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DIVISION FORMULAS FOR HOLOMORPHIC MAPPINGS WITH VALUES IN A BANACH ALGEBRA

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Contents.

Abstract.

Analogues of the division and preparation theorems for holomorphic mappings are proved in which the mappings take their values in a non-commutative Banach algebra. The results are applied to deformations of such mappings.

1. Divisors of first degree.

Consider a holomorphic mapping f from the disc $D_\rho = \{z \in \mathbf{C}: |z| < \rho\}$ into a complex Banach algebra Y. In this section we shall derive formulas of the kind

$$f(z) = q(z,b)(z - b) + r(b) \tag{1}$$

where q and r are holomorphic mappings with values in Y. We refer to formulas such as (1) as division formulas in which the quantities q and r are respectively quotient and remainder, while $z - b$ is the divisor. It will not be necessary to be very precise about what we mean by a division formula. The essential point is that the remainder should be simpler than the divisor; in our example it is of lower degree in z. We shall be concerned with far-reaching generalizaations of (1).

Formula (1) is easily verified if z is restricted to the disc D_ρ and b is restricted by

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the condition that $\operatorname{spr} b < \rho$, where $\operatorname{spr} b$ denotes the spectral radius of b. Indeed if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for $|z| < \rho$, with coefficients $a_n \in Y$, then $r(b)$ is given by

$$r(b) = \sum_{n=0}^{\infty} a_n b^n.$$

This series is convergent if $\operatorname{spr} b < \rho$. We have that

$$\begin{aligned} f(z) - r(b) &= \textstyle\sum_{n=0}^{\infty} a_n(z^n - b^n) \\ &= \left(\textstyle\sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{n} z^{k-1} b^{n-k}\right)\right)(z - b) \end{aligned}$$

so that $q(z, b) = \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{n} z^{k-1} b^{n-k}\right)$.

By exactly the same method we can prove a version of formula (1) in which all the terms are allowed to depend on an additional variable ξ in a complex Banach space X. Let X_σ be the ball of radius σ and centre 0 in the space X. Let S_ρ be the set $\{b \in Y\colon \operatorname{spr} b < \rho\}$.

THEOREM 1. *Let X be a Banach space, Y a Banach algebra and F a Banach space which is a right module over Y. We assume that the multiplication defined for elements of F and elements of Y is continuous. Let the mapping $f\colon D_\rho \times X_\sigma \to F$ be holomorphic. Then there exist holomorphic mappings $q\colon D_\rho \times X_\sigma \times S_\rho \to F$ and $r\colon X_\sigma \times S_\rho \to F$ such that*

$$f(z, \xi) = q(z, \xi, b)(z - b) + r(\xi, b).$$

PROOF. Suppose that

$$f(z, \xi) = \sum_{n=0}^{\infty} a_n(\xi) z^n$$

for $|z| < \rho$ and $\xi \in X_\sigma$. Then we have $r(\xi, b) = \sum_{n=0}^{\infty} a_n(\xi) b^n$ and $q(z, \xi, b) = \sum_{n=0}^{\infty} a_n(\xi)\left(\sum_{k=1}^{n} z^{k-1} b^{n-k}\right)$.

As an example of what can be derived from theorem 1 let us take $Y = L(\mathbf{C}^n, \mathbf{C}^n)$ and $F = \mathbf{C}^n$. We think of Y as consisting of $n \times n$ matrices which multiply elements of F, thought of as row matrices, on the right. Take

$$b = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_1 & -a_2 & \dots & -a_n \end{bmatrix}$$

and denote row matrices by bold type, as, for example, $\mathbf{a} = [a_1, a_2, \ldots, a_n]$. All unspecified entries of b (except on the superdiagonal and bottom row) are 0. We note that $\operatorname{spr} b \to 0$ as $\mathbf{a} \to 0$. If $f\colon D_\rho \to \mathbf{C}$ is a holomorphic function we have

$$[f, 0, \ldots, 0] = [q_1, \ldots, q_n]\begin{bmatrix} z & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & z & -1 \\ a_1 & a_2 & \ldots & a_{n-1} & z + a_n \end{bmatrix} + [r_1, \ldots, r_n]$$

In this formula, for convenience, the arguments have been suppressed, except for z in the square matrix, but f is $f(z)$, q_k is $q_k(z, \mathbf{a})$, and r_k is $r_k(\mathbf{a})$, where $1 \leqq k \leqq n$. A few lines of elimination lead to the formula

$$f(z) = q_n(z, \mathbf{a})(z^n + a_n z^{n-1} + \ldots + a_1) + \sum_{k=0}^{n-1} p_k(\mathbf{a}) z^k$$

where $p_k(\mathbf{a})$ depends only on $\mathbf{a}$. This is the classical polynomial division theorem.

One of the most surprising results of recent decades was the extension of the preparation theorem to C^∞ functions of real variables, conjectured by Thom and proved by Malgrange [3]. We conjecture that theorem 1 holds, with minor modifications, if the mappings are C^∞ and the spaces are real; and that this is easy to prove using known methods if the spaces are *finite-dimensional*, but requires new methods if the spaces are *infinite-dimensional*.

2. Divisors of higher degree.

Our object is to allow more general divisors than appear in theorem 1. We first prove a result, analogous to the classical preparation theorem of Weierstrass. An actual generalization of that theorem will be proved later. In what follows we shall make use of quasi-nilpotent elements of a Banach algebra. These are elements that have spectral radius 0. Equivalently they satisfy $\lim_{n\to\infty} \|x^n\|^{1/n} = 0$.

THEOREM 2. *Let $n \in Y$ be quasi-nilpotent and let $f\colon D_\rho \times X_\sigma \to Y$ be such that $f(z, 0) = z - n$. Then there exist ρ', σ' and holomorphic mappings $g\colon D_{\rho'} \times X_{\sigma'} \to Y$ and $\mathscr{b}\colon X_{\sigma'} \to Y$ such that*

$$f(z, \xi) = g(z, \xi)(z - n - \mathscr{b}(\xi))$$

Moreover $\mathscr{b}(0) = 0$ and $g(z, 0) = 1$.

For the proof we shall need two lemmas.

LEMMA 1. *Let $n \in Y$ be quasi-nilpotent and let $t \in Y$. Then* $\operatorname{spr}(n + t) \to 0$ *as* $t \to 0$.

PROOF. This is a consequence of the upper semi-continuity of the spectrum. It is proved in [1].

LEMMA 2. *Let $n \in Y$ be quasi-nilpotent and let $c\colon D_\rho \to Y$ be a holomorphic mapping such that $c(z)(z-n)$ is constant. Then c is identically* 0.

PROOF. Let $c(z)(z-n) = t$. Then

$$c(z) = t(z-n)^{-1} = t \sum_{j=0}^{\infty} z^{-j-1} n^j$$

where the series is convergent for all $z \neq 0$. By uniqueness of Laurent series we have $t = 0$ and $c(z)$ is identically 0.

PROOF OF THEOREM 2. By theorem 1 and lemma 1 we have a division formula

$$f(z,\xi) = q(z,\xi,b)(z-n-b) + r(\xi,b) \tag{2}$$

valid for all b with $\|b\|$ sufficiently small. Putting $b = 0$ and $\xi = 0$ we find

$$z - n = q(z,0,0)(z-n) + r(0,0).$$

By lemma 2 we have $q(z,0,0) = 1$ and $r(0,0) = 0$. Differentiating (2) with respect to b at $b = 0$, $\xi = 0$ gives

$$0 = (D_3 q(z,0,0)h)(z-n) - h - D_2 r(0,0)h$$

for all h. By lemma 2 this implies

$$D_2 r(0,0)h = h.$$

Hence $D_2 r(0,0)$ is invertible and the implicit function theorem yields a holomorphic mapping $\hat{b}$ such that $\hat{b}(0) = 0$ and $r(\xi, \hat{b}(\xi)) = 0$ for all ξ in a sufficiently small neighbourhood of 0 in X. But then we have

$$f(z,\xi) = q(z,\xi,\hat{b}(\xi))(z-n-\hat{b}(\xi))$$

and the theorem is proved.

For any complex Banach space E and Banach algebra Y let $\mathscr{A}_0(E, Y)$ denote the ring of germs at 0 of holomorphic mappings from E to Y. In the ensuing discussion Y is a fixed Banach algebra, and X a fixed Banach space.

THEOREM 3. *Let M be a left module over the ring $\mathscr{A}_0(\mathbf{C} \times X, Y)$ generated by finitely many elements x_k, $k = 1,2,\ldots,m$. Suppose that there exist elements n_{ij} of Y and germs $p_{ij} \in \mathscr{A}_0(\mathbf{C} \times X, Y)$ such that (in an obvious notation)*

$$zx_i = \sum_{j=1}^{m} (n_{ij} + p_{ij})x_j$$

for $1 \leqq i \leqq m$. *Suppose further that each germ* p_{ij} *has the property that* $p_{ij}(z, 0) = 0$ *and that the matrix* $\mathbf{N} = (n_{ij})$ *defines a quasi-nilpotent element in the space* $Y^{m \times m}$ *of* $m \times m$ *matrices over* Y. *Then* M, *considered as a left module over the ring* $\mathscr{A}_0(X, Y)$, *is also generated by the elements* x_k, $1 \leqq k \leqq m$.

PROOF. Consider the elements of M^m as column matrices. Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We shall use bold type to denote matrices. We can multiply elements of M^m on the left by $m \times m$ matrices with entries in $\mathscr{A}_0(\mathsf{C} \times X, Y)$, Among these are the matrices $\mathbf{I}$ (identity matrix), $\mathbf{N}$ (already introduced and consisting of constant germs), and the matrix $\mathbf{P} = (p_{ij})$. Then we can write

$$(z\mathbf{I} - \mathbf{N} - \mathbf{P})\mathbf{x} = 0$$

By theorem 2

$$z\mathbf{I} - \mathbf{N} - \mathbf{P}(z, \xi) = \mathbf{Q}(z, \xi)(z\mathbf{I} - \mathbf{N} - \hat{\mathbf{S}}(\xi))$$

for certain matrix germs $\hat{\mathbf{S}}$ and $\mathbf{Q}$ which satisfy $\hat{\mathbf{S}}(0) = 0$ and $\mathbf{Q}(z, 0) = \mathbf{I}$. If $\mathbf{F} \in \mathscr{A}_0(\mathsf{C} \times A, Y^{m \times m})$ is any matrix germ, we can write, by theorem 1 and the last formula

$$\mathbf{F}(z, \xi) = \mathbf{K}(z, \xi)(z\mathbf{I} - \mathbf{N} - \mathbf{P}(z, \xi)) + \mathbf{R}(\xi)$$

for certain matrix germs $\mathbf{K}$ and $\mathbf{R}$. Multiplying on the right by $\mathbf{x}$ we obtain $\mathbf{F}(z, \xi)\mathbf{x} = \mathbf{R}(\xi)\mathbf{x}$ and so M is generated by $x_1, \ldots, x_m$ over germs depending on ξ alone.

We can now extend the division formula (1) in order to allow divisors of higher degree. Let π_k, $1 \leqq k \leqq n$, be idempotent elements of Y and denote the germ $z\pi_k + 1 - \pi_k$ by P_k. Let b_k, $1 \leqq k \leqq n$, be generic elements of Y and write $\mathbf{b} = (b_1, \ldots, b_n)$. The divisor which we shall consider is

$$D(z, \mathbf{b}) = P_1(z)P_2(z)\ldots P_n(z) + b_1\pi_1 P_2(z)\ldots P_n(z) + b_2\pi_2 P_3(z)\ldots P_n(z) + \cdots + b_n\pi_n.$$

We can think of this as a generalization of the generic nth degree polynomial $z^n + b_1 z^{n-1} + \cdots + b_n$ with scalar coefficients. If, in this polynomial, we allow the coefficients to be elements of the algebra Y we obtain a special case of the divisor defined above in which all the idempotents are unity. This special case is not, however, very useful.

Let M be the quotient of $\mathscr{A}_0(\mathbf{C} \times Y^n, Y)$ by the left ideal consisting of all germs which can be written in the form $T(z, \mathbf{b})D(z, \mathbf{b})$ for some T. It is obvious that M is a left module over $\mathscr{A}_0(\mathbf{C} \times Y^n, Y)$. If $f \in \mathscr{A}_0(\mathbf{C} \times Y^n, Y)$ we denote the image of f in M by $[f]$.

LEMMA 3. *M is generated by the elements $[f_k]$, $1 \leqq k \leqq n$, where*

$$f_k(z) = \pi_k P_{k+1}(z) \ldots P_n(z).$$

PROOF. Let $g \in \mathscr{A}_0(\mathbf{C} \times Y^n, Y)$. Then we can write

$$\begin{aligned} g(z, \mathbf{b}) &= g(z, \mathbf{b}) - g(0, \mathbf{b}) + g(0, \mathbf{b})(1 - \pi_n) + g(0, \mathbf{b})\pi_n \\ &= g_1(z, \mathbf{b})z + g(0, \mathbf{b})(1 - \pi_n) + g(0, \mathbf{b})\pi_n \\ &= \big(g_1(z, \mathbf{b})(\pi_n + z(1 - \pi_n)) + g(0, \mathbf{b})(1 - \pi_n)\big)P_n(z) + g(0, \mathbf{b})\pi_n \\ &= q_1(z, \mathbf{b})P_n(z) + r_n(\mathbf{b})\pi_n \end{aligned}$$

with holomorphic quotient q_1 and remainder $r_n(\mathbf{b})\pi_n$. Repeating this we get the formula

$$g(z, \mathbf{b}) = q_2(z, \mathbf{b})P_{n-1}(z)P_n(z) + r_{n-1}(\mathbf{b})\pi_{n-1}P_n(\mathrm{z}) + \mathrm{r}_n(\mathbf{b})\pi_n.$$

Eventually we obtain

$$\begin{aligned} g(z, \mathbf{b}) &= q_n(z, \mathbf{b})P_1(z) \ldots P_n(z) + r_1(\mathbf{b})\pi_1 P_2(z) \ldots P_n(z) \\ &\qquad + r_2(\mathbf{b})\pi_2 P_3(z) \ldots P_n(z) + \cdots + r_n(\mathbf{b})\pi_n \\ &= q_n(z, b)P_1(z) \ldots P_n(z) + r_1(\mathbf{b})f_1(z) + \cdots + r_n(\mathbf{b})f_n(z) \\ &= q_n(z, \mathbf{b})D(z, \mathbf{b}) + T_1(z, \mathbf{b})f_1(z) + \cdots + T_n(z, \mathbf{b})f_n(z) \end{aligned}$$

Hence $[g] = \sum T_k[f_k]$ and the proof is complete.

Next we compute $z[f_k]$.

LEMMA 4. *There exist elements c_{kj}, $1 \leqq j \leqq k - 1$, $1 \leqq k \leqq n$, in Y and germs g_{kj}, such that $g_{kj}(z, \mathbf{0}) = 0$ and*

$$z[f_k] = \sum_{j=1}^{k-1} c_{kj}[f_j] + \sum_{j=1}^{n} g_{kj}[f_j]$$

for $k = 1, \ldots, n$. [Actually it turns out that the g_{kj} do not depend on z.]

PROOF. We shall first prove that

$$P_k(z) \ldots P_n(z) = \sum_{j=1}^{k-1} (1 - z)f_j(z) + P_1(z) \ldots P_n(z) \tag{3}$$

for each k. This is obvious for $k = 1$. Suppose that it holds for a given k. Then we have

$$\begin{aligned} P_{k+1}(z)\dots P_n(z) &= \pi_k P_{k+1}(z)\dots P_n(z) + (1-\pi_k)P_{k+1}(z)\dots P_n(z) \\ &= f_k(z) + P_k(z)P_{k+1}(z)\dots P_n(z) - z\pi_k P_{k+1}(z)\dots P_n(z) \\ &= (1-z)f_k(z) + P_k(z)P_{k+1}(z)\dots P_n(z) \\ &= \textstyle\sum_{j=1}^{k}(1-z)f_j(z) + P_1(z)\dots P_n(z) \end{aligned}$$

This establishes (3). We can now conclude that

$$f_k(z) = \pi_k P_{k+1}(z)\dots P_n(z) = \sum_{j=1}^{k}(1-z)\pi_k f_j(z) + \pi_k P_1(z)\dots P_n(z)$$

which leads to

$$zf_k(z) = \sum_{j=1}^{k-1}(1-z)\pi_k f_j(z) + \pi_k P_1(z)\dots P_n(z).$$

Using this equation we can recursively compute constants c_{kj} and q_k in Y such that

$$zf_k(z) = \sum_{j=1}^{k-1} c_{kj} f_j(z) + q_k P_1(z)\dots P_n(z).$$

But then it follows that

$$zf_k(z) = q_k D(z, \mathbf{b}) + \sum_{j=1}^{k-1} c_{kj} f_j(z) - \sum_{j=1}^{n} q_k b_j f_j(z)$$

Hence

$$z[f_k] = \sum_{j=1}^{k-1} c_{kj}[f_j] - \sum_{j=1}^{n} q_k b_j [f_j]$$

We can now apply theorem 3 to obtain the result that M is generated by the elements $[f_k]$ over the ring $\mathscr{A}_0(Y^n, Y)$. However we need a slightly more general conclusion for the applications to follow. We need the possibility of a further variable ξ belonging to a Banach space X. The calculations just shown are also valid if M is the quotient of $\mathscr{A}_0(\mathbf{C} \times X \times Y^n, Y)$ by the left ideal consisting of all germs of the form $T(z, \xi, \mathbf{b})D(z, \mathbf{b})$. Then we conclude that M is generated by the elements $[f_k]$ over $\mathscr{A}_0(X \times Y^n, Y)$. Our conclusion is embodied in the following theorem as a division formula.

THEOREM 4. *For each* $f \in \mathscr{A}_0(\mathbf{C} \times X \times Y^n, Y)$ *there exist germs* $q \in \mathscr{A}_0(\mathbf{C} \times X \times Y^n, Y)$ *and* $r_k \in \mathscr{A}_0(X \times Y^n, Y)$, $k = 1, \dots, n$, *such that*

$$f(z,\xi,\mathbf{b}) = q(z,\xi,\mathbf{b})D(z,\mathbf{b}) + \sum_{k=1}^{n} r_k(\xi,\mathbf{b})f_k(z).$$

In the above division formula the divisor $D(z,\mathbf{b})$ stands on the right. We can also prove a parallel formula in which the divisor, suitably modified, stands on the left. For left division we use the divisor:

$$\tilde{D}(z,\mathbf{b}) = P_1(z)P_2(z)\ldots P_n(z) + P_1(z)\ldots P_{n-1}(z)\pi_n b_n + P_1(z)\ldots P_{n-2}(z)\pi_{n-1}b_{n-1} + \cdots + \pi_1 b_1$$

or more simply

$$\tilde{D}(z,\mathbf{b}) = P_1(z)P_2(z)\ldots P_n(z) + \sum_{k=1}^{n} \tilde{f}_k(z)b_k$$

where $\tilde{f}_k(z) = P_1(z)\ldots P_{k-1}(z)\pi_k$. We then have the division formula:

$$f(z,\xi,\mathbf{b}) = \tilde{D}(z,\mathbf{b})\tilde{q}(z,\xi,\mathbf{b}) + \sum_{k=1}^{n} \tilde{f}_k(z)\tilde{r}_k(\xi,\mathbf{b}).$$

3. Deformations and left equivalence.

In this section we use theorem 4 to study deformations of mappings taking values in the Banach algebra Y and prove a generalization of the classical Weierstrass Preparation Theorem. As in the last section X is a Banach space and $\pi_1,\ldots,\pi_n$ are idempotent elements of Y. The quantities $P_k(z)$, $f_k(z)$ and the divisor $D(z,\mathbf{b})$ will be as defined in section 2.

We first need a lemma.

LEMMA 5 (Stripping lemma). *Suppose that $g \in \mathscr{A}_0(\mathrm{C}, Y)$ is such that $g(z)P_1(z)\ldots P_n(z) = \sum_{k=1}^{n} r_k f_k(z)$, where $r_1,\ldots,r_n$ are elements of Y. Then $g = 0$ and $r_k\pi_k = 0$ for $k = 1,\ldots,n$.*

PROOF. Suppose that $h \in \mathscr{A}_0(\mathrm{C}, Y)$ has the property that $h(z)P_n(z) = r_n f_n(z)$, that is, $h(z)(z\pi_n + 1 - \pi_n) = r_n\pi_n$. From this we deduce that $h(z)(1-\pi_n) = 0$ and $h(z)z\pi_n = r_n\pi_n$. The last equation implies $r_n\pi_n = 0$ and $h(z)\pi_n = 0$; so that $h(z) = 0$.

We now apply this to the situation of the lemma. We use backward induction, stripping off one $P_k(z)$ at a time and beginning with $P_n(z)$. We deduce successively that $r_k\pi_k = 0$ and that

$$g(z)P_1(z)\ldots P_{k-1}(z) = \sum_{j=1}^{k-1} r_j\pi_j P_{j+1}(z)\ldots P_{k-1}(z)$$

for $k = n, n-1,\ldots,1$.

THEOREM 5. *Let* $f \in \mathscr{A}_0(\mathbf{C} \times X, Y)$ *be such that* $f(z,0) = P_1(z) \ldots P_n(z)$. *Then there exist germs* $U \in \mathscr{A}_0(\mathbf{C} \times X, Y)$ *and* $\mathfrak{b} \in \mathscr{A}_0(X, Y^n)$ *such that*

$$f(z,\xi) = U(z,\xi)D(z,\mathfrak{b}(\xi)).$$

Moreover $U(0,0) = 1$ *and* $\mathfrak{b}(0) = \mathbf{0}$.

PROOF. We have the division formula

$$f(z,\xi) = q(z,\xi,\mathbf{b})D(z,\mathbf{b}) + \sum_{k=1}^{n} r_k(\xi,\mathbf{b})f_k(z).$$

Putting $\mathbf{b} = 0$ and $\xi = 0$ we find that

$$P_1(z)\ldots P_n(z) = q(z,0,0)P_1(z)\ldots P_n(z) + \sum_{k=1}^{n} r_k(0,0)f_k(z)$$

The stripping lemma now implies that $q(z,0,0) = 1$ and $r_k(0,0)\pi_k = 0$ for each k. Let F be the Banach space $Y\pi_1 \times \cdots \times Y\pi_n$ and define the mapping $\mathbf{s}: X \times F \to F$ by $s_k(\xi,\mathbf{b}) = r_k(\xi,\mathbf{b})\pi_k$. Consider $\mathbf{b}$ to be henceforth restricted to the space F. Differentiating the division formula with respect to $\mathbf{b}$ at $\mathbf{b} = 0$, $\xi = 0$ gives

$$0 = (D_3 q(z,0,0)\mathbf{h})P_1(z)\ldots P_n(z) + \sum_{k=1}^{n} h_k f_k(z) + \sum_{k=1}^{n} (D_2 r_k(0,0)\mathbf{h})f_k(z)$$

where $\mathbf{h} = (h_1,\ldots,h_n) \in F$. Hence by the stripping lemma $D_2 s_k(0,0)\mathbf{h} = -h_k$, that is $D_2\mathbf{s}(0,0)\mathbf{h} = -\mathbf{h}$. By the implicit function theorem there is a germ $\mathfrak{b} \in \mathscr{A}_0(X,F)$ such that $\mathfrak{b}(0) = 0$ and $\mathbf{s}(\xi,\mathbf{b}(\xi)) = 0$. We conclude that

$$f(z,\xi) = q(z,\xi,\mathfrak{b}(\xi))D(z,\mathfrak{b}(\xi)).$$

It is natural to interpret theorem 5 in terms of deformations and left equivalence. We define left equivalence of germs as follows. Let E be a Banach space. Let f and g be elements of $\mathscr{A}_0(E, Y)$. Then f and g are left equivalent if there exists a germ $u \in \mathscr{A}_0(E, Y)$ such that $f = ug$ and $u(0)$ is invertible. Similarly we can define right equivalence. A deformation of a germ $f \in \mathscr{A}_0(E, Y)$ with parameter space X is a germ $F \in \mathscr{A}_0(E \times X, Y)$ such that F restricted to $E \times \{0\}$ is f. We say that one deformation $F_1 \in \mathscr{A}_0(E \times X_1, Y)$ induces another $F_2 \in \mathscr{A}_0(E \times X_2, Y)$ if there exist germs $U \in \mathscr{A}_0(E \times X_2, Y)$ and $h \in \mathscr{A}_0(X_2, X_1)$ such that $U(0,0)$ is invertible and

$$F_2(v,\xi) = U(v,\xi)F_1(v,h(\xi))$$

Theorem 5 describes all deformations of the germ $P_1(z)\ldots P_n(z)$ up to left equivalence in the sense that any deformation of $P_1(z)\ldots P_n(z)$ is induced by the deformation

$$F(z,\mathbf{b}) = P_1(z)\dots P_n(z) + \sum_{k=1}^{n} b_k f_k(z)$$

The classical preparation theorem is obtained by taking $Y = \mathrm{C}$ and $\pi_k = 1$ for $k = 1,\dots,n$.

A case of importance is when Y is the full space $L(F,F)$ of bounded linear operators on a Banach space F. In the finite-dimensional case the theorem shows how to find a complete set of parameters to describe up to left equivalence all deformations of a matrix-valued holomorphic function of z. This is examined in more detail in section 5.

4. Divisors with quasi-nilpotent elements.

Theorem 5 is nearly a generalization of theorem 2 to the case of finitely many factors; nearly, but not quite, because in theorem 2 we are deforming $z - n$ where n is a quasi-nilpotent element. We therefore seek a common generalization of theorem 5 and theorem 2. This is provided by the following result.

THEOREM 6. *Let $\pi_1,\dots,\pi_n$ be idempotent elements of Y, let $t_1,\dots,t_n$ be elements of Y and assume that the elements $\pi_k t_k \pi_k$ are quasi-nilpotent. Let $Q_k(z) = (z - \pi_k t_k)\pi_k + 1 - \pi_k$ for each k. Then every deformation of $Q_1(z)\dots Q_n(z)$ is induced by the deformation*

$$E(z,\mathbf{b}) = Q_1(z)\dots Q_n(z) + \sum_{k=1}^{n} b_k g_k(z)$$

where $g_k(z) = \pi_k Q_{k+1}(z)\dots Q_n(z)$.

PROOF. This parallels the proof of theorem 5. First we establish a division rule (extending theorem 4)

$$f(z,\xi) = q(z,\xi,\mathbf{b})E(z,\mathbf{b}) + \sum_{k=1}^{n} r_k(\xi,\mathbf{b})g_k(z). \tag{4}$$

Once this is done the proof proceeds as for the proof of theorem 5 except that lemma 2 is needed to carry out the task of the stripping lemma with Q_k replacing P_k and g_k replacing f_k.

Let us now prove the division formula (4). Let M be the quotient of $\mathscr{A}_0(\mathrm{C} \times X \times Y^n, Y)$ by the left ideal consisting of all germs which can be written in the form $T(z,\xi,\mathbf{b})E(z,\mathbf{b})$ for some T. We denote the image of a germ in M by enclosing the symbol which represents it in square brackets. We shall show that M is generated by the elements $[g_k]$ and that

$$z[g_k] = \sum_{j=1}^{k} c_{kj}[g_j] + \sum_{j=1}^{n} q_k b_j [g_j] \tag{5}$$

for certain constants c_{kj}, $1 \leqq j \leqq k$, and q_k, $1 \leqq k \leqq n$.

As in the proof of theorem 4 we drop the variable ξ from our formulas to make them more readable. Let $f \in \mathscr{A}_0(\mathrm{C} \times Y^n, Y)$. By division formula (1) there exist $f_1 \in \mathscr{A}_0(\mathrm{C} \times Y^n, Y)$ and $r_n \in \mathscr{A}_0(Y^n, Y)$ such that

$$\begin{aligned} f(z, \mathbf{b}) &= f_1(z, \mathbf{b})(z - \pi_n t_n \pi_n) + r_n(\mathbf{b}) \\ &= \big(f_1(z, \mathbf{b})\pi_n + f_1(z, \mathbf{b})z(1 - \pi_n) + r_n(\mathbf{b})(1 - \pi_n)\big)Q_n(z) + r_n(\mathbf{b})\pi_n \\ &= q_1(z, \mathbf{b})Q_n(z) + r_n(\mathbf{b})\pi_n \end{aligned}$$

The proof that M is generated by $[g_k]$ is now completed as in the proof of lemma 3.

We turn now to the proof of (5). The analogue of equation (3) is

$$Q_k(z)\dots Q_n(z) = \sum_{j=1}^{k-1} (1 - z + \pi_j t_j \pi_j)g_j(z) + Q_1(z)\dots Q_n(z)$$

leading to

$$zg_k(z) = \pi_k t_k \pi_k g_k(z) + \sum_{j=1}^{k-1} \big((1 - z)\pi_k + \pi_k \pi_j t_j \pi_j\big)g_j(z) + \pi_k Q_1(z)\dots Q_n(z)$$

Note the on-diagonal term on the right-hand side. Using this formula recursively we obtain formula (5) with the on-diagonal coefficient $c_{kk} = \pi_k t_k \pi_k$.

When we come to applying theorem 3 to complete the proof we find that the matrix $\mathbf{N}$ is triangular with non-vanishing diagonal. However the diagonal entries are $\pi_k t_k \pi_k$ and since these are known to be quasi-nilpotent we conclude that $\mathbf{N}$ too is quasi-nilpotent.

5. Finite ascent and finite quasi-ascent.

The concept of multiplicity of an operator-valued mapping has been studied from various points of view during the last thirty years. Several different definitions have been given. In particular the author gave a definition in 1974 [2] in which matrices do not appear. The definition would be applicable to functions with values in a Banach algebra were it not for the need for rank of an idempotent element. We recapitulate the definition here.

Let F be a Banach space which we suppose to be complex, although the definition works as well for real spaces. Let $A(z)$ be a holomorphic mapping from the disc D_ρ into the operator algebra $L(F, F)$. We define recursively a sequence of operator-valued mappings $A_n(z)$ as follows. Firstly $A_0 = A$. If $A_n(z)$ has been defined examine the kernel of $A_n(0)$. If it is nontrivial let π_n be a bounded projection with range $\ker A_n(0)$. Set $A_{n+1}(z) = A_n(z)(z^{-1}\pi_n + I - \pi_n)$.

One possible difficulty is apparent. It could be that the closed space $\ker A_n(0)$

admits no topological complement and therefore the projection π_k does not exist. In [2] it was generally assumed that $A(z)$ was a Fredholm operator of index 0, which implies that $\ker A_n(0)$ is finite-dimensional. The existence of a bounded projection is then automatic. But now we wish to proceed without any assumptions of finite-dimensionality and at the same time not to get involved in the thorny question of topological complements. We shall say that a germ $A(z)$ is *admissible* if for each k we can find a continuous projection π_k with range $\ker A_k(0)$.

It is plain that the projections π_k can be determined in different ways. However – and this was shown in [2] – if different projections π'_k are chosen fitting the above prescription, and if this leads to a different sequence $A'_k(z)$ of operator-valued functions, then $A'_k(z)$ and $A_k(z)$ are right equivalent and the projections π_k and π'_k are conjugate. In particular they have the same rank. There is therefore a well-determined numerical sequence $(\text{rank}\ \pi_k)_{k=0}^{\infty}$. The number $\sum_{k=0}^{\infty} \text{rank}\ \pi_k$ is called the multiplicity of $A(z)$ at 0 and was denoted by $\mu[A;0]$ in [2].

It was shown also in [2] that the sequence $(\text{rank}\ \pi_k)$ is non-increasing. If, for some n, we have $\text{rank}\ \pi_n = 0$, we call the smallest such n the ascent of A at 0, denoted in [2] by $\alpha[A;0]$. This means that $A_n(0)$ has trivial kernel. *We shall only speak of finite ascent and finite multiplicity on the assumption that $A_n(0)$ is actually invertible, that is, it has an inverse in $L(F, F)$.* This is the convention used in [2]. If A has finite ascent at 0 then A is plainly left equivalent to the germ

$$(z\pi_{n-1} + I - \pi_{n-1})(z\pi_{n-2} + I - \pi_{n-2})\dots(z\pi_0 + I - \pi_0)$$

Thus we see that the deformation results of section 3 apply to germs of finite ascent. If F is finite-dimensional and $\det A(z)$ is not identically 0, then $A(z)$ necessarily has finite multiplicity. The deformation results therefore apply to all but an exceptional class of matrix functions.

Another result proved in [2] concerns the mapping $A(z) = zI - T$ where T is a fixed operator. It was shown, as a corollary to a more general theorem, that provided T is a Fredholm operator of index 0, the germ $zI - T$ has finite multiplicity at 0 if there exists n such that $F = \ker T^n \oplus \text{ran}\ T^n$ and the restriction of T to $\text{ran}\ T^n$ is invertible as a linear mapping from $\text{ran}\ T^n$ to itself. Moreover the ascent is the least such n and the multiplicity is the dimension of $\ker T^n$. This result will be reproved in the next section without the need for a Fredholm hypothesis. It follows that the concepts of multiplicity and ascent coincide in this case with the traditional notions applied to a single operator (see [7, page 271]; for an operator the ascent is often called the index). In fact ascent can be defined in the context where Y is a Banach algebra not necessarily of the form $L(F, F)$. Let $f \in \mathscr{A}_0(\mathbf{C}, Y)$. The ascent f at 0 could simply be defined as the least number n such that there exist idempotents $\pi_1, \pi_2, \dots, \pi_n$ such that f is left equivalent to $P_1(z)\dots P_n(z)$, where $P_k(z) = z\pi_k + 1 - \pi_k$. This is consistent with the definition of ascent for operator-valued functions. However the definition is certainly of little use unless the algebra is rich in idempotents. There is even another candidate for a definition of ascent: the order of the pole $z = 0$ of $f(z)^{-1}$.

In section 4 we saw that as far as deformations are concerned factors of the form $z\pi + 1 - \pi$ can be replaced by factors of the more general form $z\pi - \pi t\pi + 1 - \pi$ where $\pi t\pi$ is quasi-nilpotent. This suggests that we define a new kind of ascent which we shall call quasi-ascent.

We define the quasi-ascent as the least number n such that there exist idempotents $\pi_1, \pi_2, \ldots, \pi_n$ and elements $t_1, t_2, \ldots, t_n$ in Y such that $\pi_k t_k \pi_k$ is a quasi-nilpotent element for each k and $f(z)$ is left equivalent to the product $Q_1(z) \ldots Q_n(z)$ where $Q_k(z) = z\pi_k - \pi_k t_k \pi_k + 1 - \pi_k$. Clearly the quasi-ascent is less than or equal to the ascent. For example if $f(z) = z - c$ where c is quasi-nilpotent then the quasi-ascent is 1 but the ascent could be infinite.

For the germ $z - a$ we can characterize finite quasi-ascent.

THEOREM 7. *Let $a \in Y$. The germ $z - a$ has finite quasi-ascent at* 0 *if and only if* 0 *is in the resolvent set or is an isolated point of the spectrum of a. In these cases the quasi-ascent is* 0 *or* 1 *respectively.*

PROOF. Suppose that 0 is an isolated point of the spectrum of a. Let P be the spectral projection (idempotent) associated with the point 0. The element PaP is quasi-nilpotent since its spectrum consists of the point 0 alone. Let $f(z) = (z - a)(zP - PaP + 1 - P)^{-1}$. It is enough to show that $f(z)$ is holomorphic at $z = 0$ and that $f(0)$ is invertible. Let $k(\zeta)$ be defined in the complex ζ-plane, equal to 1 on an small disc which contains 0 and no other spectral point of a, and equal to 0 elsewhere. Then $P = k(a)$ (using the operational calculus for holomorphic functions of a) and $f(z) = F(z, a)$ where

$$F(z,\zeta) = (z - \zeta)\big(zk(\zeta) - k(\zeta)\zeta k(\zeta) + 1 - k(\zeta)\big)^{-1}$$

$$= \begin{cases} 1 & \text{on a neighbourhood of } 0 \\ z - \zeta & \text{on the rest of the spectrum} \end{cases}$$

$$= (z - \zeta)(1 - k(\zeta)) + k(\zeta)$$

This is just a lengthy way of showing that $f(z) = (z - a)(1 - P) + P$. It follows that $f(z)$ is holomorphic at $z = 0$ and $f(0)$ is invertible.

This enables us to prove a result which generalizes theorem 3.

THEOREM 8. *Let M be a left module over the ring $\mathscr{A}_0(\mathrm{C} \times X, Y)$ generated by finitely many elements x_k, $k = 1, 2, \ldots, m$. Suppose that there exist elements n_{ij} of Y and germs $p_{ij} \in \mathscr{A}_0(\mathrm{C} \times X, Y)$ such that*

$$zx_i = \sum_{j=1}^{m} (n_{ij} + p_{ij})x_j$$

for $1 \leqq i \leqq m$. *Suppose further that each germ* p_{ij} *has the property that* $p_{ij}(z, 0) = 0$ *and that either* 0 *is in the resolvent set or it is an isolated point of spectrum of the matrix* $\mathbf{N} = (n_{ij})$ *considered as an operator on* Y^m. *Then* M, *considered as a left module over the ring* $\mathscr{A}_0(X, Y)$, *is generated by the elements* $x_1, \ldots, x_m$.

PROOF. As in the proof of theorem 3 we have

$$(z\mathbf{I} - \mathbf{N} - \mathbf{P})\mathbf{x} = 0$$

The matrix-germ $z\mathbf{I} - \mathbf{N}$ has quasi-ascent 0 or 1 so there exists a projection matrix $\mathbf{\Pi}$ and a matrix $\mathbf{T}$ in $Y^{m \times m}$, such that $\mathbf{\Pi T \Pi}$ is quasi-nilpotent and $z\mathbf{I} - \mathbf{N}$ is left equivalent to $z\mathbf{\Pi} - \mathbf{\Pi T \Pi} + \mathbf{I} - \mathbf{\Pi}$. Let $\mathbf{F} \in \mathscr{A}_0(\mathrm{C} \times X, Y^{m \times m})$. By theorems 4 and 5 we have

$$\mathbf{F}(z, \xi) = \mathbf{U}(z, \xi)(z\mathbf{I} - \mathbf{N} - \mathbf{P}(z, \xi)) + \mathbf{R}(\xi)\mathbf{\Pi}$$

for certain germs $\mathbf{U} \in \mathscr{A}_0(\mathrm{C} \times X, Y^{m \times m})$ and $\mathbf{R} \in \mathscr{A}_0(X, Y^{m \times m})$. Hence $\mathbf{F}(z, \xi)\mathbf{x} = \mathbf{R}(\xi)\mathbf{\Pi x}$ and so M is generated over $\mathscr{A}_0(X, Y)$ by the entries of $\mathbf{x}$. This concludes the proof.

The argument of the preceeding proof can be made to yield a more general (and probably less useful) theorem.

THEOREM 9. *Let* M *be a left module over the ring* $\mathscr{A}_0(\mathrm{C} \times X, Y)$ *generated by finitely many elements* x_k, $k = 1, 2. \ldots, m$. *Suppose that there exists a matrix-germ* $\mathbf{W} \in \mathscr{A}_0(\mathrm{C} \times X, Y^{m \times m})$ *such that* $\mathbf{Wx} = 0$ *and* $\mathbf{W}(z, 0)$ *has finite quasi-ascent* l *at* 0. *Then* M *is generated over* $\mathscr{A}_0(X, Y)$ *by at most* lm *elements which include* $x_1, \ldots, x_m$.

PROOF. There exist projections (that is idempotents) $\mathbf{\Pi}_1, \ldots, \mathbf{\Pi}_l$ in $Y^{m \times m}$ and matrices $\mathbf{T}_1, \ldots, \mathbf{T}_l$ such that $\mathbf{\Pi}_k \mathbf{T}_k \mathbf{\Pi}_k$ is quasi-nilpotent for each k and $\mathbf{W}(z, 0)$ is left equivalent to the product $\mathbf{Q}_1(z) \ldots \mathbf{Q}_l(z)$ where $\mathbf{Q}_k(z) = z\mathbf{\Pi}_k - \mathbf{\Pi}_k \mathbf{T}_k \mathbf{\Pi}_k + \mathbf{I} - \mathbf{\Pi}_k$. Let $\mathbf{f}_k(z) = \mathbf{\Pi}_k \mathbf{Q}_{k+1}(z) \ldots \mathbf{Q}_l(z)$. Let $\mathbf{F} \in \mathscr{A}_0(\mathrm{C} \times X, Y^{m \times m})$. By theorem 6 and division rule (4) we have

$$\mathbf{F}(z, \xi) = \mathbf{U}(z, \xi)\mathbf{W}(z, \xi) + \sum_{j=1}^{l} \mathbf{R}_j(\xi)\mathbf{f}_j(z)$$

for certain germs $\mathbf{U} \in \mathscr{A}_0(\mathrm{C} \times X, Y^{m \times m})$ and $\mathbf{R}_j \in \mathscr{A}_0(X, Y^{m \times m})$. Hence $\mathbf{F}(z, \xi)\mathbf{x} = \Sigma \mathbf{R}_j(\xi)\mathbf{f}_j(z)\mathbf{x}$ and so M is generated over $\mathscr{A}_0(X, Y)$ by the lm entries of $\mathbf{f}_j(z)\mathbf{x}$. This concludes the proof.

6. Well-behaved germs and left-right equivalence.

The material of this section is largely independent of that of preceding sections. We are not here concerned with division formulas, but with developing certain properties of operator germs from where the subject was allowed to rest in the author's paper [2]. The reason for including here material that could stand alone is that it forms the basis of an application of the division formulas in the next section.

Let F be a Banach space and consider a germ A with values in $L(F, F)$. Suppose that the ascent is a finite number n. We can then find projections $\pi_0, \ldots, \pi_{n-1}$, chosen according to the prescription from [2] summarized in section 5, such that $A(z)$ is left equivalent to the product

$$(6) \qquad (z\pi_{n-1} + I - \pi_{n-1})(z\pi_{n-2} + I - \pi_{n-2}) \ldots (z\pi_0 + I - \pi_0)$$

Note that the order of the subscripts is different from what it was in previous sections. This is to bring it into line with the conventions of multiplicity theory.

As we have seen the projections are not uniquely determined. Now it always happens that $\ker \pi_k \cap \ker A_{k+1}(0) = \{0\}$. The question arises whether we can exploit this by choosing π_{k+1} so that $\ker \pi_k \subset \ker \pi_{k+1}$. If this is possible for each k and if $\ker \pi_k$ is complemented in $\ker \pi_{k+1}$ we shall say that the germ $A(z)$ is *well-behaved*. Given that $A(z)$ is admissible it can be blocked from being well-behaved if for some k it is impossible to choose $\pi_0, \ldots, \pi_k$ so that the sum $\ker \pi_k \oplus \ker A_{k+1}(0)$ admits a topological complement.

If $A(z)$ is well-behaved we may always choose the multiplicity projections to satisfy the conditions

$$(7) \qquad \ker \pi_0 \subset \ker \pi_1 \subset \cdots \subset \ker \pi_{n-1}$$

and this choice simplifies some proofs. (For an example see the part of [2] that deals with bifurcation theory.) In fact this choice enables us to simplify and transform the product (6) into the sum

$$\sum_{k=0}^{n} z^k(\pi_{k-1} - \pi_k)$$

where we use the convention that $\pi_{-1} = I$ and $\pi_n = 0$. To see this note that the conditions (7) are equivalent to the algebraic conditions: $\pi_i(I - \pi_j) = 0$ for all $i \geqq j$. The product (6) therefore reduces to

$$\begin{aligned}\textstyle\sum_{k=0}^{n} z^k(I - \pi_{n-1}) \ldots (I - \pi_k)\pi_{k-1} \ldots \pi_0 &= \textstyle\sum_{k=0}^{n} z^k(I - \pi_k)\pi_{k-1} \\ &= \textstyle\sum_{k=0}^{n} z^k(\pi_{k-1} - \pi_k)\end{aligned}$$

It would clearly be useful to have conditions that guarantee that $A(z)$ is well-behaved. The case considered in [2] contains one such: $A(0)$ is a Fredholm operator of index 0. If we wish to avoid finite-dimensionality conditions the question is more difficult, even in the comparative comfort of Hilbert space.

In the author's paper [2] it was shown that the spaces $U_k = \sum_{i=0}^{k} \operatorname{ran} \pi_i$ and $V_k = \pi_0 \ldots \pi_{k-1} \ker A_k(0)$ are independent of the choice of projections. In fact they are invariants of the left equivalence class (by abuse of language) of $A(z)$. The spaces V_k form a decreasing sequence of subspaces of $\ker A(0)$.

THEOREM 10. (i) *$A(z)$ is well-behaved if and only if V_{k+1} is complemented in V_k for each k.*

(ii) *If $A(z)$ is well-behaved then, however the multiplicity projections π_k are chosen, the sum* $\ker A_k(0) \oplus \ker \pi_{k-1}$ *is complemented for each k.*

PROOF. First suppose that $A(z)$ is well-behaved. Choose the projections π_k so that $\ker \pi_k \subset \ker \pi_{k+1}$ and $\ker \pi_k$ is complemented in $\ker \pi_{k+1}$ for each k. By way of an induction hypothesis we suppose that V_i is complemented in V_{i-1} for $i \leqq k-1$. Then V_{k-1} is closed so that the restriction of $\pi_0 \ldots \pi_{k-2}$ to $\ker A_{k-1}(0)$ is an isomorphism of that space onto V_{k-1}. Now $\ker A_k(0) \oplus \ker \pi_{k-1}$ is complemented. Suppose that $F = W \oplus \ker A_k(0) \oplus \ker \pi_{k-1}$. Applying the projection π_{k-1} we find that $\ker A_{k-1}(0) = \pi_{k-1} W \oplus \pi_{k-1} \ker A_k(0)$, that is, $\pi_{k-1} \ker A_k(0)$ is complemented in $\ker A_{k-1}(0)$. Applying $\pi_0 \ldots \pi_{k-2}$ we find that V_k is complemented in V_{k-1}. To begin the induction note that since V_0 is closed the same argument shows that $\pi_0 \ker A_1(0)$ is complemented in $\ker A(0)$.

Conversely suppose that V_{k+1} is complemented in V_k for each k. Then we can find a sequence of spaces W_k such that $V_k = W_{k+1} \oplus V_{k+1}$ for each k. We define W_0 so that $F = W_0 \oplus V_0$. All the sums are intended to be topological. Now we claim that for each j we can choose π_j so that $\ker \pi_j = W_0 \oplus \cdots \oplus W_j$. Let us assume that this has been done for $j = 0, \ldots, k-1$. We can choose π_k with the required kernel if

$$F = W_0 \oplus \cdots \oplus W_k \oplus \ker A_k(0)$$

Now we have that

$$\begin{aligned} F &= W_0 \oplus \cdots \oplus W_k \oplus V_k \\ &= W_0 \oplus \cdots \oplus W_k \oplus \pi_0 \ldots \pi_{k-1} \ker A_k(0) \\ &= W_0 \oplus \cdots \oplus W_k \oplus \pi_1 \ldots \pi_{k-1} \ker A_k(0) \\ &= W_0 \oplus \cdots \oplus W_k \oplus \pi_2 \ldots \pi_{k-1} \ker A_k(0) \\ &\ \ \vdots \qquad\qquad \vdots \\ &= W_0 \oplus \cdots \oplus W_k \oplus \ker A_k(0) \end{aligned}$$

where we use repeatedly the equations

$$\begin{aligned} W_0 \oplus \cdots \oplus W_j \oplus \pi_j \ldots \pi_{k-1} \ker A_k(0) &= \ker \pi_j \oplus \pi_j \ldots \pi_{k-1} \ker A_k(0) \\ &= \ker \pi_j \oplus \pi_{j+1} \ldots \pi_{k-1} \ker A_k(0) \\ &= W_0 \oplus \cdots \oplus W_j \oplus \pi_{j+1} \ldots \pi_{k-1} \ker A_k(0) \end{aligned}$$

This concludes the proof of (i).

To proof (ii) we note that, by (i), if $A(z)$ is well-behaved the space V_{k+1} is complemented in V_k for each k. It follows that $\pi_0 \ldots \pi_{k-1}$ maps $\ker A_k(0)$ isomorphically onto V_k for each k and hence $\pi_k \ker A_{k+1}(0)$ is complemented in $\ker A_k(0)$. Since $\ker \pi_k \oplus \ker A_{k+1}(0) = \ker \pi_k \oplus \pi_k \ker A_{k+1}(0)$ it follows that $\ker \pi_k \oplus \ker A_{k+1}(0)$ is complemented.

We see from the last theorem that $A(z)$ is well-behaved if and only if $\ker \pi_k \oplus \ker A_{k+1}(0)$ is complemented for each k whereas it is admissible if and only if $\ker A_k(0)$ is complemented for each k. Well-behavedness is a strictly stronger property than admissible as the following instructive example shows.

Let F be a Hilbert space, let U and V be closed subspace such that $U \cap V = \{0\}$ and the sum $U \oplus V$ is not closed (see [6]). Let P and Q be bounded projections such that $\ker P = U$ and $\operatorname{ran} Q = V$, and define $A(z) = (zQ + I - Q)(zP + I - P)$. Then $A(0) = (I - Q)(I - P)$ and so $\ker A(0) = \operatorname{ran} P$. We may therefore take $\pi_0 = P$. Then $A_1(z) = zQ + I - Q$, $A_1(0) = I - Q$ and $\ker A_1(0) = \operatorname{ran} Q$. So $\ker \pi_0 \oplus \ker A_1(0)$ is not closed and hence cannot be complemented.

The advantage (to the theorem prover) of well-behavedness appears when considering left-right equivalence, a notion which we now define. Let $A \in \mathscr{A}_0(\mathrm{C}, L(F,F))$ and $B \in \mathscr{A}_0(\mathrm{C}, L(F,F))$. We say that A and B are *left-right equivalent* at $z = 0$ if there exist operator-valued germs U and V such that $U(0)$ and $V(0)$ are invertible, and $A = UBV$.

The definition clearly makes sense for germs taking values in a Banach algebra, and it is not hard to show that ascent is in this case a left-right invariant. However the proof of the next result uses the special choice of multiplicity projections discussed above, available for well-behaved operator-valued germs. The conclusion is known for matrix functions and its usual proof in that context makes use of a matrix concept, the Smith form [4]. Our proof is completely matrix free.

THEOREM 11. *Let two admissible operator germs $A(z)$ and $B(z)$ have finite ascent n and let $\pi_k, \rho_k, k = 0, 1, 2, \ldots$ be multiplicity sequences for $A(z)$ and $B(z)$ respectively. Let $V_k^A = \pi_0 \ldots \pi_{k-1} \ker A_k(0)$ and $V_k^B = \rho_0 \ldots \rho_{k-1} \ker B_k(0)$.*
Then:

(i) *If $A(z)$ and $B(z)$ are left-right equivalent there exists an invertible linear mapping $T \in L(F, F)$ such that $T(V_k^A) = V_k^B$ for each $k \geqq 0$.*

(ii) *If $A(z)$ and $B(z)$ are well-behaved and there exists an invertible linear mapping $T \in L(F, F)$ such that $T(V_k^A) = V_k^B$ for each $k \geqq 0$ then $A(z)$ and $B(z)$ are left-right equivalent.*

PROOF. (i) Since the spaces V_k are invariants for left equivalence we may assume that $A(z)$ and $B(z)$ are right equivalent: let $A(z) = B(z)U(z)$ where $U(0)$ is invertible. Then there exists for each k a germ $U_k(z)$ such that $U_k(0)$ is invertible and $A_k(z) = B_k(z)U_k(z)$. From this follows $\rho_k U_k(0)\pi_k = U_k(0)\pi_k$. Moreover we have that

$$A_{k+1}(z) = B_{k+1}(z)(z\rho_k + I - \rho_k)U_k(z)(z^{-1}\pi_k + I - \pi_k)$$

so that

$$U_{k+1}(z) = (z\rho_k + I - \rho_k)U_k(z)(z^{-1}\pi_k + I - \pi_k).$$

Hence $\rho_k U_{k+1}(0) = \rho_k U_k(0)\pi_k = U_k(0)\pi_k$. Iterating backwards we obtain

$$\rho_0 \dots \rho_k U_{k+1}(0) = U(0)\pi_0 \dots \pi_k,$$

and this holds for all $k \geqq 0$. Hence

$$\begin{aligned} U(0)V_k^A &= U(0)\pi_0 \dots \pi_{k-1} \ker A_k(0) \\ &= \rho_0 \dots \rho_{k-1} U_k(0) \ker A_k(0) \\ &= \rho_0 \dots \rho_{k-1} \ker B_k(0) = V_k^B. \end{aligned}$$

(ii) By (i) it suffices to consider the case $T = I$. Suppose then that $V_k^A = V_k^B$ for all k and (dropping the superscripts A and B) suppose that V_k is complemented in V_{k-1} for each k. By the proof of the second part of theorem 10 we may choose the projections π_k and ρ_k so that $\ker \pi_k = \ker \rho_k$ for each k and, denoting these spaces by K_k, the inclusions $K_k \subset K_{k+1}$ hold.

We may assume, without loss of generality, that

$$A(z) = (z\pi_{n-1} + I - \pi_{n-1})(z\pi_{n-2} + I - \pi_{n-2})\dots(z\pi_0 + I - \pi_0)$$

and

$$B(z) = (z\rho_{n-1} + I - \rho_{n-1})(z\rho_{n-2} + I - \rho_{n-2})\dots(z\rho_0 + I - \rho_0)$$

We now claim that both $B(z)^{-1}A(z)$ and $A(z)^{-1}B(z)$ are polynomials, thus establishing the left-right equivalence of A and B. In fact we have

$$\begin{aligned} B(z)^{-1}A(z) &= (z^{-1}\rho_0 + I - \rho_0)(z^{-1}\rho_1 + I - \rho_1)\dots(z^{-1}\rho_{n-1} + I - \rho_{n-1}) \\ &\qquad (z\pi_{n-1} + I - \pi_{n-1})(z\pi_{n-2} + I - \pi_{n-2})\dots(z\pi_0 + I - \pi_0) \\ &= (z^{-1}\rho_0 + I - \rho_0)(z^{-1}\rho_1 + I - \rho_1)\dots(z^{-1}\rho_{n-1} + I - \rho_{n-1}) \sum_{k=0}^{n} z^k(\pi_{k-1} - \pi_k) \end{aligned}$$

We have that $\rho_i(I - \pi_j) = 0$, that is, $\rho_i = \rho_i\pi_j$, if $i \geqq j$. This leads to $\rho_i\pi_{j-1} = \rho_i\pi_j\pi_{j-1} = \rho_i\pi_j$. Hence $\rho_i(\pi_{j-1} - \pi_j) = 0$ if $i \geqq j$. Hence

$B(z)^{-1}A(z) =$

$$\sum_{k=0}^{n} z^k(z^{-1}\rho_0 + I - \rho_0)(z^{-1}\rho_1 + I - \rho_1)\dots(z^{-1}\rho_{k-1} + I - \rho_{k-1})(\pi_{k-1} - \pi_k)$$

$$= \sum_{k=0}^{n} (\rho_0 + z(I - \rho_0))\dots(\rho_{k-1} + z(I - \rho_{k-1}))(\pi_{k-1} - \pi_k)$$

and this is plainly a polynomial. The proof that $A^{-1}B$ is a polynomial is similar. This concludes the proof.

A corollary of the theorem is that the numerical sequence rank π_k, if consisting of finite numbers ending in 0, completely determines the left-right equivalence class of $A(z)$ at 0.

Given $A(z)$ well-behaved and of finite ascent n we can now find a germ of a particularly simple form which is left-right equivalent to A at 0. Let $\rho_0, \rho_1, \dots, \rho_{n-1}$ be projections which commute with each other such that for each k the range of ρ_k is V_k. The germ $A(z)$ is then left-right equivalent to the product

$$(z\rho_{n-1} + I - \rho_{n-1})(z\rho_{n-2} + I - \rho_{n-2})\dots(z\rho_0 + I - \rho_0)$$

which equals

$$\sum_{k=0}^{n} z^k(\rho_{k-1} - \rho_k) = \sum_{k=0}^{n} z^k\sigma_k \tag{8}$$

where $\sigma_k = \rho_{k-1} - \rho_k$ and is a projection. Note that $\sigma_i\sigma_j = 0$ if $i \neq j$ and that $\sum_{k=0}^{n}\sigma_k = I$. Furthermore we can recover ρ_k from the formula $\rho_k = \sum_{j=k+1}^{n}\sigma_j$. We have proved:

THEOREM 12. *Any well-behaved germ of finite ascent is left-right equivalent to a germ of the form* $\sum_{k=0}^{n} z^k\sigma_k$ *for certain commuting projections* σ_k *satisfying* $\sigma_i\sigma_j = 0$ *if* $i \neq j$ and $\sum_{k=0}^{n}\sigma_k = I$.

In the case where the germ has finite multiplicity at 0 this is a case of a theorem of Gohberg and Segal (theorem 3.1 of [5]). Our proof is, however, quite different.

In the author's paper [2] there was some discussion of commuting families (theorem 2.3 of [2]). A stronger result is possible. We are able to identify both sequences of spaces U_k and V_k.

THEOREM 13. *Let A be an operator germ and suppose that:*

(i) *The space* $\ker A(0)^k$ *is complemented for each k;*

(ii) $A(z)A(0) = A(0)A(z)$ *for all z;*

(iii) *For each k the restriction of* $DA(0)$ *to* $\ker A(0)^k$ *is an isomorphism of* $\ker A(0)^k$ *onto itself.*

Then $A(z)$ *is admissible, and for each k we have*

(9) $$\ker A(0)^k = \ker A_0(0) \oplus \ker A_1(0) \oplus \cdots \oplus \ker A_{k-1}(0) = U_{k-1}$$

where the sum is topological; and

(10) $$V_k = T^{-k} A(0)^k \ker A(0)^{k+1}$$

where T is the restriction of $DA(0)$ to $\ker A(0)$. *Next assume that:*

(iv) $A(0)^k \ker A(0)^{k+1}$ *is complemented in $TA(0)^{k-1} \ker A(0)^k$ for each k.*
Then $A(z)$ is well-behaved.

PROOF. We preface the proof with some general remarks. Note that $DA(0)$ commutes with $A(0)$ and so it automatically maps each space $\ker A(0)^k$ into itself. In [2] it was assumed, in addition to the present hypothesis (ii), that $A(0)$ was a Fredholm operator of index zero and that $\ker A(0)$ intersected $\ker DA(0)$ only at 0. These hypotheses imply the present hypothesis (iii). Another hypothesis that implies (iii) and is interesting because it makes sense in a Banach algebra is: $(0, 0)$ *does not belong to the joint spectrum of the commuting pair* $(A(0), DA(0))$.

We now proceed with the proof proper. Initially we follow the proof of theorem 2.3 of [2]. Thus we quote the following consequences of (i) and (ii), referring the reader to [2]:

$$A(0)^k A_k(z) = A(z) A(0)^k$$

$$A(0)^k DA(0) = DA(0) A(0)^k$$

$$\ker A(0)^k \cap \ker A_k(0) = \{0\}.$$

From these relations we conclude that $A_k(0)$ maps the space $\ker A(0)^k$ injectively into itself, and that the sum of $\ker A(0)^k$ and $\ker A_k(0)$ is direct, though we cannot as yet conclude that it is topological.

We use induction to prove the sum formula (9). Let $(H)_k$ refer to the conjunction of the two statements:

$$\ker A(0)^k = \ker A_0(0) \oplus \ker A_1(0) \oplus \cdots \oplus \ker A_{k-1}(0);$$

$$A_{k-1}(0) \text{ maps } \ker A(0)^{k-1} \textit{ onto } \text{itself.}$$

The statement $(\mathrm{H})_1$ is trivially true. Now assume, by way of an induction hypothesis, that $(\mathrm{H})_j$ is true for all natural numbers $j \leqq k$. We note that, by the proposition on page 255 of [2], the sum of spaces in equation (9) is independent of the way the projections are chosen. We may therefore make a special choice of π_{k-1} by requiring that it should map $\ker A(0)^{k-1}$ to 0. Now we can show that $A_k(0)$ maps $\ker A(0)^k$ onto itself. We have that $\ker A(0)^k = \ker A(0)^{k-1} \oplus \ker A_{k-1}(0)$. Since $A_k(0) = DA_{k-1}(0)\pi_{k-1} + A_{k-1}(0)$ the restriction of $A_k(0)$ to $\ker A(0)^{k-1}$ is $A_{k-1}(0)$, known already by the induction hypothesis to map $\ker A(0)^{k-1}$ onto itself. It is enough therefore to show that $A_k(0)$ maps $\ker A_{k-1}(0)$

to a complement of $\ker A(0)^{k-1}$ in $\ker A(0)^k$. The mapping $A_k(0)$ induces a mapping $\overline{A_k(0)}$ from $\ker A(0)^k/\ker A(0)^{k-1}$ into itself. We have to show that this induced mapping is surjective. The mapping $DA(0)$ also induces a mapping $\overline{DA(0)}$ from $\ker A(0)^k/\ker A(0)^{k-1}$ into itself. Now this mapping is surjective by (ii). We shall show that $\overline{A_k(0)} = \overline{DA(0)}$. Let $x \in \ker A_{k-1}(0)$. We have that

$$\begin{aligned} A(0)^{k-1}(A_k(0)x - DA(0)x) &= A(0)^{k-1}(DA_{k-1}(0)x - DA(0)x) \\ &= DA(0)A(0)^{k-1}x - A(0)^{k-1}DA(0)x = 0 \end{aligned}$$

Hence $A_k(0)x - DA(0)x \in \ker A(0)^{k-1}$ and so $\overline{A_k(0)} = \overline{DA(0)}$. This establishes half of $(\mathrm{H})_{k+1}$.

We have still to show that $\ker A(0)^{k+1} = \ker A(0)^k \oplus \ker A_k(0)$. Now it is clear that $\ker A(0)^k \oplus \ker A_k(0)$ is a subspace of $\ker A(0)^{k+1}$. We do not know whether or not the sum is topological but this will follow once we prove that the inclusion is in fact equality. The argument for equality is purely algebraic. Let S be the restriction of $A_k(0)$ to $\ker A(0)^{k+1}$. By what we have proved the range of S is $\ker A(0)^k$, its null-space is $\ker A_k(0)$, its range and null-space meet only at 0, and it maps its range *onto* itself. It follows that the domain of S is the sum of the range and null-space of S.

To prove (10) we note that by (9)

$$A(0)^k \ker A(0)^{k+1} = A(0)^k (\ker A(0)^k \oplus \ker A_k(0)) = A(0)^k \ker A_k(0)$$

We therefore have to show that

$$(DA(0)^k \pi_0 \dots \pi_{k-1} \ker A_k(0) = A(0)^k \ker A_k(0)$$

To prove this we shall establish a formula interesting in its own right:

$$(11) \qquad (DA(0))^k \pi_0 \dots \pi_{k-1} x = (-1)^k A(0)^k x$$

for all $x \in \ker A_k(0)$. Again we use induction. The case $k = 0$ reduces to $\pi_0 x = x$ for $x \in \ker A(0)$. Suppose that (11) holds for a given k. Let $x \in \ker A_{k+1}(0)$. Then $\pi_k x \in \ker A_k(0)$ and so

$$\begin{aligned} DA(0)^{k+1}\pi_0 \dots \pi_{k-1}\pi_k x &= (-1)^k DA(0)A(0)^k \pi_k x \\ &= (-1)^k A(0)^k DA_k(0)\pi_k x \\ &= (-1)^{k+1} A(0)^k A_k(0)x \\ &\qquad \text{(since } A_{k+1}(0)x = 0) \\ &= (-1)^{k+1} A(0)^{k+1} x \end{aligned}$$

This concludes the proof.

For a germ $A(z)$ satisfying the conditions of the last theorem, except possibly condition (iv), there is a special choice of projections available whether or not it is well-behaved. We can choose the projections π_k so that π_k maps all spaces

$\ker A_j(0)$ to 0 if $j < k$. It follows that $\pi_i \pi_j = 0$ if $i > j$ and an easy calculation reduces the product (5) to

$$(z-1)\sum_{k=0}^{n-1} \pi_k + I = (z-1)S + I$$

so that $A(z)$ is left equivalent to the germ $(z-1)S + I$, where $S = \sum_{k=0}^{n-1} \pi_k$. In fact this germ provides a simple example of a germ which satisfies the conditions of the theorem.

THEOREM 14. *Let $A(z)$ satisfy the conditions of theorem 13, suppose that $A(0)$ is not invertible but has finite ascent m. Then $A(z)$ has quasi-ascent* 1.

PROOF. We have that $A(z)$ is left equivalent to $(z-1)S + I = (z-1)(S - I + tI)$ where $t = (z-1)^{-1} - 1$. Since reparametrization does not affect ascent, the result follows from theorem 7.

We end this section with some additional results concerning the property of well-behavedness.

THEOREM 15. *Let $A(z)$ be an admissible germ.*

(i) *If* $\operatorname{ran} A(0)$ *is closed then the sum* $\ker \pi_0 \oplus \ker A_1(0)$ *is closed.*

(ii) *If the sum* $\ker A_1(0)$ *is complemented and* $\operatorname{ran} A_1(0)$ *is closed then* $\operatorname{ran} A(0)$ *is closed.*

(iii) *If $A(z)$ is well-behaved then* $\operatorname{ran} A_k(0)$ *is closed for each k.*

(iv) *If the space F is isomorphic to a Hilbert space and $A(z)$ has finite ascent then it is sufficient for $A(z)$ to be well-behaved that* $\operatorname{ran} A_k(0)$ *should be closed for each k.*

PROOF. (i) Let $\operatorname{ran} A(0)$ be closed. Let $u_n \in \ker \pi_0$ and $v_n \in \ker A_1(0)$ be such that $u_n + v_n$ converges to a point y. It suffices to show that u_n converges. We have that $\pi_0 v_n \to \pi_0 y$. Furthermore $A(0)(u_n + v_n) \to A(0)y$. But $A(0)(u_n + v_n) = A(0)u_n - DA(0)\pi_0 v_n$ and $DA(0)\pi_0 v_n$ converges. Hence $A(0)u_n$ converges, and since $\operatorname{ran} A(0)$ is closed and $u_n \in \ker \pi_0$ we have that u_n converges.

(ii) Assume that the sum $\ker \pi_0 + \ker A_1(0)$ is complemented and that $\operatorname{ran} A_1(0)$ is closed. Choose π_1 so that $\ker \pi_0 \subset \ker \pi_1$. Now let x_n be a sequence such that $A(0)x_n$ converges. We may suppose that $x_n \in \ker \pi_0$. But then $A_1(0)x_n = A(0)x_n$ so that $A_1(0)x_n$ converges. Now x_n lies also in $\ker \pi_1$ and $\operatorname{ran} A_1(0)$ is closed. We conclude that x_n converges. Hence $\operatorname{ran} A(0)$ is closed.

Assertions (iii) and (iv) are obvious corollaries of (i) and (ii).

7. Deformations and left-right equivalence.

In this section we will describe the deformations of well-behaved operator germs of finite ascent up to left-right equivalence. Since left-right equivalence is weaker than left equivalence, we can expect that in general fewer parameters will be needed than for left equivalence. For example if $A(z)$ is an operator-germ of finite multiplicity and ascent n at 0, with multiplicity sequence $\pi_0, \dots, \pi_{n-1}$, then by theorem 5 we can describe all deformations of $A(z)$, up to left equivalence, using parameters in the space $L(F,F)\pi_0 \times \cdots \times L(F,F)\pi_{n-1}$. This space is infinite-dimensional if F is infinite-dimensional. If left-right equivalence is used then we can use a finite-dimensional parameter space provided the multiplicity is finite, and, what is more, we obtain a precise formula for its dimension.

Theorem 12 provides a canonical form with respect to left-right equivalence for well-behaved operator germs of finite ascent. This form is $\sum_{k=0}^{n} \sigma_k z^k$ where σ_k $(k = 0, \dots, n)$ are commuting projections such that $\sigma_i \sigma_j = 0$ if $i \neq j$ and $\sum_{k=0}^{n} \sigma_k = I$. We shall consider deformations of a germ in canonical form and allow values in a Banach algebra Y.

First we observe that for a germ in canonical form the division formula of theorem 4 takes a rather simple form:

$$f(z, \xi, \mathbf{a}) = U(z, \xi, \mathbf{a}) \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k \rho_k z^k \right) + \sum_{k=0}^{n-1} r_k(\xi, \mathbf{a}) \rho_k z^k$$

where $\rho_k = \sum_{i=k+1}^{n} \sigma_i$. This formula is matched by another division formula in which the divisor is on the left:

$$f(z, \xi, \mathbf{b}) = \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k z^k \right) V(z, \xi, \mathbf{b}) + \sum_{k=0}^{n-1} \rho_k s_k(\xi, \mathbf{b}) z^k$$

We proceed to combine these two formulas into one.

LEMMA 6. *Let Y be a Banach algebra, X a Banach space, σ_k $(k = 0, \dots, n)$ commuting idempotent elements of Y such that $\sigma_i \sigma_j = 0$ if $i \neq j$ and $\sum_{k=0}^{n} \sigma_k = 1$. Let $\rho_k = \sum_{i=k+1}^{n} \sigma_i$. Then for any germ $f \in \mathscr{A}_0(\mathbf{C} \times X \times Y^n \times Y^n, Y)$ there exist U and V in $\mathscr{A}_0(\mathbf{C} \times X \times Y^n \times Y^n, Y)$ and $g_k \in \mathscr{A}_0(X \times Y^n \times Y^n, Y)$ such that*

$$f(z, \xi, \mathbf{a}, \mathbf{b}) = U(z, \xi, \mathbf{a}, \mathbf{b}) \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k \rho_k z^k \right)$$

$$+ \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k z^k \right) V(z, \xi, \mathbf{a}, \mathbf{b}) + \sum_{k=0}^{n-1} \rho_k g_k(\xi, \mathbf{a}, \mathbf{b}) \rho_k z^k$$

PROOF. Let M_r be the vector-space of all germs expressible in the form

$$U(z, \xi, \mathbf{a}, \mathbf{b}) \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k \rho_k z^k \right)$$

and let M_l be the vector-space of all germs expressible in the form

$$\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k z^k \right) V(z, \xi, \mathbf{a}, \mathbf{b})$$

for some U and V. Let $M = M_r + M_l$. Let N_k be the vector-space of all germs of the form $\rho_k g(\xi, \mathbf{a}, \mathbf{b})\rho_k z^k$ for some g. Let L_k and R_k be the vector-spaces of all germs of the form $\rho_k s(\xi, \mathbf{a}, \mathbf{b}) z^k$ and $t(\xi, \mathbf{a}, \mathbf{b})\rho_k z^k$ respectively. We shall show using backward induction that $L_k \subset M + N_0 + \cdots + N_{n-1}$ and $R_k \subset M + N_0 + \cdots + N_{n-1}$ for $k = 0, 1, \ldots, n-1$. We begin with $n - 1$. Using right division we can find $t_k \in \mathscr{A}_0(X \times Y^n \times Y^n, Y)$ such that

$$z^{n-1} \in M_r + \sum_{k=0}^{n-1} t_k \rho_k z^k$$

Hence, for $s \in \mathscr{A}_0(X \times Y^n \times Y^n, Y)$ we have

$$\begin{aligned} \rho_{n-1} s z^{n-1} &\in M_r + \sum_{k=0}^{n-1} \rho_{n-1} s t_k \rho_k z^k \\ &= M_r + \sum_{k=0}^{n-1} \rho_k \rho_{n-1} s t_k \rho_k z^k \subset M + N_0 + \cdots + N_{n-1} \end{aligned}$$

Hence $L_{n-1} \subset M + N_0 + \cdots + N_{n-1}$. Similarly $R_{n-1} \subset M + N_0 + \cdots + N_{n-1}$.

Next we suppose that the inclusions are known for $k + 1, k + 2, \ldots, n - 1$. By right division we can find $t_j \in \mathscr{A}_0(X \times Y^n \times Y^n, Y)$ such that

$$z^k \in M_r + \sum_{j=0}^{n-1} t_j \rho_j z^j$$

Hence, for $s \in \mathscr{A}_0(X \times Y^n \times Y^n, Y)$ we have

$$\begin{aligned} \rho_k s z^k &\in M_r + \sum_{j=0}^{n-1} \rho_k s t_j \rho_j z^j \\ &= M_r + \sum_{j=0}^{k} \rho_j \rho_k s t_j \rho_j z^j + \sum_{j=k+1}^{n-1} \rho_k s t_j \rho_j z^j \\ &\in M_r + N_0 + \cdots + N_k + R_{k+1} + \cdots + R_{n-1} \\ &\subset M + N_0 + \cdots + N_{n-1} \end{aligned}$$

by the induction hypothesis. Hence $L_k \subset M + N_0 + \cdots + N_{n-1}$ and by similar reasoning $R_k \subset M + N_0 + \cdots + N_{n-1}$. This concludes the proof.

LEMMA 7. *Let $f \in \mathscr{A}_0(\mathsf{C} \times X, Y)$ be such that $f(z, 0) = \sum_{k=0}^{n} \sigma_k z^k$. Then for any h there exist U and V in $\mathscr{A}_0(\mathsf{C} \times X, Y)$ and $g_k \in \mathscr{A}_0(X, Y)$ such that*

$$h(z, \xi) = U(z, \xi) f(z, \xi) + f(z, \xi) V(z, \xi) + \sum_{k=0}^{n-1} \rho_k g_k(\xi) \rho_k z^k$$

If, moreover, X is a product of Banach spaces $X_1 \times X_2$ and h restricted to $X_1 \times 0$ vanishes, then we can choose U, V and g so that they too vanish on $X_1 \times 0$.

PROOF. By theorem 5 and its analogue for left division we have

$$f(z,\xi) = q_1(z,\xi)\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k(\xi)\rho_k z^k\right)$$

and

$$f(z,\xi) = \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k(\xi) z^k\right) q_2(z,\xi)$$

for certain a_k and b_k. By the last lemma we have

$$(12)\quad h(z,\xi) = U_1(z,\xi,\mathbf{a},\mathbf{b})\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k\rho_k z^k\right) + \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k z^k\right) V_1(z,\xi,\mathbf{a},\mathbf{b}) + \sum_{k=0}^{n-1} \rho_k g_k(\xi,\mathbf{a},\mathbf{b})\rho_k z^k$$

for certain germs U_1, V_1 and g_k. Now put $\mathbf{a} = \mathbf{a}(\xi)$, $\mathbf{b} = \mathbf{b}(\xi)$, $U = U_1 q_1^{-1}$ and $V = q_2^{-1} V_1$.

Now for the last part. Let $(\xi)_1$ denote the projection of ξ to $X_1 \times 0$. By (12) we have

$$0 = U_1(z,(\xi)_1,\mathbf{a},\mathbf{b})\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} a_k\rho_k z^k\right) + \left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k b_k z^k\right) V_1(z,(\xi)_1,\mathbf{a},\mathbf{b}) + \sum_{k=0}^{n-1} \rho_k g_k((\xi)_1,\mathbf{a},\mathbf{b})\rho_k z^k$$

Now we set $\mathbf{a} = \mathbf{a}(\xi)$, $\mathbf{b} = \mathbf{b}(\xi)$,

$$U(z,\xi) = \big(U_1(z,\xi,\mathbf{a}(\xi),\mathbf{b}(\xi)) - (U_1(z,(\xi)_1,\mathbf{a}(\xi),\mathbf{b}(\xi))\big) q_1(z,\xi)^{-1}$$

and

$$V(z,\xi) = q_2(z,\xi)^{-1}\big(V_1(z,\xi,\mathbf{a}(\xi),\mathbf{b}(\xi)) - (V_1(z,(\xi)_1,\mathbf{a}(\xi),\mathbf{b}(\xi))\big).$$

THEOREM 16. *Let $f(z,\xi)$ be such that $f(z,0) = \sum_{k=0}^{n} \sigma_k z^k$. Then there exist invertible germs E and F such that*

$$f(z,\xi) = E(z,\xi)\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k g_k(\xi)\rho_k z^k\right) F(z,\xi)$$

for certain germs $g_k \in \mathscr{A}_0(X,Y)$. Moreover $E(z,0) = F(z,0) = 1$ and $g_k(0) = 0$ for each k.

PROOF. The idea is to find invertible germs E and F, and germs g_k, depending on an additional global scalar parameter p such that the function H given by

(13)
$$H(z,\xi,p) = E(z,\xi,p)\left((1-p)f(z,\xi) + p\sum_{k=0}^{n}\sigma_k z^k + \sum_{k=0}^{n-1}\rho_k g_k(\xi,p)\rho_k z^k\right)F(z,\xi,p)$$

is independent of p and $g_k(\xi,0) = 0$. Equality between the cases $p = 0$ and $p = 1$ gives the required result.

Let $\mathbf{c} \in Y^n$, and consider the expression

$$K(z,\xi,\mathbf{c},p) = (1-p)f(z,\xi) + p\sum_{k=0}^{n}\sigma_k z^k + \sum_{k=0}^{n-1}\rho_k c_k \rho_k z^k$$

Note that $K(z,0,0,p_0) = f(z,0) = \sum_{k=0}^{n}\sigma_k z^k$. Applying lemma 7 we can find U, V and r_k such that

(14)
$$f(z,\xi) - \sum_{k=0}^{n}\sigma_k z^k = U(z,\xi,\mathbf{c},p)K(z,\xi,\mathbf{c},p) + K(z,\xi,\mathbf{c},p)V(z,\xi,\mathbf{c},p) + \sum_{k=0}^{n-1}\rho_k r_k(\xi,\mathbf{c},p)\rho_k z^k$$

for $(z,\xi,\mathbf{c},p)$ in a neighbourhood of $(0,0,0,p_0)$. Moreover by the last part of lemma 7 we can ensure that $U(z,0,\mathbf{c},p) = V(z,0,\mathbf{c},p) = 0$ and $r_k(0,\mathbf{c},p) = 0$. We can do the same for any p_0 in the unit interval $[0,1]$. By choosing finitely many neighbourhoods covering the unit interval, and a partition of unity, we can define U, V and r_k for all p in the interval $[0,1]$ and all $(z,\xi,\mathbf{c})$ in a neighbourhood of $(0,0,0)$ so as to satisfy (14). They will be analytic in $(z,\xi,\mathbf{c})$ but only C^∞ in p.

Let E and F (taking values in Y), and $\mathbf{g}$ (taking values in Y^n) satisfy the differential equations

$$\frac{dE}{dp} = EU(z,\xi,\mathbf{g},p); \quad \frac{dF}{dp} = V(z,\xi,\mathbf{g},p)F; \quad \frac{d\mathbf{g}}{dp} = \mathbf{r}(\xi,\mathbf{g},p)$$

with the initial conditions

$$E|_{p=0} = 1; \quad F|_{p=0} = 1; \quad \mathbf{g}|_{p=0} = 0$$

In these equations z and ξ play the role of parameters. The first two equations are linear and therefore have unique solutions, which have invertible elements as their values, defined over the whole interval $0 \leqq p \leqq 1$. The third equation has a unique solution on this interval if ξ is sufficiently small, since, by the lemma, the equation is satisfied by $\mathbf{g} = 0$ in case $\xi = 0$. The solutions are analytic in z and ξ. From the vanishing of the right-hand sides of the differential equations when $\xi = 0$ we deduce that $E(z,0,p) = F(z,0,p) = 1$ and $g_k(0,p) = 0$.

Consider the mapping H defined by equation (13). By the differential equations satisfied by E, F and $\mathbf{g}$ we find that $dH/dp = 0$. Hence H is independent of p and comparing $p = 0$ and $p = 1$ we deduce

$$f(z, \xi) = E(z, \xi, 1)\left(\sum_{k=0}^{n} \sigma_k z^k + \sum_{k=0}^{n-1} \rho_k g_k(\xi, 1)\rho_k z^k \right) F(z, \xi, 1)$$

This concludes the proof.

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