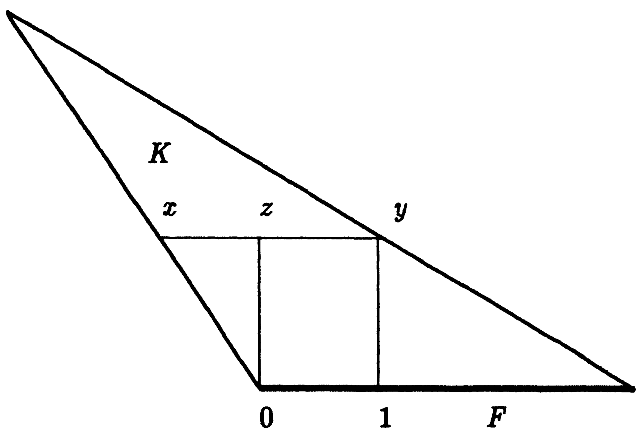
Therefore $\|Th_0\|^{-1}\|\mu\| \leqq 1$ and $\|Th_0\|^{-1}\|T\| \leqq 1$, and an application of Lemma 2.4 yields a sequence $(L_n)_{n\in\mathbf{N}}$ of positive linear contractions $L_n: A(K) \to A(K)$ with the properties:

$$\lim_{n\to\infty} \|L_n h - \|Th_0\|^{-1}T(h)\| = 0 \text{ and } \delta_{x_n} \circ L_n = \|Th_0\|^{-1}\mu \forall n \in \mathbf{N},$$

because $\delta_{x_n} \to \delta_x$ and the δ_{x_n} are extreme points in the positive part of the unit ball of $A'(K)$. The $T_n := \|Th_0\| L_n$ is the sequence we seeked. In the case $Th_0 = 0$ it follows that $T = 0$ and $\mu = 0$, so we can choose $T_n = 0 \,\forall n \in \mathbf{N}$.

Now by an application of the claim we obtain as in the proof of Theorem 2.5 that $\mu(f) = \delta_x \circ T(f)$. This completes the proof.

2.8. Remark. G. M. Ustinov also states Theorem 2.7 for more special metrizable Choquet simplices K, which he calls simple, [U1], [U3]. In his proof he needs the assertion that for a compact Choquet simplex K in a Hilbert space and a closed face F of K the best approximation $\sigma: K \to F$, defined by $\|\sigma(x) - x\| = \min_{y\in F} \|x - y\|$, is affine (continuous). But this is unfortunately not true. To see that σ is not affine in general, consider a triangle K as below:



We have $\sigma(z) = 0 = \sigma(x)$ and $\sigma(y) = 1$. With $z = \lambda x + (1-\lambda)y$, $\lambda \in (0,1)$, we get the contradiction: $0 = \sigma(z) = \lambda\sigma(x) + (1-\lambda)\sigma(y) = 1 - \lambda$.

2.9. For the next section we recall some properties of Lindenstrauss spaces.

(i) Let E be a real or complex Lindenstrauss space and $H \subseteq E$ be a separable subspace. Then there exists a real or complex closed separable Lindenstrauss space Z such that $H \subseteq Z \subseteq E$.

(ii) Let K be a Choquet simplex and H a separable subspace of $A(K)$ containing the constant functions. If we choose a separable closed Lindenstrauss space Z as in (i), $H \subseteq Z \subseteq A(K)$, then Z is isometrically and order isomorphic to $A(K_1)$ where K_1 is a metrizable Choquet simplex.

(i) is Lemma 1 in [Lac, Chap. 7, §23]. To prove (ii), define

$$K_1 := \{\psi \in Z' : \psi \geqq 0, \|\psi\| = 1 = \psi(1)\}.$$

As an archimedean ordered space with order unit 1 that is complete in the order (= sup-) norm, Z is isometrically isomorphic to $A(K)$ via $f \mapsto \tilde{f}$, $f \in Z, \tilde{f}(\psi) := \psi(f)$, $\psi \in K_1$, and K_1 is a metrizable Choquet simplex because Z' is a separable AL-space (cf. [Alf, Th. II.1.8] and the definition of a Choquet simplex).

In the situation of (i) let Z be infinite-dimensional. Then there exists a sequence $(Z_n)_{n \in \mathsf{N}}$ of finite-dimensional subspaces $Z_n \subseteq Z$, $Z_n \subseteq Z_{n+1}$ $\forall n \in \mathsf{N}$ such that Z_n is isometrically isomorphic to $l_\infty(n, \mathsf{K})$, $\mathsf{K} = \mathsf{R}$ or C, and $Z = \overline{\bigcup_{n \in \mathsf{N}} Z_n}$ ([Lac], [M-P], [N-O]). The subspaces Z_n have a so called admissible basis, i.e. there exist vectors $\{z_i^n\}_{i=1}^n \subseteq Z_n$, $n \in \mathsf{N}$ and scalars $\{a_i^n\}_{i=1}^n$, $n \in \mathsf{N}$, such that

$$z_i^n = z_i^{n+1} + a_i^n z_{n+1}^{n+1},\ 1 \leqq i \leqq n,\ \sum_{i=1}^n |a_i^n| \leqq 1,$$

for every $n \in \mathsf{N}$. Now we can construct contractive projections $P_n: E \to Z_n$ as follows. Choose $\varphi_i^n \in Z'$ with $\varphi_i^n(z_j^n) = \delta_{ij}$ and put $\varphi_j^n(x) := a_j$ for $x = \sum_{i=1}^n a_i z_i^n \in Z_n$. Then $\|\varphi_j^n\| = 1$. Extend the φ_i^n's with preservation of the norm to E (we use the same notation) and define:

$$P_n(x) = \sum_{i=1}^n \varphi_i^n(x) z_i^n \in Z_n$$

for $x \in E$. P_n is a projection with $\|P_n\| = 1$ $(Z_n \simeq l_\infty(n, \mathsf{K}))$ and we have $\lim_{n\to\infty} \|P_n(z) - z\| = 0\ \forall z \in Z$.

Analogously, in the situation of (ii) there exist subspaces $Z_n \subseteq Z \subseteq A(K)$, $Z = \overline{\bigcup_{n \in \mathsf{N}} Z_n}$, $Z_n \simeq l_\infty(n, \mathsf{R})\ \forall n \in \mathsf{N}$, and the spaces Z_n are spanned linearly by "peaked partitions of unity", i.e. there exist functions $f_1^{(n)}, \ldots, f_n^{(n)} \in A(K)$ with the properties:

$$f_i^{(n)} \geqq 0,\ \sum_{i=1}^n f_i^{(n)} = 1,\ \|f_i^{(n)}\| = 1$$

(cf. [M-P, Prop. 5.1]). Choose $k_i^{(n)} \in \partial_e K, i = 1, \ldots, n$, with $f_i^{(n)}(k_j^{(n)}) = \delta_{ij}$. Then $P_n\colon A(K) \to Z_n$,

$$(P_n f)(x) := \sum_{i=1}^{n} f(k_i^{(n)}) f_i^{(n)}(x), \quad x \in K, \ f \in A(K),$$

is a positive projection with $\|P_n\| = 1$.

After these lengthy preparations we obtain

2.10. THEOREM. *Let K be a Choquet simplex and $H \subseteq A(K)$ a separable subspace with the following properties:*

(i) *$1 \in H$ and H separates the points of $\partial_e K$.*

(ii) *For any $x \in \partial_e K$, $\varepsilon > 0$ and $f \in H$ with $f(x) = 0$ there exists $g \in H_+$ such that $g \geqq f$ and $g(x) < \varepsilon$.*
We then have:

$$\operatorname{Kor}^+(H, \mathrm{id}) = U_+(H, \mathrm{id}).$$

PROOF. We need only prove the inclusion "$\subseteq$". Assume, $f_0 \in \operatorname{Kor}^+(H, \mathrm{id})$, but $f_0 \notin U_+(H, \mathrm{id})$. Then there exists $x_0 \in \overline{\partial_e K}$ and a positive linear functional $\varphi \in A'_+(K)$ with the properties:

$$\varphi_{|H} = \delta_{x|H} \text{ and } \varphi(f_0) \neq f_0(x_0). \tag{3}$$

Let $[H \cup \{f_0\}]$ be the linear hull of H and f_0. To this separable subspace of $A(K)$ choose as in 2.9 (i) a separable closed Lindenstrauss space Z with $[H \cup \{f_0\}] \subseteq Z \subseteq A(K)$ such that $Z \simeq A(K_1)$ via $f \mapsto \tilde{f}$, $\tilde{f}(\psi) = \psi(f)$, $f \in Z$, $\psi \in K_1$. Define $\tau\colon K \to K_1$, $\tau(x) := \delta_{x|Z} \in K_1$. Obviously, τ is affine continuous and surjective.

Claim: $\tau(\partial_e K) \subseteq \partial_e K_1$. To see this, consider for $x \in \partial_e K$:

$$Q_x^H := \{y \in K\colon h(x) = h(y)\, \forall\, h \in H\}.$$

By Prop. 4 of [E-VS], Q_x^H is a closed face of K. Therefore $Q_x^H = \overline{\mathrm{co}}(\partial_e Q_x^H)$ (Krein-Milman) and $\partial_e Q_x^H \subseteq \partial_e K$. Assumption (i) then implies $Q_x^H = \{x\}$. To prove the claim, consider $\tau(x) = \lambda\tau(x_1) + (1 - \lambda)\tau(x_2)$ with $\tau(x_i) \in K$, $i = 1, 2$, $\lambda \in (0, 1)$ (τ is surjective). This obviously implies $\lambda x_1 + (1 - \lambda)x_2 \in Q_x^H$, so that $x = x_1 = x_2$ because $x \in \partial_e K$. Therefore $\tau(x) = \tau(x_1) = \tau(x_2)$, and the claim is proved.

Now put $\varepsilon_0 := |\varphi(f_0) - f_0(x_0)| > 0$ and choose a net $(x_\alpha)_{\alpha \in A}$ in $\partial_e K$ with $x_\alpha \to x_0$. There exists an $\alpha_0 \in A$ such that for every $\alpha \geqq \alpha_0$ we have

$$|f_0(x_0) - f_0(x_\alpha)| < \frac{\varepsilon_0}{4}. \tag{4}$$

The net $(\tau(x_\alpha))_{\alpha \geqq \alpha_0}$ in $\partial_e K_1$ converges to $\tau(x_0)$, and because K_1 is metrizable we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \{x_\alpha : \alpha \geqq \alpha_0\}$ such that $\tau(x_n) \to \tau(x_0)$. An application of the claim in the proof of Theorem 2.7 yields a sequence $(L_n)_{n\in\mathbb{N}}$ of positive linear contractions ($T = \mathrm{id}$, $h_0 = 1$) $L_n : A(K_1) \to A(K_1)$ with the properties:

$$(5) \qquad \lim_{n\to\infty} \|L_n\tilde{h} - \tilde{h}\| = 0 \; \forall h \in H \text{ and } L_n\tilde{f}_0(\tau(x_n)) = \varphi(f_0)$$

for every $n \in \mathbb{N}$. With the positive projections $P_n : A(K) \to Z_n$, $\|P_n\| = 1$, of 2.9 we define positive linear operators $U_n : A(K) \to A(K)$ by $(U_n f)(x) := L_n(P_n f)^{\sim}(\tau(x))$, $n \in \mathbb{N}$, $x \in K$, $f \in A(K)$. For $h \in H$ we obtain:

$$\begin{aligned}
\sup_{x\in K} |U_n h(x) - h(x)| &= \sup_{\tau(x)\in K_1} |L_n(P_n h)^{\sim}(\tau(x)) - \tilde{h}(\tau(x))| \\
&= \|L_n(P_n h)^{\sim} - \tilde{h}\| \\
&\leqq \|L_n(P_n h)^{\sim} - L_n\tilde{h}\| + \|L_n\tilde{h} - \tilde{h}\| \\
&\leqq \|L_n\| \, \|(P_n h)^{\sim} - \tilde{h}\| + \|L_n\tilde{h} - \tilde{h}\| \\
&\leqq \|P_n h - h\| + \|L_n\tilde{h} - \tilde{h}\| \xrightarrow{n} 0.
\end{aligned}$$

But $f_0 \in \mathrm{Kor}^+(H, \mathrm{id})$ and therefore we get $\lim_{n\to\infty} \|U_n f_0 - f_0\| = 0$, in particular $\lim_{n\to\infty} |U_n f_0(x_n) - f_0(x_n)| = 0$. Then

$$\begin{aligned}
\varepsilon_0 &= |\varphi(f_0) - f_0(x_0)| \\
&\leqq |\varphi(f_0) - f_0(x_n)| + |f_0(x_n) - f_0(x_0)| \\
&\leqq |\varphi(f_0) - f_0(x_n)| + \frac{\varepsilon_0}{4} \text{ by definition of } x_n, (4) \\
&\overset{(5)}{\leqq} |L_n\tilde{f}_0(\tau(x_n)) - L_n(P_n f_0)^{\sim}(\tau(x_n))| + |U_n f_0(x_n) - f_0(x_n)| + \frac{\varepsilon_0}{4} \\
&\leqq \|L_n\| \, \|\tilde{f}_0 - (P_n f_0)^{\sim}\| + |U_n f_0(x_n) - f_0(x_n)| + \frac{\varepsilon_0}{4},
\end{aligned}$$

a contradiction for sufficiently large $n \in \mathbb{N}$.

REMARK. The sequence $(P_n)_{n\in\mathbb{N}}$ in the above proof converges exactly on the space Z to the identity. Therefore $\mathrm{Kor}^+(H, \mathrm{id}) \subseteq Z$.

3. Weak approximation in Lindenstrauss spaces.

3.1. In this section we want to characterize the weak sequential Korovkin closure $\sigma\omega\text{-}\mathrm{Kor}^1(H, T)$ of a subspace H of a real or complex Lindenstrauss space E, where $T: E \to E$ is a linear contraction. We show that this closure is equal to an appropriate uniqueness closure $V_1(H, T)$. Let us give the relevant definitions. If

E is a real or complex Banach space and $T: E \to E$ a linear contraction, then the weak sequential contractive Korovkin closure $\sigma\omega\text{-Kor}^1(H, T)$ of a subspace $H \subseteq E$ with respect to T is the space:

$$\sigma\omega\text{-Kor}^1(H, T) := \{y \in E\colon \text{ If } (L_n)_{n \in \mathbb{N}} \text{ is a sequence of linear contractions } L_n\colon E \to E \text{ such that } \lim_{n\to\infty} p(L_n h - Th) = 0 \forall p \in E' \forall h \in H \text{ holds, then } \lim_{n\to\infty} p(L_n y - Ty) = 0 \forall p \in E'\}.$$

The appropriate uniqueness closure is:

$$V_1(H, T) := \{y \in E\colon \text{ If } p \in B_1(E') \text{ and } q \in \partial_e B_1(E') \text{ such that } p(h) = q(Th) \forall h \in H, \text{ then } p(y) = q(Ty)\}.$$

The inclusion $V_1(H, T) \subseteq \sigma\omega\text{-Kor}^1(H, T)$ holds in general. To see this (cf. 3.4), it is useful to make the observation that Rainwater's theorem also holds for complex Banach spaces:

RAINWATER'S THEOREM. Let E be a real or complex Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence in E. Then $x_n \to x$ weakly, if and only if $(x_n)_{n\in\mathbb{N}}$ is norm-bounded and $q(x_n) \to q(x)$ for every $q \in \partial_e B_1(E')$.

R. R. Phelps proved this theorem in the real case using the integral representation theorem of Bishop-de Leeuw, [P, p. 33–34]. In the complex case, consider E' as a real space. Then $B_1(E')$ is a w*-compact convex set in E' and $\partial_e B_1(E')$ remains the same set in this point of view. By Bishop-de Leeuw's theorem there exists a σ-ring $\mathcal{S}$ with $\partial_e B_1(E') \in \mathcal{S}$ and a probability measure μ on $\mathcal{S}$ with $\mu(B_1(E') \backslash \partial_e B_1(E')) = 0$ such that

$$L(p) = \int L d\mu$$

holds for all w*-continuous affine real-valued functions L on $B_1(E')$. But the real parts $\mathcal{R}\hat{x}_n$, $\mathcal{R}\hat{x}$ of $\hat{x}_n$, $\hat{x}$ are functions of this kind ($\hat{}$ is the embedding of E in E''). Lebesgue's dominated convergence yields

$$\mathcal{R}p(x_n) = \mathcal{R}\hat{x}_n(p) = \int \mathcal{R}\hat{x}_n d\mu \to \int \mathcal{R}\hat{x}\, d\mu = \mathcal{R}\hat{x}(p) = \mathcal{R}p(x)$$

for $p \in B_1(E')$, if we assume $(x_n)_{n\in\mathbb{N}}$ to be a norm bounded sequence in E such that $q(x_n) \to q(x) \forall q \in \partial_e B_1^+(E')$. Do the same with the imaginary parts; this proves the non-trivial implication. The rest is clear.

Recall, that an L-projection $P: E \to E$ is a linear map such that $P^2 = P$ and $\|x\| = \|Px\| + \|x - Px\| \; \forall x \in E$. A subspace $F \subseteq E$ is called an L ideal, if there exists and L-projection P with $P(E) = F$. Let E be a Lindenstrauss space and q be an extreme point of $B_1(E')$. Then the subspace $[q]$ spanned by q in E' is a w*-closed L-ideal.

To prove a substitute of Lemma 2.4, we need a suitable selection theorem for symmetric or T-symmetric correspondences $\Psi: B_1(E') \to 2^{E'}$. Ψ is symmetric or T-symmetric, if $\Psi(-p) = -\Psi(p)$ or $\Psi(tp) = t\Psi(p)$ holds for $p \in B_1(E')$ and $t \in \mathsf{T} := \{z \in \mathsf{C}: |z| = 1\}$, respectively. Lazar-Lindenstrauss (real case) and Olsen (complex case) proved the following selection theorem.

3.2. THEOREM ([L-L], [O]). *Let E be a real or complex Lindenstrauss space and F a real or complex Fréchet space. If $\Psi: B_1(E') \to \bar{c}(F)$ is a convex symmetric or* T-*symmetric* w*-*lower semicontinuous correspondence, then there exists a* w*-*continuous affine symmetric or* T-*symmetric, respectively, selection $\varphi: B_1(E') \to F$, $\varphi(p) \in \Psi(p) \forall p \in B_1(E')$.*

Let S be an essentially closed face of $B_1(E')$ (i.e. *a face of $B_1(E')$ such that* $\mathrm{co}(S \cup -S)$ *is* w*-*closed*), $V := \mathrm{co}(S \cup -S)$ *and $f: V \to F$ a* w*-*continuous affine symmetric selection of $\Psi_{|V}$. Then φ can be chosen such that $\varphi_{|V} = f$ holds (real case).*

Let N be a w*-*closed L-ideal and $f: N \cap B_1(E') \to F$ a* w*-*continuous affine* T-*symmetric selection of $\Psi_{|N \cap B_1(E')}$. Then φ can be chosen such that $\varphi_{|N \cap B_1(E')} = f$ holds.*

3.3. LEMMA. *Let E be a real or complex separable Lindenstrauss space, $H \subseteq E$ a subspace and $L: E \to E$ a linear contraction. If $p \in B_1(E')$ and $q \in \overline{\partial_e B_1(E')}$ are elements such that $p(h) = q(Lh)$ holds for any $h \in H$, then for any sequence $(q_n)_{n \in \mathsf{N}}$ in $\partial_e B_1(E')$ with limit q there exists a sequence $(L_n)_{n \in \mathsf{N}}$ of linear contractions $L_n: E \to E$ with the properties:*

$$\lim_{n \to \infty} \|L_n h - Lh\| = 0 \forall h \in H \quad \text{and} \quad q_n(L_n y) = p(y) \forall n \in \mathsf{N} \forall y \in E.$$

SKETCH OF A PROOF. The proof follows the same lines as that of Lemma 2.4. Choose a dense sequence $(h_n)_{n \in \mathsf{N}}$ in H and define for $n \in \mathsf{N}$ the correspondence $\Psi_n: B_1(E') \to c(B_1(E'))$ by

$$\Psi_n(v) := \{w \in B_1(E'): |w(h_i) - v(Lh_i)| < \frac{1}{n} \forall n \in \mathsf{N}_n\} \neq \emptyset.$$

Ψ_n is convex symmetric (T-symmetric) and w*-lower semicontinuous. The same is true for $\Phi_n(v) := \overline{\Psi_n(v)} \subseteq B_1(E')$, $v \in B_1(E')$. Because of $p \in \Phi_n(q) \forall n \in \mathsf{N}$, after passing to a subsequence, one can assume $p \in \Phi_n(q_n) \forall n \in \mathsf{N}$. Theorem 3.2 yields a w*-continuous affine symmetric (T-symmetric) selection $\varphi_n: B_1(E') \to B_1(E')$ with the properties:

$$\varphi_n(v) \in \Phi_n(v) \forall v \in B_1(E') \quad \text{and} \quad \varphi_n(q_n) = p \forall n \in \mathsf{N}.$$

In the same way as in the proof of Lemma 2.4 one can now define the sequence $(L_n)_{n\in\mathbb{N}}$ with the desired properties.

We now give the promised characterization of $\sigma\omega$-$\mathrm{Kor}^1(H,T)$ for separable Lindenstrauss spaces.

3.4. THEOREM. *Let E be a real or complex Banach space, $H \subseteq E$ a subspace and $T: E \to E$ a linear contraction. Then*

$$V_1(H,T) \subseteq \sigma\omega\text{-}\mathrm{Kor}^1(H,T).$$

If E is a separable real or complex Lindenstrauss space, then we have the equality:

$$V_1(H,T) = \sigma\omega\text{-}\mathrm{Kor}^1(H,T).$$

PROOF. "$\subseteq$". Assume $y \in V_1(H,T)$ but $y \notin \sigma\omega$-$\mathrm{Kor}^1(H,T)$. Then there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of linear contractions such that $T_n h \to Th$ weakly for all $h \in H$ but $T_n y \nrightarrow Ty$ weakly. $(T_n)_{n\in\mathbb{N}}$ is norm-bounded, therefore by Rainwater's theorem there exists an element $q \in \partial_e B_1(E')$ such that $q(T_n y) \nrightarrow q(Ty)$, i.e., there exists $\varepsilon > 0$ such that

$$|q(T_n y) - q(Ty)| \geqq \varepsilon \tag{6}$$

holds for infinitely many $n \in \mathbb{N}$. Alaoglu-Bourbaki yields a subnet $(\varphi_i)_{i\in I}$ of $(q \circ T_n)_{n\in\mathbb{N}}$ with functionals φ_i satisfying $|\varphi_i(y) - q(Ty)| \geqq \varepsilon$ for every $i \in I$ that converges to $p \in B_1(E')$. Of course, $p(h) = q(Th)\forall h \in H$, but this implies $p(y) = q(Ty)$ because $y \in V_1(H,T)$. For sufficiently large $i \in I$ we get a contradiction to the definition of the φ_i and (6).

"$\supseteq$". This is an easy application of Lemma 3.3.

We close this section with an analogon to Theorem 2.10.

3.5. THEOREM. *Let E be a real or complex Lindenstrauss space, H in E a subspace. If H is separable, separates the points of $\partial_e B_1(E')$ and if the map $\tau: E' \to H'$, $\tau(p) := p_{|H}$, $p \in E'$ has the property $\tau(\partial_e B_1(E')) \subseteq \partial_e B_1(H')$, then the equation*

$$V_1(H,\mathrm{id}) = \sigma\omega\text{-}\mathrm{Kor}^1(H,\mathrm{id})$$

holds.

PROOF. It suffices to prove the inclusion "$\supseteq$". Let $y \in \sigma\omega$-$\mathrm{Kor}^1(H,\mathrm{id})$, $p \in B_1(E')$ and $q \in \partial_e B_1(E')$ with $p(h) = q(h)\forall h \in H$. $[H \cup \{y\}]$ is separable, and using 2.9 (i) we can find a separable Lindenstrauss space Z such that $[H \cup \{y\}] \subseteq Z \subseteq E$. With the inclusion $\tau(\partial_e B_1(E')) \subseteq \partial_e B_1(H')$ and the assumption that H separates the points of $\partial_e B_1(E')$, one can show just in the same way as in the proof of Theorem 2.10: $q_{|Z} \in \partial_e B_1(Z')$. By Lemma 3.3 there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of linear contractions $S_n: Z \to Z$ with the properties:

$$\lim_{n\to\infty} \|S_n h - h\| = 0 \forall h \in H \quad \text{and} \quad q_{|Z}(S_n y) = p_{|Z}(y) \forall n \in \mathbb{N}.$$

As in 2.9 there exist subspaces Z_n of Z and linear projections P_n: E $\to$ Z_n such that $\|P_n\| = 1$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|P_n z - z\| = 0 \forall z \in Z$. Now define $L_n: E \to E$, $L_n := S_n \circ P_n$, $n \in \mathbb{N}$. L_n is a linear contraction, and for $h \in H \subseteq Z$ we have:

$$\|L_n h - h\| \leqq \|S_n\| \, \|P_n h - h\| + \|S_n h - h\| \xrightarrow{n} 0.$$

$y \in \sigma\omega\text{-Kor}^1(H, \mathrm{id})$ now implies $\lim_{n\to\infty} q(L_n y - y) = 0$. Because of

$$\begin{aligned} |p(y) - q(y)| &\leqq |q(S_n y) - q(L_n y)| + |q(L_n y) - q(y)| \\ &\leqq \|S_n\| \, \|y - P_n y\| + |q(L_n y - y)| \to 0 \end{aligned}$$

we get $p(y) = q(y)$. This completes the proof.

4. Remarks.

4.1. One can also consider Korovkin closures with respect to contractions and the strong topology (cf. [Alt], formulas (2.1) and (2.3); there $T = \mathrm{id}$). Using Lemma 3.3 it is now easy to prove, that for separable Lindenstrauss spaces E and linear contractions $T: E \to E$ the equation

$$\text{Kor}^1(H, T) = U_1(H, T)$$

holds for a subspace $H \subseteq E$. But "in general" we have $\overline{\partial_e B_1(E')} = B_1(E')$ (cf. [K] for a more precise statement), in particular $0 \in \overline{\partial_e B_1(E')}$. Hahn-Banach therefore yields $\bar{H} = U_1(H, T) = \text{Kor}^1(H, T)$.

4.2. Let K be a compact convex set in a locally convex Hausdorff topological vector space and $T: A(K) \to A(K)$ a positive linear operator. For a subspace H of $A(K)$ define:

$$\begin{aligned} \sigma\omega\text{-Kor}^+(H, T) := \{ f \in A(K) \colon &\text{ If } (L_n)_{n\in\mathbb{N}} \text{ is a sequence of positive linear} \\ &\text{ operators } L_n\colon A(K) \to A(K) \text{ such that} \\ &\lim_{n\to\infty} \varphi(L_n h - Th) = 0 \forall \varphi \in A'(K) \forall h \in H, \\ &\text{ then } \lim_{n\to\infty} \varphi(L_n f - Tf) = 0 \forall \varphi \in A'(K)\} \end{aligned}$$

and

$$\begin{aligned} V_+(H, T) := \{ f \in A(K) \colon &\text{ If } \varphi \in A'_+(K) \text{ and } x \in \partial_e K \text{ such that } \varphi(h) = \\ &\delta_x \circ T(h) \forall h \in H, \text{ then } \varphi(f) = \delta_x \circ T(f)\}. \end{aligned}$$

Let H be a cofinal subspace of $A(K)$ and K a metrizable Choquet simplex. Using the claim in the proof of Theorem 2.7 and Rainwater's theorem it is now easy to show that

$$\sigma\omega\text{-Kor}^{+}(H,T) = V_{+}(H,T)$$

holds (cf. [Alt, Prop. 2.3, Th. 2.4]).

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