MATH. SCAND. 75 (1994), 263–279

# INTEGRABILITY AND REMOVABILITY RESULTS FOR QUASIREGULAR MAPPINGS IN HIGH DIMENSIONS

T. IWANIEC, L. MIGLIACCIO, L. NANIA and C. SBORDONE[1]

## 1. Introduction.

This paper is dedicated to a detailed study of the integrability and removability results for quasiregular mappings, see [IM1], [I1] and the references given there. Our arguments are based on new estimates for singular integrals [IM2] and non-linear commutators [IS] adapted from the Rochberg-Weiss interpolation theory [RW]. We are thus led to various refinements of the results in [IM] and [I].

Let us begin by recalling the following Caccioppoli type inequality:

THEOREM A ([IM], [I]). *For each dimension $n = 2, 3, \ldots,$ and $K \geqq 1$, there exist exponents $q(n, K) < n < p(n, K)$ such that if $f\colon \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular (in a weak sense) and belongs to the Sobolev class $W^{1,s}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ for some $q(n, K) \leqq s \leqq p(n, K)$, then*

$$(1.1)\qquad \int_\Omega |\varphi(x)|^s |Df(x)|^s\, ds \leqq C(n, K) \int_\Omega |\nabla\varphi(x)|^s\, |f(x)|^s\, dx$$

*for each test function $\varphi \in C_0^\infty(\Omega)$.*

Caccioppoli's inequality seems to be fundamental for the regularity properties of quasiconformal mappings. Quite a few of these properties depend rather strongly on $q(n, K)$ and $p(n, K)$. These exponents are at present far from being identified, even in two-dimensional case. In the present paper we are particularly interested in finding how $q(n, K)$ and $p(n, K)$ depend on the dimension. In recent years the theory of quasiregular mappings has progressed in many directions. One of these is the nonlinear elasticity theory of John Ball, which deals with mappings of *finite dilatation*. With these future developments in mind we shall recall basic notions of such mappings.

[1] The first author was supported by GNAFA-CNR and partially by N.S.F. Grant DMS-9208296; the other authors were supported by M.U.R.S.T. (40%, 1992).

Received August 13, 1993.

There are a few definitions of the dilatation of a mapping. It often does not matter which of them is chosen. The success of our work, however, will depend on the following one.

For $\Omega$ an open subset of $\mathbb{R}^n$, we consider a mapping $f\colon \Omega \to \mathbb{R}^n$ of Sobolev class $W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $1 \leqq s < \infty$. The differential, denoted by $Df(x)$, and its Jacobian $J(x, f) = \det Df(x)$ are defined at almost every $x \in \Omega$. Throughout we assume that $f$ preserves the orientation of $\mathbb{R}^n$, that is, $J(x, f) \geqq 0$. Define the operator norm of $Df(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ by $|Df(x)| = \sup\{|Df(x)\xi|;\ \xi \in S^{n-1}\}$.

Then $f$ is said to have finite dilatation if

$$|Df(x)|^n \leqq K(x)J(x,f) \tag{1.2}$$

where $1 \leqq K(x) < \infty$, for almost every $x \in \Omega$. For such mappings, the dilatation function is defined by

$$K(x) = \begin{cases} \dfrac{|Df(x)|^n}{J(x,f)} & \text{if } Df(x) \text{ exists and } J(x,f) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Now, a mapping $f \in W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be *weakly $K$-quasiregular* if its maximal dilatation $K = \operatorname{ess\,sup}\{K(x); x \in \Omega\}$ is finite. If $s = n$, our definition agrees with the one given in [Re] for a $K$-quasiregular mapping, see also [BI] and [V]. This choice of the maximal dilatation seems to be the best adapted to our study of the integrability exponents $q(n, K) < n < p(n, K)$, see also [IM1].

We prove the following strengthening of Caccioppoli's inequality

THEOREM 1. *For any $K \geqq 1$ there exists a positive number $\gamma = \gamma(K)$ such that inequality* (1.1) *is true with*

$$q(n, K) = n\left(1 - \frac{\gamma}{\log n}\right) \tag{1.3}$$

*and*

$$p(n, K) = n\left(1 + \frac{\gamma}{\log n}\right). \tag{1.4}$$

It may very well be that Theorem 1 still holds if we drop $\log n$. This would follows from dimension free estimates of a singular integral, which we have not been able to prove yet, see [IM2] for partial results.

Essential to the proof of Theorem 1 are new equations for quasiregular mappings, which also hold for mappings with finite dilatation. Some of these equations are simply relations between $l \times l$-minors of the Jacobian matrix $Df(x)$, with $l$ near half of the dimension. The advantage of using such equations is that for $l \approx \frac{n}{2}$ we need $L^p$-estimates of a singular integral with $p$ close to 2 only. The $L^2$-theory, together with standard interpolation arguments, enables us to establish sufficient estimates.

Our proof makes appeal to precise inequalities concerning non-linear commutators, see inequality (3.5).

Two consequences of Theorem 1 merit mentioning here.

THEOREM 2 (The Regularity Theorem). *Let $\gamma = \gamma(K)$, $q = n - n\gamma/\log n$ and $p = n + n\gamma/\log n$, be as in Theorem 1. Then every weakly K-quasiregular mapping of class $W^{1,q}_{loc}(\Omega, \mathbb{R}^n)$ actually belongs to $W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ and, therefore, is K-quasiregular.*

In even dimensions, this result has previously been established in [IM1]. It was shown, in particular, that every weakly 1-quasiregular mapping of Sobolev class $W^{1,\frac{n}{2}}_{loc}(\Omega, \mathbb{R}^n)$ must be Möbius and the Sobolev exponent $q = n/2$ is the lowest possible for such a conclusion to be true. In odd dimensions, however, our estimate for $q(n, K)$ is the best known even in the case of $K = 1$. One interesting question, still unanswered, is whether Theorem 2 holds with $q(n, 1) = n/2$ for $n = 3, 5, 7, \ldots$

A geometric consequence of the Caccioppoli estimate, which in fact motivated our work, is the following improvement of the removability theorem.

THEOREM 3 (The Removability Theorem). *Set $\alpha = \alpha(n, K) = n\gamma/\log n, \gamma = \gamma(K)$ being determined in Theorem 1. Let $E \subset \mathbb{R}^n$ be a closed set of Hausdorff dimension $\dim_H(E) < \alpha$. Then, every bounded K-quasiregular mapping $f\colon \Omega \backslash E \to \mathbb{R}^n$ extends to a K-quasiregular mapping on $\Omega$.*

The classical result here is the theorem of P. Painlevé and A. S. Besicovitch which states that a bounded analytic function $f\colon \Omega\backslash E \to C$ extends analytical to $\Omega$ if the one dimensional Hausdorff measure of $E$ equals zero. For a recent account for the removability results we refer the reader to [I2], [JV], [KMa], [Ri1] and [Ri2].

We give the proof only for Theorem 1, the other follow by the same arguments as in [I1]. However, for the convenience of the reader we repeat the relevant material from [IM1] and [I1] without proofs, thus making our paper self-contained.

This paper grew out of our attempt to simplify arguments from [I1]. The estimates we obtain in the course of this work seem to be interesting on its own right.

REMARK. Quite recently, Kari Astala [A] proved that in dimension 2 the assertion of Theorem 2 holds with any $q > \frac{2\sqrt{K}}{\sqrt{K}+1}$ and $p < \frac{2\sqrt{K}}{\sqrt{K}-1}$ and that these bounds for the integrability exponents are sharp. He also identified the largest number $\alpha = \alpha(2, K)$ for the removability theorem to be true, namely

$$\alpha(2, K) = \frac{2}{1+\sqrt{K}}.$$

We conjecture that Caccioppoli's inequality (1.1) (in dimension 2) holds if

$$\frac{2\sqrt{K}}{\sqrt{K}+1} < s < \frac{2\sqrt{K}}{\sqrt{K}-1}$$

Unfortunately, Astala's arguments do not apply to this problem. However, our conjecture on sharp Caccioppoli inequality would follow if the $p$-norms of the complex Hilbert transform were equal to $A_p = \max\left\{p-1, \frac{1}{p-1}\right\}$. This question is of independent interest in harmonic analysis.

## 2. The Signature Operator.

We follow the notation of [IL], which is slightly different from that of [IM1] and [I1]. Accordingly, the linear space of all $l$-covectors in $\mathbb{R}^n$ is denoted by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$. The standard inner product in $\mathbb{R}^n$, denoted by $\langle\cdot|\cdot\rangle$, induces an inner product in $\Lambda^l(\mathbb{R}^n)$. We use the same angular brackets $\langle\alpha|\beta\rangle$ to designate the inner product $\sum\alpha_I\beta_I$ of $\alpha = \sum\alpha_I e^I$ and $\beta = \sum\beta_I e^I$.

The Hodge star operator $*\colon \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n)$ is defined by the rules: $*1 = e^1 \wedge \ldots \wedge e^n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = *\langle\alpha|\beta\rangle$, for $\alpha, \beta \in \Lambda^l(\mathbb{R}^n)$.

If $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping, then its algebraic dual is denoted by $G_\#\colon \Lambda^1(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)$, which extends naturally to a linear mapping of $\Lambda^l(\mathbb{R}^n)$. For abbreviation, we continue to write $G_\#\colon \Lambda^l \to \Lambda^l$ for this extension.

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the singular values of $G$, then the products $\lambda_{i_1}\cdot\lambda_{i_2}\cdot\ldots\cdot\lambda_{i_l}$ are the singular values of $G_\#\colon \Lambda^l \to \Lambda^l$, for each ordered $l$-tuple $1 \leqq i_1 < \ldots < i_l \leqq n$.

A differential form $\omega$ (of degree $l$) on $\Omega \subset \mathbb{R}^n$ is simply a locally integrable function or a Schwartz distribution with values in $\Lambda^l(\mathbb{R}^n)$. We shall use few spaces of differential forms whose notation is self-explanatory.

For example, $L^p(\Omega, \Lambda^l)$ denotes the usual Lebesgue space of differential forms such that

$$\|\alpha\|_p = \left(\int_\Omega (\sum|\alpha_I(x)|^2)^{\frac{p}{2}}\,dx\right)^{\frac{1}{p}} < \infty.$$

The class $L_1^p(\Omega, \Lambda^l)$ consists of $l$-forms $\omega$ for which $\nabla\omega$ is a regular distribution in $L^p(\Omega, \mathbb{R}^n \otimes \Lambda^l)$. Similarly, $W^{1,p}(\Omega, \Lambda^l)$ stands for the usual Sobolev space of differential $l$-forms.

Next, recall the exterior derivative $d\colon W^{1,p}(\Omega, \Lambda^l) \to L^p(\Omega, \Lambda^{l+1})$ and its formal adjoint operator (Hodge codifferential), defined by

$$d^* = (-1)^{1+nl} * d * \colon W^{1,p}(\Omega, \Lambda^{l+1}) \to L^p(\Omega, \Lambda^l) \tag{2.1}$$

Of special importance to our arguments will be the following Hodge decomposition in $\mathbb{R}^n$.

THEOREM B ([IM1]). *For each* $\omega \in L^p(\mathbb{R}^n, \Lambda^l)$, $1 < p < \infty$, *there exist forms* $\alpha \in L_1^p(\mathbb{R}^n, \Lambda^{l-1})$ *and* $\beta \in L_1^p(\mathbb{R}^n, \Lambda^{l+1})$ *such that*

$$\omega = d\alpha + d^*\beta \tag{2.2}$$

$$\|\nabla\alpha\|_p + \|\nabla\beta\|_p \leqq C(n,p)\,\|\omega\|_p \tag{2.3}$$

Note that $d\alpha$ and $d^*\beta$ can be expressed by means of singular integrals of $\omega$. To this effect we introduce the signature operator $S\colon L^p(\mathbb{R}^n, \Lambda^l) \to L^p(\mathbb{R}^n, \Lambda^l)$, defined by $S\omega = d\alpha - d^*\beta$, for $\omega = d\alpha + d^*\beta$. Thus $S$ acts as identity on exact forms and minus identity on coexact forms. It is fairly easy to see that $S$ is an isometry in $L^2(\mathbb{R}^n, \Lambda^l)$, for each $l = 1, 2, \dots, n-1$. For a thorough discussion of the operator $S$ we refer the reader to [IM1-2], where some estimates for the $p$-norms of $S$ are given. These norms are expected to be dimension free. If so, we would obtain qualitatively sharp estimates for the exponents $q(n,K)$ and $p(n,K)$; namely $q(n,K) = [1 - \gamma(K)]n$ and $p(n,K) = [1 + \gamma(K)]n$.

The following estimate of the $p$-norm of $S$ gives a linear growth with respect to the dimension

$$\|S\colon L^p(\mathbb{R}^n, \Lambda^l) \to L^p(\mathbb{R}^n, \Lambda^l)\| = S_p(n,l) \leqq (n+1)A_p \tag{2.4}$$

where $A_p = S_p(2,1)$ denotes the $p$-norm of $S$ in dimension $n = 2$. In terms of the familiar Riesz-transforms the operator $S\colon L^p(\mathbb{R}^2, \Lambda^1) \to L^p(\mathbb{R}^2, \Lambda^1)$, also known as the complex Hilbert transform or the Beurling-Ahlfors transform, can be identified with $S = (R_1 + iR_2)^2\colon L^p(C) \to L^p(C)$.

Inequality (2.4) might not suffice, but it does so after an interpolation.

PROPOSITION 2.1. *Given any integer* $n \geqq 2$ *suppose that* $\left|1 - \frac{2}{p}\right| \leqq \frac{1}{4\log n}$.

*Then*

$$S_p(n,l) \leqq 72 \tag{2.5}$$

*for every* $l = 1, 2, \dots, n-1$.

PROOF. The conditions stated above imply that $p \in [\sigma, \tau]$, where $\sigma = \dfrac{8\log n}{4\log n + 1}$ and $\tau = \dfrac{8\log n}{4\log n - 1}$. Note that $\sigma$ and $\tau$ are Hölder conjugate. By duality arguments and convexity property of the $p$-norms it follows that $S_p(n,l) \leqq S_\sigma(n,l) = S_\tau(n,l)$. Therefore, we shall have established inequality (2.5) if we prove it for $p = \tau$. Let us first estimate the 4-norm of the complex Hilbert transform. Writing $S = (R_1 + iR_2)^2$, by triangle inequality, we obtain

$$A_4 \leqq (\|R_1\|_4 + \|R_2\|_4)^2 = 4\cot^2\frac{\pi}{8} = 4(1+\sqrt{2})^2 \leqq 24.$$

(Very recently, Rodrigo Banuelos and Gang Wang [BW] established the estimate $A_p \leqq 4\max\left\{p-1, \dfrac{1}{p-1}\right\}$.)

Here we have used precise formulas for the $p$-norms of the Riesz-transforms, namely

$$\|R_i\|_p = \begin{cases} \cot\dfrac{\pi}{2p} & \text{for} \quad p \geqq 2 \\ \tan\dfrac{\pi}{2p} & \text{for} \quad 1 < p \leqq 2 \end{cases}$$

for any dimension $n$ and all $i = 1, 2, \ldots, n$, see [IM2]. Now, (2.4) yields

$$S_4(n,l) \leqq 24(n+1).$$

We next sharpen this result by using Riesz-Thorin convexity theorem [BS]. To this effect, we decompose

$$\frac{1}{\tau} = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{4}, \quad \text{where} \quad \varepsilon = \frac{1}{2\log n}.$$

Accordingly,

$$S_\tau(n,l) \leqq S_2(n,l)^{1-\varepsilon} S_4(n,l)^{\varepsilon} \leqq (24n+24)^{\frac{1}{2\log n}} < 72.$$

This completes the proof, the detailed verification of the last step being left to the reader.

## 3. A Nonlinear Commutator.

Given $\omega \in L^2(\mathrm{R}^n, \Lambda^l)$, formula (2.2) provides us with the exact form $d\alpha \in L^2(\mathrm{R}^n, \Lambda^l)$, which is the nearest one to $\omega$. Then, Calderòn-Zygmund theory of singular integrals gives $L^p$-estimates for $d^*\beta = \omega - d\alpha$

$$\|\omega - d\alpha\|_p \leqq C(n,p)\,\|\omega\|_p. \tag{3.1}$$

Obviously, if $\omega$ is an exact form, this holds with constant $C(n, p) = 0$. Now, one may ask whether an $\omega = |du|^{\varepsilon} du$, being a nonlinear perturbation of an exact form $du$, is nearer to the space of exact forms than the generic $\omega \in L^p(\mathbb{R}^n, \Lambda^l)$. An affirmative answer to this question and rather fairly complete discussion of the nonlinear $L^p$-projections onto the space of exact forms is presented in [IL].

The following special case of the result therein is the key tool in our present work.

THEOREM 3.1. *Given any* $u \in L_1^{p+\varepsilon}(\mathbb{R}^n, \Lambda^{l-1})$, *where* $n > 33$, $2 \leqq p \leqq \dfrac{2n}{n-1}$, $|\varepsilon| \leqq \delta \log^{-1} n$, $0 < \delta < \frac{1}{4}$, *consider the Hodge decomposition*

$$|du|^{\varepsilon} du = d\alpha + d^*\beta. \tag{3.2}$$

*Then*

$$\|d^*\beta\|_{\frac{p+\varepsilon}{1+\varepsilon}} \leqq 10^4 \delta \, \|du\|_{p+\varepsilon}^{1+\varepsilon}. \tag{3.3}$$

We only outline the main idea of the proof. For this, recall the signature operator $S$: $L^p(\mathbb{R}^n, \Lambda^l) \to L^p(\mathbb{R}^n, \Lambda^l)$, $1 < p < \infty$, defined in the previous section. Accordingly,

$$d^*\beta = T(|du|^{\varepsilon} du) \tag{3.4}$$

where $T = \frac{1}{2}(\mathrm{Id} - S)$.

An important point here is that $T$ vanishes on exact forms, which follows from the uniqueness of the Hodge decomposition. For abbreviation, we denote $f = du$, thus $Tf = 0$. When this is substituted in (3.4) $d^*\beta$ takes the form

$$d^*\beta = T(|f|^{\varepsilon} f) - |Tf|^{\varepsilon}\, Tf$$

where we think of the right hand side as a commutator of $T$ and the non-linear mapping $f \mapsto |f|^{\varepsilon} f$, see [IS].

It is worthwhile discussing here more general situation.

Let $(X, \mu)$ be a measure space and let $\mathscr{H}$ be a separable complex Hilbert space. Denote by $L^p(X, \mathscr{H})$, $1 \leqq p \leqq \infty$, the usual Lebesgue space of $\mathscr{H}$-valued functions on $X$. Furthermore, suppose that $T$: $L^p(X, \mathscr{H}) \to L^p(X, \mathscr{H})$ is a bounded linear operator with norm $\|T\|_p$, for all $p$ from an interval $[\sigma, \tau]$, $1 \leqq \sigma < \tau < \infty$. Finally, consider the non-linear transformation

$$A^{\varepsilon}: L^s(X, \mathscr{H}) \to L^{\frac{s}{1+\varepsilon}}(X, \mathscr{H})$$

given by $A^{\varepsilon}(f) = \left(\dfrac{|f|}{\|f\|_s}\right)^{\varepsilon} f$, where both $s$ and $\dfrac{s}{1+\varepsilon}$ belong to the interval $[\sigma, \tau]$. In [IS] we have shown that

$$(3.5)\qquad \|TA^{\varepsilon}(f) - A^{\varepsilon}(Tf)\|_{\frac{s}{1+\varepsilon}} \leqq |\varepsilon| \frac{2s(\tau - \sigma)}{(s - \sigma)(\tau - s)} (\|T\|_{\sigma} + \|T\|_{\tau}) \|f\|_s^{1+\varepsilon}$$

However, in this paper we only need the following special case of this result.

PROPOSITION 3.2. *Under the above hypotheses, if moreover $f \in \ker T$, then*

$$(3.6)\qquad \|T(|f|^{\varepsilon} f)\|_{\frac{s}{1+\varepsilon}} \leqq |\varepsilon| \frac{2s(\tau - \sigma)}{(s - \sigma)(\tau - s)} (\|T\|_{\sigma} + \|T\|_{\tau}) \|f\|_s^{1+\varepsilon}$$

PROOF OF THEOREM 3.1. Denote by $\sigma = \dfrac{8\log n}{1 + 4\log n}$ and $\tau = \dfrac{8\log n}{4\log n - 1}$. An easy computation shows that $p + \varepsilon \in [\sigma, \tau]$ and $\dfrac{p+\varepsilon}{1+\varepsilon} \in [\sigma, \tau]$, because $8\log n \leqq n - 1$ for $n \geqq 33$. It then follows that

$$(3.7)\qquad \|d^{*}\beta\|_{\frac{p+\varepsilon}{1+\varepsilon}} \leqq |\varepsilon| \frac{2(p+\varepsilon)(\tau - \sigma)}{(p + \varepsilon - \sigma)(\tau - p - \varepsilon)} (\|T\|_{\sigma} + \|T\|_{\tau}) \|du\|_{p+\varepsilon}^{1+\varepsilon}$$

Before making some other estimates we recall inequality (2.5), which yields

$$(3.8)\qquad \|T\|_{\sigma} + \|T\|_{\tau} \leqq \tfrac{1}{2}(1 + 72) + \tfrac{1}{2}(1 + 72) = 73$$

It remains to estimate each factor of the right hand side of (3.7). We easily find that

$$\begin{cases} |\varepsilon| < \dfrac{\delta}{\log n} \\ p + \varepsilon \leqq 5 \\ \tau - \sigma < \dfrac{2}{\log n} \\ p + \varepsilon - \sigma \geqq 2 - \dfrac{\delta}{\log n} - \left(2 - \dfrac{2}{4\log n + 1}\right) > \dfrac{1}{8\log n} \\ \tau - p - \varepsilon \geqq 2 + \dfrac{2}{4\log n - 1} - \left(2 + \dfrac{2}{n - 1}\right) - \dfrac{\delta}{\log n} > \dfrac{1}{8\log n} \end{cases}$$

Inequality (3.3) is now immediate.

## 4. A Lemma.

In this section we prove two elementary inequalities that will be used to show Caccioppoli type inequality.

LEMMA 4.1. *Suppose $X$ and $Y$ are vectors of an inner product space. Then*

$$(4.1) \qquad ||X|^{\varepsilon} X - |Y|^{\varepsilon} Y| \leqq \frac{1-\varepsilon}{1+\varepsilon} 2^{-\varepsilon} |X - Y|^{1+\varepsilon}$$

*for* $-1 < \varepsilon \leqq 0$, *and*

$$(4.2) \qquad ||X|^{\varepsilon} X - |Y|^{\varepsilon} Y| \leqq (1 + \varepsilon)(|Y| + |X - Y|)^{\varepsilon} |X - Y|$$

*for* $\varepsilon \geqq 0$.

PROOF. We can certainly assume that $X \neq Y$. First we find that for $0 \leqq t \leqq 1$

$$\left| \frac{d}{dt} |tX - tY + Y|^{\varepsilon} (tX - tY + Y) \right| =$$

$$||tX - tY + Y|^{\varepsilon} (X - Y) + \varepsilon |tX - tY + Y|^{\varepsilon - 2} \times$$

$$\langle tX - tY + Y | X - Y \rangle (tX - tY + Y)| \leqq$$

$$(1 + |\varepsilon|)\, |tX - tY + Y|^{\varepsilon} |X - Y| \leqq$$

$$\begin{cases} (1 - \varepsilon)\, |t|X - Y| - |Y||^{\varepsilon} |X - Y| & -1 < \varepsilon \leqq 0 \\ (1 + \varepsilon)(|Y| + |X - Y|)^{\varepsilon} |X - Y| & \varepsilon \geqq 0. \end{cases}$$

The case $\varepsilon > 0$ is immediate, by Mean-Value Theorem. For $-1 < \varepsilon \leqq 0$, integrating with respect to the parameter $t \in [0, 1]$ yields

$$||X|^{\varepsilon} X - |Y|^{\varepsilon} Y| \leqq (1 - \varepsilon)\, |X - Y|^{1+\varepsilon} \int_0^1 |t - a|^{\varepsilon}\, dt$$

where

$$a = \frac{|Y|}{|X - Y|}.$$

By an elementary geometric argument we obtain

$$\int_0^1 |t - a|^{\varepsilon}\, dt \leqq \max_b \int_0^{b+1} |\tau|^{\varepsilon}\, d\tau = \int_{-1/2}^{1/2} |\tau|^{\varepsilon}\, d\tau = \frac{1}{2^{\varepsilon}(1 + \varepsilon)}.$$

Hence the lemma follows.

## 5. $\mathcal{A}$-Harmonic Equation for Mappings with Finite Dilatation.

Let $f\colon \Omega \to \mathbf{R}^n$ be a mapping of finite dilatation, so its differential $Df(x)\colon \mathbf{R}^n \to \mathbf{R}^n$ is nonsingular except at those points $x \in \Omega$ where $Df(x) = 0$. The matrix dilatation of $f$ is a function $G\colon \Omega \to GL(n)$ defined by the Beltrami equation

$$(5.1) \qquad D^t f(x) Df(x) = J(x, f)^{\frac{2}{n}} G(x)$$

where $D^t f(x)$ stands for the transpose to $Df(x)$. In case $J(x,f) = 0$ we choose the identity matrix to define $G(x)$. Thus $G(x)$ is a symmetric positive definite matrix of determinant 1. If $G(x)$ is the identity everywhere the Beltrami equation reduces to the $n$-dimensional Cauchy-Riemann system:

$$D^t f(x) Df(x) = J(x,f)^{\frac{2}{n}} I. \tag{5.2}$$

Denote the eigenvalues of $G(x)$ by $0 < \lambda_1^2(x) \leqq \lambda_2^2(x) \leqq \ldots \leqq \lambda_n^2(x) < \infty$, thus $\lambda_1(x)\lambda_2(x)\ldots\lambda_n(x) \equiv 1$.

It follows from the Beltrami equation that

$$\begin{aligned} |Df(x)|^n &= \sup_{|h|=1} |Df(x)h|^n = J(x,f) \sup_{|h|=1} \langle G(x)h \mid h \rangle^{\frac{n}{2}} \\ &= J(x,f)\lambda_n^n(x). \end{aligned}$$

where we recall that $f$ has finite dilatation. Therefore, the dilatation of $f$ can be expressed in terms of the largest eigenvalue of $G(x)$, namely

$$K(x) = \lambda_n^n(x). \tag{5.3}$$

Next, we consider the $l$-th exterior power $G_{\#}^{-1}(x)$: $\Lambda^l(\mathrm{R}^n) \to \Lambda^l(\mathrm{R}^n)$, $l = 1, 2, \ldots, n-1$, of $G^{-1}(x)$. Recall that eigenvalues of $G_{\#}^{-1}$ are the products $(\lambda_{i_1} \cdot \lambda_{i_2} \cdot \ldots \cdot \lambda_{i_l})^{-2}$ corresponding to all ordered $l$-tuples $1 \leqq i_1 < \ldots < i_l \leqq n$. This clearly yields

$$\lambda_n^{-2}(x) \ldots \lambda_{n-l+1}^{-2}(x)\, |\xi|^2 \leqq \langle G_{\#}^{-1}(x)\xi \mid \xi \rangle \leqq \lambda_n^2(x) \ldots \lambda_{n-l+1}^2(x)\, |\xi|^2$$

for every $\xi \in \Lambda^l(\mathrm{R}^n)$. In particular,

$$K^{-\frac{2l}{n}}(x)\, |\xi|^2 \leqq \langle G_{\#}^{-1}(x)\xi \mid \xi \rangle \leqq K^{\frac{2l}{n}}(x)\, |\xi|^2. \tag{5.4}$$

Similar arguments yield

$$|G_{\#}^{-1}(x)\xi| \leqq K^{\frac{l}{n}}(x)\, |\xi|. \tag{5.4'}$$

Some first order differential systems follow from the Beltrami equation. To see them fix an integer $l = 1, 2, \ldots, n-1$ and assume that the mapping $f = (f^1, f^2, \ldots, f^n)$: $\Omega \to \mathrm{R}^n$ belongs to the Sobolev class $W^{1,s}_{\mathrm{loc}}(\Omega, \mathrm{R}^n)$, where $s = \max\{l, n-l\}$. This assumption makes it legitimate to apply the operators $d$ and $d^*$ to the differential forms

$$u = f^l df^1 \wedge \ldots \wedge df^{l-1} \quad \text{and} \quad v = * f^{l+1} df^{l+2} \wedge \ldots \wedge df^n \tag{5.5}$$

respectively. This results in the following formulas

$$du = (-1)^{l-1} df^1 \wedge \ldots \wedge df^l \quad \text{and} \quad d^*v = (-1)^{l+1} * df^{l+1} \wedge \ldots \wedge df^n. \tag{5.6}$$

Note that the coefficients of $du$, being $l \times l$-minors of $Df$, are locally integrable.

Similarly, the coefficients of $d^*v$ are $(n-l) \times (n-l)$-minors of $Df$, thus locally integrable as well.

In conformal case, corresponding to $G(x) = I$, it was shown [I1] that

$$|du|^{p-2}du = d^*v \qquad p = \frac{n}{l} \tag{5.7}$$

or, equivalently

$$|d^*v|^{q-2}d^*v = du \qquad q = \frac{n}{n-l}. \tag{5.8}$$

Here we notice that $(p, q)$ is a Hölder conjugate pair.

In even dimensions, of special interest is the case $l = \frac{n}{2}$, because it leads to a linear Cauchy-Riemann system $du = d^*v$. For more general mappings with finite dilatation, we have the following linear Beltrami type equation

$$du = G_{\#}(x)d^*v \tag{5.9}$$

See [IM1] for a more complete theory of this equation. In odd dimensions, however, we need to consider a non-linear mapping $A\colon \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ given by

$$A(x, \xi) = \langle G_{\#}^{-1}(x)\xi \mid \xi\rangle^{\frac{p-2}{2}} G_{\#}^{-1}(x)\xi \qquad p = \frac{n}{l}. \tag{5.10}$$

Its inverse with respect to $\xi$ can be computed by solving the equation $\zeta = A(x, \xi)$ for $\xi$;

$$A^{-1}(x, \zeta) = \langle G_{\#}(x)\zeta \mid \zeta\rangle^{\frac{q-2}{2}} G_{\#}(x)\zeta \qquad q = \frac{n}{n-l}.$$

Then, equations analogous to (5.7) and (5.8) for a general mapping are the $A$-harmonic equations

$$A(x, du) = d^*v \tag{5.11}$$

or, equivalently

$$A^{-1}(x, d^*v) = du. \tag{5.12}$$

Of course, for a given mapping $f\colon \Omega \to \mathbb{R}^n$, there are more $A$-harmonic equations. They all are obtained from these particular ones by permuting the coordinate functions $(f^1, f^2, \ldots, f^n)$.

We finish this section with the following

LEMMA 5.1. *Let* $1 \leqq K(x) < \infty$ *denote the dilatation function of a mapping* $f \in W^{1,s}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, $s \geqq \max\{l, n-l\}$. *Then A, defined by* (5.10), *satisfies*

(5i) $$|A(x,\xi)| \leqq M(x)\,|\xi|^{p-1}, \qquad M(x) \leqq K^{\frac{n-l}{n}}(x)$$

(5ii) $$\langle A(x,\xi) \,|\, \xi\rangle \geqq m(x)\,|\xi|^{p}, \qquad m(x) \geqq K^{-1}(x)$$

(5iii) $$A(x,t\xi) = |t|^{p-2} t A(x,\xi), \qquad t \in \mathbf{R}.$$

This Lemma is a straightforward consequence of the definition of $A(x, \xi)$ and estimates (5.4) and (5.4′).

As a corollary, for $f$ a weakly $K$-quasiregular mapping, we obtain the following dimension free estimates

(5.13) $$K^{-1}|\xi|^p \leqq \langle A(x,\xi) \,|\, \xi\rangle \leqq |A(x,\xi)|\,|\xi| \leqq K\,|\xi|^p.$$

## 6. Caccioppoli Inequality.

Having disposed of the preliminary steps, we can now prove Theorem 1. We need only consider $n \geqq 33$; the other cases of dimension less than 33 are covered by Theorem A. Let us reveal in advance that our arguments will work for all exponents $s$ such that

(6.1) $$n - \frac{\gamma(K)n}{\log n} \leqq s \leqq n + \frac{\gamma(K)n}{\log n}$$

where

(6.2) $$\gamma(K) = \frac{1}{6 \cdot 10^4 K^2}$$

We choose integer $l = \frac{n}{2}$ if $n$ is even and $l = \frac{n-1}{2}$ if $n$ is odd.

Let $f = (f^1, \ldots, f^n)\colon \Omega \to \mathbf{R}^n$ be a weakly $K$-quasiregular mapping of class $W^{1,s}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$. Observe that $s \geqq \max\{l, n-l\}$. We want to estimate the differential forms $u$ and $v$ introduced by (5.5). These forms solve the $A$-harmonic equation (5.11), that is

(6.3) $$A(x, du) = d^*v$$

where $du \in L^{\frac{s}{l}}_{\mathrm{loc}}(\Omega, \Lambda^l)$ and $d^*v \in L^{\frac{s}{n-l}}_{\mathrm{loc}}(\Omega, \Lambda^l)$. Let $\varphi \in C_0^\infty(\Omega)$ be an arbitrary non-negative test function. Multiplying (6.3) by $\varphi^{p-1}$, in view of the homogeneity property (5iii) we obtain

(6.4) $$A(x, \varphi du) = \varphi^{p-1} d^*v, \qquad p = \frac{n}{l}.$$

Our nearest goal is to derive the following estimate

$$\int_\Omega |\varphi du|^{p+\varepsilon} \leqq C_K \int_\Omega |u|^{p+\varepsilon} |d\varphi|^{p+\varepsilon} + C_K \int_\Omega (|v|\,|d\varphi^{p-1}|)^{\frac{p+\varepsilon}{p-1}}, \tag{6.5}$$

where $\varepsilon = \dfrac{s}{l} - p$, which in view of (6.1)–(6.2) verifies the condition

$$10^4 K^2 |\varepsilon| \log n \leqq \tfrac{1}{2}.$$

Denote $\delta = |\varepsilon| \log n$ to obtain

$$10^4 K^2 \delta \leqq \tfrac{1}{2}. \tag{6.6}$$

We only give the main steps of the proof of (6.5); the details are standard and left to the reader.

First, we split the differential form $|d(\varphi u)|^\varepsilon d(\varphi u)$ by using Hodge decomposition as in Theorem 3.1

$$|d(\varphi u)|^\varepsilon d(\varphi u) = d\alpha + d^*\beta \tag{6.7}$$

where, in view of (3.3), we can estimate the terms $d\alpha$ and $d^*\beta$ as follows

$$\|d^*\beta\|_{\frac{p+\varepsilon}{1+\varepsilon}} \leqq 10^4 \delta \, \|d(\varphi u)\|_{p+\varepsilon}^{1+\varepsilon}. \tag{6.8}$$

Hence

$$\|d\alpha\|_{\frac{p+\varepsilon}{1+\varepsilon}} \leqq 10^4 \, \|d(\varphi u)\|_{p+\varepsilon}^{1+\varepsilon}. \tag{6.9}$$

Next, we compute the inner product of the forms in the left hand side of (6.4) and (6.7);

$$\int_{\mathbf{R}^n} \langle A(x, \varphi du) \,|\, |d\varphi u|^\varepsilon d(\varphi u)\rangle = \int_{\mathbf{R}^n} \langle \varphi^{p-1} d^* v \,|\, d\alpha + d^*\beta\rangle = \int_{\mathbf{R}^n} \langle v \,|\, d\varphi^{p-1} \wedge d\alpha\rangle + \int_{\mathbf{R}^n} \langle A(x, \varphi du) \,|\, d^*\beta\rangle. \tag{6.10}$$

Writing $|d\varphi u|^\varepsilon d(\varphi u) = |\varphi du|^\varepsilon \varphi du + B(\varphi, du)$ we find from Lemma 4.1 that

$$|B(\varphi, du)| \leqq \frac{1-\varepsilon}{1+\varepsilon} 2^{-\varepsilon} |d\varphi \wedge u|^{1+\varepsilon} \tag{6.11}$$

for $-1 < \varepsilon \leqq 0$ and

$$|B(\varphi, du)| \leqq (1+\varepsilon)(|\varphi du| + |d\varphi \wedge u|)^{\varepsilon} |d\varphi \wedge u|$$

for $\varepsilon > 0$. Identity (6.10) yields

$$\int_{\mathbf{R}^n} \langle A(x, \varphi du) \,|\, |\varphi du|^{\varepsilon} \varphi du \rangle \leqq \int_{\mathbf{R}^n} |v|\, |d\varphi^{p-1}|\, |d\alpha| + \int_{\mathbf{R}^n} |A(x, \varphi du)| \,(|d^*\beta| + |B(\varphi, du)|). \tag{6.12}$$

The task is now to estimate each of the above integrals. From now on we continue the proof only for $\varepsilon$ negative. The other case of $\varepsilon > 0$, being similar, is left to the reader.

On account of conditions (5.13), we have

$$K^{-1} \int_{\mathbf{R}^n} |\varphi du|^{p+\varepsilon} \leqq \int_{\mathbf{R}^n} |v|\, |d\varphi^{p-1}|\, |d\alpha| + K \int_{\mathbf{R}^n} |\varphi du|^{p-1} |d^*\beta| + \frac{(1-\varepsilon)K}{(1+\varepsilon)2^{\varepsilon}} \int_{\mathbf{R}^n} |\varphi du|^{p-1} (|u|\, |d\varphi|)^{1+\varepsilon}.$$

Then, by Hölder's inequality

$$K^{-1} \|\varphi du\|_{p+\varepsilon}^{p+\varepsilon} \leqq \|\, |v|\, |d\varphi^{p-1}| \,\|_{\frac{p+\varepsilon}{p-1}} \|d\alpha\|_{\frac{p+\varepsilon}{1+\varepsilon}} + K \|\varphi du\|_{p+\varepsilon}^{p-1} \|d^*\beta\|_{\frac{p+\varepsilon}{1+\varepsilon}} + \frac{(1-\varepsilon)K}{(1+\varepsilon)2^{\varepsilon}} \|\varphi du\|_{p+\varepsilon}^{p-1} \|\, |u|\, d\varphi| \,\|_{p+\varepsilon}^{1+\varepsilon}.$$

Next, we use estimates (6.8) and (6.9)

$$K^{-1} \|\varphi du\|_{p+\varepsilon}^{p+\varepsilon} \leq 10^4 \|\, |v|\, |d\varphi^{p-1}| \,\|_{\frac{p+\varepsilon}{p-1}} \|d(\varphi u)\|_{p+\varepsilon}^{1+\varepsilon} + 10^4 \delta K \|\varphi du\|_{p+\varepsilon}^{p-1} \|d(\varphi u)\|_{p+\varepsilon}^{1+\varepsilon} + \frac{(1-\varepsilon)K}{(1+\varepsilon)2^{\varepsilon}} \|\varphi du\|_{p+\varepsilon}^{p-1} \|\, |u|\, |d\varphi| \,\|_{p+\varepsilon}^{1+\varepsilon}.$$

Writing

$$\|d(\varphi u)\|_{p+\varepsilon}^{1+\varepsilon} = \|d\varphi \wedge u + \varphi du\|_{p+\varepsilon}^{1+\varepsilon} \leqq \|d\varphi \wedge u\|_{p+\varepsilon}^{1+\varepsilon} + \|\varphi du\|_{p+\varepsilon}^{1+\varepsilon}$$

we obtain

$$\|\varphi du\|_{p+\varepsilon}^{p+\varepsilon} \leqq 10^4 K^2 \|\varphi du\|_{p+\varepsilon}^{p+\varepsilon} + \\ + K^2\left[\frac{(1-\varepsilon)}{(1+\varepsilon)2^\varepsilon} + 10^4\delta\right]\|\varphi du\|_{p+\varepsilon}^{p-1} \|\, |u|\, |d\varphi|\, \|_{p+\varepsilon}^{1+\varepsilon} + \\ + 10^4 \|\, |v|\, |d\varphi^{p-1}|\, \|_{\frac{p+\varepsilon}{p-1}} [\|\, |u|\, |d\varphi|\, \|_{p+\varepsilon}^{1+\varepsilon} + \|\varphi du\|_{p+\varepsilon}^{1+\varepsilon}],$$

where we notice that $10^4\delta K^2 \leqq \frac{1}{2} < 1$.

What remains is to separate $\varphi du$ in the right hand side from the other terms. This can be done routinely with the aid of Young's inequality. We then conclude with estimate (6.5).

Recalling that $p + \varepsilon = \dfrac{s}{l}$ estimate (6.5) takes the form

$$\int_\Omega |\varphi du|^{\frac{s}{l}} \leqq C_K \int_\Omega |u|^{\frac{s}{l}} |d\varphi|^{\frac{s}{l}} + C_K \int_\Omega |v|^{\frac{s}{n-l}} |d\varphi^{\frac{n-l}{l}}|^{\frac{s}{n-l}}. \tag{6.13}$$

What is left is to relate inequality (6.13) to (1.1). From (5.5) we have point-wise inequalities

$$|u| \leqq |f|\, |Df|^{l-1} \quad \text{and} \quad |v| \leqq |f|\, |Df|^{n-l-1}.$$

We need to replacee $\varphi$ in (6.5) by $\varphi^l$. After making such a replacement, by Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega |\varphi|^s |du|^{\frac{s}{l}} &\leqq C \int_\Omega |f|^{\frac{s}{l}} |Df|^{\frac{(l-1)s}{l}} |\varphi|^{\frac{s(l-1)}{l}} |d\varphi|^{\frac{s}{l}} \\ &\quad + C \int_\Omega |f|^{\frac{s}{n-l}} |Df|^{\frac{(n-l-1)s}{n-l}} |\varphi|^{\frac{s(n-l-1)}{n-l}} |d\varphi|^{\frac{s}{n-l}} \\ &\leqq C\left(\int_\Omega |f|^s |d\varphi|^s\right)^{\frac{1}{l}} \left(\int_\Omega |\varphi|^s |Df|^s\right)^{\frac{l-1}{l}} \\ &\quad + C\left(\int_\Omega |f|^s |d\varphi|^s\right)^{\frac{1}{n-l}} \left(\int_\Omega |\varphi|^s |Df|^s\right)^{\frac{n-l-1}{n-l}}. \end{aligned} \tag{6.14}$$

Recall that the coefficients of the form $du$ are precisely the $l \times l$-minors of $Df(x)$, that is:

$$\frac{\partial(f^1, f^2, \ldots, f^l)}{\partial(x_{j_1}, x_{j_2}, \ldots, x_{j_l})}$$

corresponding to all $l$-tuples $1 \leqq j_1 < \ldots < j_l \leqq n$.

On the other hand, we could replace $f^1, f^2, \ldots, f^l$ by any other set of $l$-coordinate function of $f$. Thus (6.14) actually provides us with estimates for all possible $l \times l$-minors of $Df(x)$. Finally, we need an inequality which is well known in the theory of determinants

$$J(x,f) = \det Df(x) \leqq C(n) \cdot \sum_{\substack{1 \leqq i_1 < \ldots < i_l \leqq n \\ 1 \leqq j_1 < \ldots < j_l \leqq n}} \left| \frac{\partial(f^{i_1}, \ldots, f^{i_l})}{\partial(x_{j_1}, \ldots, x_{j_l})} \right|^{\frac{n}{l}}$$

see [IL] for the precise constant.

The above arguments show that estimate (6.14) remains valid if the term $|du|^{\frac{s}{l}}$ is replaced by $|J(x,f)|^{\frac{s}{n}}$, that is

$$\int_\Omega |\varphi|^s J(x,f)^{\frac{s}{n}} \leqq C\left(\int_\Omega |f|^s |d\varphi|^s\right)^{\frac{1}{l}} \left(\int_\Omega |\varphi Df|^s\right)^{\frac{l-1}{l}} +$$

$$C\left(\int_\Omega |f|^s |d\varphi|^s\right)^{\frac{1}{n-l}} \left(\int_\Omega |\varphi Df|^s\right)^{\frac{n-l-1}{n-l}}.$$

The last step is to use the dilatation condition, $|Df|^s \leqq K^{\frac{s}{n}} J(x,f)^{\frac{s}{n}}$. Hence, by Young's inequality, we conclude with (1.1) completing the proof of Theorem 1.

## REFERENCES


[A] K. Astala, *Area distortion of quasiconformal mappings*, to appear in Acta Math.

[BI] B. Bojarski, T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in* $R^n$, Ann. Acad. Sci. Fenn., Ser. A.I. 8 (1983), 257–324.

[BS] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Inc. Pure and Applied Mathematics; vol. 129, 1988.

[BW] R. Banuelos, G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms*, preprint 1993.

[CJMR] M. Cwikel, B. Jawerth, M. Milman, R. Rochberg, *Differential estimates and commutators in interpolation theory*, Analysis at Urbana II, London Math. Soc., Cambridge Univ. Press, 1989, 170–220.

[CRW] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611–635.

[G] F. W. Gehring, *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–277.

[GVä] F. W. Gehring, J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. 6 (1973), 504–512.

[I1] T. Iwaniec, *p-Harmonic tensors and quasiregular mappings*, Ann. of Math. 136 (1992), 589–624.

[I2] T. Iwaniec, *$L^p$-theory of quasiregular mappings*, Lecture Notes in Math. 1508 (1992), 39–64.

[I3] T. Iwaniec, *On $L^p$-integrability in* PDEs *and quasiregular mappings for large exponents*, Ann. Acad. Sci. Fenn., Ser. A. I, 7 (1982), 301–322.

[IL] T. Iwaniec, A. Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. 125 (1993), 25–79.

[IM1] T. Iwaniec, G. Martin, *Quasiregular mappings in even dimension*, Acta Math. 170 (1992), 29–81.

[IM2] T. Iwaniec, G. Martin, *The Beurling-Ahlfors transform in* $\mathbb{R}^n$ *and related singular integrals*, Preprint of Inst. Hautes Etudes Sci., 1990.

[IS] T. Iwaniec, C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. 454, (1994), 143–161.

[JRW] B. Jawerth, R. Rochberg, G. Weiss, *Commutator and other second order estimates in real interpolation theory*, Ark. Mat. 24 (1986), 191–219.

[JV] P. Järvi, M. Vuorinen, *Self-similar Cantor sets and quasiregular mappings*, J. Reine Angew. Math. 424 (1992), 31–45.

[KMa] P. Koskela, O. Martio, *Removability theorems for quasiregular mappings*, Preprint, Mittag-Leffler Institute, Report No. 22, 1989/1990. Ann. Acad. Sci. Fenn., Ser. A.I. Math. 15 (1990), 381–399.

[MaRVä] O. Martio, S. Rickman, J. Väisälä, *Distortions and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn., Ser. A.I. 465 (1970), 1–13.

[M] M. Milman, *A commutator theorem with applications*, to appear.

[P] A. Passarelli, *Nonlinear commutators for the maximal function*, Boll. Un. Mat. Ital., to appear.

[Re] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Trans. Math. Monographs, Amer. Math. Soc., vol. 73, 1989.

[Ri1] S. Rickman, *Nonremovable Cantor sets for bounded quasiregular mappings*, Mittag-Leffler Institute, Report no. 42, 1989/90.

[Ri2] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, Berlin, Heidelberg, 1993.

[RW] R. Rochberg, G. Weiss, *Derivatives of analytic families of Banach spaces*, Ann. of Math. 118 (1983), 315–347.

[S] B. Stroffolini, *On weakly A-harmonic tensors*, Studia Math., to appear.

[Vä] J. Väisälä, *Lectures on n-dimensional Quasiconformal Mappings*, Lecture Notes in Math. 1319, (1971).

[V] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319 (1988).

DEPT. OF MATHEMATICS
SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK 13244
USA

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
VIA CINTIA - 80126 NAPOLI
ITALIA