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ON CONTINUITY OF SINGULAR INTEGRAL OPERATORS IN SOBOLEV SPACES

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In this paper we give conditions for the boundedness of a singular integral operator, acting from the Sobolev class $H^m(\mathsf{R}^n)$ into $H^l(\mathsf{R}^n)$ with $m \geqq l \geqq 0$. The symbol may depend not only on the angular variable $\theta \in \mathsf{S}^{n-1}$ but also on the space variable $x \in \mathsf{R}^n$. It will be shown that the conditions, which are stated in terms of a certain space of multipliers, are precise in a sense.

1. Function spaces.

Let μ be a measurable function defined on R^{n-1} satisfying the conditions $\mu(\xi) \geqq c$ and $\mu(\xi + \eta) \leqq (1 + c|\xi|^Q)\mu(\eta)$, where c and Q are positive constants. By $\mathscr{H}_\mu(\mathsf{R}^{n-1})$ we denote the completion of $C_0^\infty(\mathsf{R}^{n-1})$ in the norm

$$(1) \qquad \|v; \mathsf{R}^{n-1}\|_{\mathscr{H}_\mu} = \left(\int_{\mathsf{R}^{n-1}} |\mu(\xi)(Fv)(\xi)|^2 \, d\xi\right)^{1/2},$$

where F is the Fourier transform in R^{n-1}. We obtain the Sobolev space $H^l(\mathsf{R}^{n-1})$, $l \in \mathsf{R}^1$, by setting $\mu(\xi) = (1 + |\xi|)^l$. The space $\mathscr{H}_\mu$ was introduced and studied in [1], [2]. In particular, in [1], [2] it was shown that $\mathscr{H}_\mu(\mathsf{R}^{n-1})$ is embedded into the space $C(\mathsf{R}^{n-1})$ of continuous and bounded functions on R^{n-1} if and only if

$$(2) \qquad \int_{\mathsf{R}^{n-1}} \frac{d\xi}{\mu(\xi)^2} < \infty.$$

We shall suppose that μ is weakly subadditive, i.e., $\mu(\xi + \eta) \leqq c(\mu(\xi) + \mu(\eta))$, $c = \text{const}$. An easy modification of the proof of a similar result for H^l given in [3] shows that the space $\mathscr{H}_\mu(\mathsf{R}^{n-1})$ is an algebra with respect to pointwise multiplication if μ satisfies (2). The contrary also holds. In fact, since $\mu(\xi) \geqq c > 0$, then for all $u \in \mathscr{H}_\mu(\mathsf{R}^{n-1})$ one has

$$c\,\|u^N; \mathsf{R}^{n-1}\|_{L_2} \leqq \|u^N; \mathsf{R}^{n-1}\|_{\mathscr{H}_\mu} \leqq c_1^N \,\|u; \mathsf{R}^{n-1}\|_{\mathscr{H}_\mu}^N,$$

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where $N = 1, 2, \ldots$ and the constants c, c_1 do not depend on N. Taking the Nth root and passing to the limit as $N \to \infty$, we arrive at

$$\|u; \mathsf{R}^{n-1}\|_{L_\infty} \leqq c_1 \|u; \mathsf{R}^{n-1}\|_{\mathscr{H}_\mu}.$$

Consequently, $\mathscr{H}_\mu(\mathsf{R}^{n-1}) \subset C(\mathsf{R}^{n-1})$, which is equivalent to (2).

In what follows we assume the condition (2) *to be always valid.*

Let S^{n-1} denote the boundary of the n-dimensional unit ball with center at the origin. We supply S^{n-1} with a structure of the class C^∞ by introducing a family of coordinate neighbourhoods $\{U_k\}$ and a family of diffeomorphisms $\phi_k\colon U_k \to \mathsf{R}^{n-1}$. Further, let $\{v_k\}$ be a smooth partition of unity on S^{n-1} subordinate to the covering $\{U_k\}$.

A function σ defined on S^{n-1} belongs to the space $\mathscr{H}_\mu(S^{n-1})$ if

$$(v_k\sigma) \circ \phi_k^{-1} \in \mathscr{H}_\mu(\mathsf{R}^{n-1})$$

for all k. The norm in $\mathscr{H}_\mu(S^{n-1})$ is introduced by

$$\|\sigma; S^{n-1}\|_{\mathscr{H}_\mu} = \left(\sum_k \|(v_k\sigma) \circ \phi_k^{-1}; R^{n-1}\|^2_{\mathscr{H}_\mu}\right)^{1/2}.$$

Similarly to $\mathscr{H}_\mu(\mathsf{R}^{n-1})$ the space $\mathscr{H}_\mu(S^{n-1})$ is an algebra with respect to multiplication if and only if (2) holds. The same condition is equivalent to the embedding $\mathscr{H}_\mu(S^{n-1}) \subset C(S^{n-1})$.

Let B denote a ball in R^n. We shall need the space $H^{l,\mu}(B \times S^{n-1})$ of functions $B \times S^{n-1} \ni (x, \theta) \to u(x, \theta)$ with the finite norm

$$\left(\int_B (\|\nabla_l u(x, \cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} + \|u(x, \cdot); S^{n-1}\|^2_{\mathscr{H}_\mu})\, dx\right)^{1/2}$$

for integer $l \geqq 0$ and

$$\left(\int_B \int_B \|\nabla_{[l],x} u(x, \cdot) - \nabla_{[l],y} u(y, \cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} \frac{dx\,dy}{|x-y|^{n+2\{l\}}} + \int_B \|u(y, \cdot), S^{n-1}\|^2_{\mathscr{H}_\mu}\, dy\right)^{1/2}$$

for fractional $l > 0$. Here $[l]$ and $\{l\}$ denote the integer and the fractional parts of l.

Further, we introduce the space $H^{l,\mu}(\mathsf{R}^n \times S^{n-1})$ of functions $\mathsf{R}^n \times S^{n-1} \ni (x, \theta) \to u(x, \theta)$ with the finite norm

$$\|u; \mathsf{R}^n \times S^{n-1}\|_{H^{l,\mu}} = \left(\int_{\mathsf{R}^n} ((\mathscr{D}_{l,\mu} u(x))^2 + (\mathscr{D}_{0,\mu} u(x))^2)\, dx\right)^{1/2},$$

where

(3) $$\mathscr{D}_{l,\mu}u(x) = \|\nabla_{l,x}u(x,\cdot); S^{n-1}\|_{\mathscr{H}_\mu}$$

for $\{l\} = 0$ and

(4) $$\mathscr{D}_{l,\mu}u(x) = \left(\int_{\mathsf{R}^n} \|\nabla_{[l],x}u(x+h,\cdot) - \nabla_{[l],x}u(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} \frac{dh}{|h|^{n+2\{l\}}}\right)^{1/2}$$

for $\{l\} > 0$.

We say that a function γ defined on $\mathsf{R}^n \times S^{n-1}$ belongs to the space of multipliers $M(H^{m,\mu} \to H^{l,\mu})$ if $\gamma u \in H^{l,\mu}(\mathsf{R}^n \times S^{n-1})$ for all $u \in H^{m,\mu}(\mathsf{R}^n \times S^{n-1})$.

Since the embedding operator

$$H^{m,\mu}(\mathsf{R}^n \times S^{n-1}) \ni u \to \gamma u \in H^{l,\mu}(\mathsf{R}^n \times S^{n-1})$$

is closed, it is bounded. As a norm in $M(H^{m,\mu} \to H^{l,\mu})$ we take the norm of the multiplication operator:

$$\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu}\to H^{l,\mu})}$$

$$= \sup\{\|\gamma u; \mathsf{R}^n \times S^{n-1}\|_{H^{l,\mu}}: \|u; \mathsf{R}^n \times S^{n-1}\|_{H^{m,\mu}} \leqq 1\}.$$

We shall use the notation $MH^{l,\mu}$ instead of $M(H^{l,\mu} \to H^{l,\mu})$.

2. Description of the space $M(H^{m,\mu} \to H^{l,\mu})$.

In order to obtain two-sided estimates for the norm in $M(H^{m,\mu} \to H^{l,\mu})$, i.e., necessary and sufficient conditions for a function to belong to this space, we need the notion of the s-capacity of a compact set e in R^n. The capacity is defined as

$$\operatorname{cap}_s(e) = \inf\{\|u; \mathsf{R}^n\|^2_{H^s}: u \in C_0^\infty(\mathsf{R}^n),\ u \geqq 1 \text{ on } e\}$$

and is equivalent to the capacity generated by the Bessel potential of order $2s$ (see [6]).

Henceforth we shall use the following well-known result.

LEMMA 1 (see [4], Ch. 8). *Let ν be a measure in R^n and let v be an arbitrary function in $C_0^\infty(\mathsf{R}^n)$. The best constant C in the inequality*

(5) $$\int_{\mathsf{R}^n} |v|^2\, d\nu \leqq C\, \|v; \mathsf{R}^n\|^2_{H^m}$$

is equivalent to

$$\sup_e \frac{\nu(e)}{\operatorname{cap}_m(e)},$$

where e is an arbitrary compact set in R^n.

We pass to a description of the space $M(H^{m,\mu} \to H^{l,\mu})$. Consider first the case $l = 0$.

LEMMA 2. *A function γ defined on $\mathsf{R}^n \times S^{n-1}$ belongs to the space $M(H^{m,\mu} \to H^{0,\mu})$ if and only if $\gamma \in H^{0,\mu}(B \times S^{n-1})$ for an arbitrary ball B, and for any compact set $e \subset \mathsf{R}^n$*

$$\|\gamma; e \times S^{n-1}\|^2_{\mathscr{H}^{0,\mu}} \leqq c\,\mathrm{cap}_m(e),$$

where c is a constant which does not depend upon e. Moreover,

$$\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu}\to H^{0,\mu})} \sim \sup_{e \subset \mathsf{R}^n} \left(\frac{\int_e \|\gamma(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} dx}{\mathrm{cap}_m(e)} \right)^{1/2}. \tag{6}$$

(Here and henceforth $a \sim b$ means that the ratio a/b is bounded and separated from zero.)

PROOF. *Necessity.* We substitute the function $u(x, \theta) = u(x)$ from $H^m(\mathsf{R}^n)$ into the inequality

$$\left(\int_{\mathsf{R}^n} \|\gamma(x,\cdot)u(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} dx \right)^{1/2} \leqq c\,\|u; \mathsf{R}^n \times S^{n-1}\|_{H^{m,\mu}}.$$

Then

$$\left(\int_{\mathsf{R}^n} \|\gamma(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} |u(x)|^2\, dx \right)^{1/2} \leqq c\,\|u; \mathsf{R}^n\|_{H^m}.$$

By Lemma 1 the exact constant in this inequality is equivalent to the right-hand side of (6).

Sufficiency. Since under the condition (2) the space $\mathscr{H}_\mu(S^{n-1})$ is an algebra, it follows that

$$\|\gamma u; \mathsf{R}^n \times S^{n-1}\|^2_{\mathscr{H}^{0,\mu}} \leqq c \int_{\mathsf{R}^n} \|\gamma; S^{n-1}\|^2_{\mathscr{H}_\mu} \|u; S^{n-1}\|^2_{\mathscr{H}_\mu} dx =$$

$$c \sum_j \int_{\mathsf{R}^{n-1}} |\mu(\xi)|^2 \int_{\mathsf{R}^n} \|\gamma; S^{n-1}\|^2_{\mathscr{H}_\mu} |F[v_j(\phi_j^{-1}(\xi))u(x, \phi_j^{-1}(\xi))]|^2\, dx d\xi.$$

Applying Lemma 1 to the internal integral one obtains

$$\|\gamma u; \mathsf{R}^n \times S^{n-1}\|^2_{H^{0,\mu}} \leqq \sup_{e \subset \mathsf{R}^n} \frac{\int_e \|\gamma(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} dx}{\mathrm{cap}_m(e)}$$

$$\times \left(\int_{\mathsf{R}^n} \int_{\mathsf{R}^n} \frac{dh}{|h|^{n+2\{m\}}} \sum_j \int_{\mathsf{R}^{n-1}} |\mu(\xi)^2 \, |F \Delta_h \nabla_{[m],x}(v_j(\phi_j^{-1}(\xi)) u(x, \phi_j^{-1}(\xi)))|^2 \, d\xi dx \right.$$

$$\left. + \int_{\mathsf{R}^n} \sum_j \int_{\mathsf{R}^{n-1}} |\mu(\xi)|^2 \, |F(v_j(\phi_j^{-1}(\xi)) u(x, \phi_j^{-1}(\xi)))|^2 \, d\xi dx \right),$$

where $\Delta_h v(x, \theta) = v(x + h, \theta) - v(x, \theta)$. Hence, using the definition of the norm in $\mathscr{H}_\mu(S^{n-1})$, we arrive at

$$\|\gamma u; \mathsf{R}^n \times S^{n-1}\|^2_{H^{0,\mu}} \leqq c \sup_{e \subset \mathsf{R}^n} \frac{\int_e \|\gamma(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} dx}{\mathrm{cap}_m(e)} \|u; \mathsf{R}^n \times S^{n-1}\|^2_{H^{m,\mu}}.$$

The proof is complete.

REMARK 1. According to [5, Sec. 1.1 and 3.2], we may restrict ourselves in Lemma 2 to compact sets e satisfying $\mathrm{diam}(e) \leqq 1$.

In order to obtain two-sided estimates for the norm in $M(H^{m,\mu} \to H^{l,\mu})$ for $m \geqq l > 0$ one should prove a few auxiliary assertions which are derived in the same way as the corresponding assertions on multipliers in Sobolev classes $M(H^m(\mathsf{R}^n) \to H^l(\mathsf{R}^n))$ (see [5], Ch. 1, 3). While doing this one should replace $|\gamma(x)|$ by $\|\gamma(x,\cdot); S^{n-1}\|_{\mathscr{H}_\mu}$ and change $D_{2,l}u(x)$ in [5] to $\mathscr{D}_{l,\mu}u(x)$ which is defined by (3), (4). As a result we arrive at the following description of the class $M(H^{m,\mu} \to H^{l,\mu})$.

THEOREM 1. *A function γ belongs to the space $M(H^{m,\mu} \to H^{l,\mu})$, $m \geqq l \geqq 0$, if and only if $\mathscr{D}_{l,\mu}\gamma \in L_{2,\mathrm{loc}}(\mathsf{R}^n)$, $\mathscr{D}_{0,\mu}\gamma \in L_{2,\mathrm{loc}}(\mathsf{R}^n)$ and for any compact set $e \subset \mathsf{R}^n$ with* $\mathrm{diam}(e) \leqq 1$ *the inequality*

$$\int_e (\mathscr{D}_{l,\mu}\gamma(x))^2 \, dx \leqq c \, \mathrm{cap}_m(e)$$

is valid.

Moreover,

$$\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu} \to H^{l,\mu})} \sim \sup_{\{e: \mathrm{diam}(e) \leqq 1\}} \left(\frac{\int_e (\mathscr{D}_{l,\mu}\gamma(x))^2 \, dx}{\mathrm{cap}_m(e)} \right)^{1/2} + \begin{cases} \sup_{x \in \mathsf{R}^n} \left(\int_{B_1^n(x)} \|\gamma(y,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu} dy\right)^{1/2} & \text{for } m > l, \\ \operatorname{ess\,sup}_{x \in \mathsf{R}^n} \|\gamma(x,\cdot); S^{n-1}\|_{\mathscr{H}_\mu} & \text{for } m = l, \end{cases} \tag{7}$$

where $B_1^n(x) = \{y \in \mathsf{R}^n : |y - x| < 1\}$. The restriction $\mathrm{diam}(e) \leqq 1$ *can be omitted.*

REMARK 2. In the same way as in the case of the space $M(H^m \to H^l)$ (cf. Sec. 1.3.2 [5]) one can check that $M(H^{m,\mu} \to H^{l,\mu})$ is continuously embedded into

$M(H^{m-l,\mu} \to H^{0,\mu})$. Since the spaces $H^{m,\mu}(\mathsf{R}^n \times S^{n-1})$ form an interpolation scale (see, for instance, [7], Sec. 1.18.5), then, for any $j \in [0, l]$,

$$\text{(8)} \qquad \|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m-j,\mu} \to H^{l-j,\mu})}$$
$$\leqq c\, \|\gamma; \mathsf{R}^n \times S^{n-1}\|^{(l-j)/l}_{M(H^{m,\mu} \to H^{l,\mu})} \|\gamma; \mathsf{R}^n \times S^{n-1}\|^{j/l}_{M(H^{m-l,\mu} \to H^{0,\mu})}.$$

The embedding $M(H^{m,\mu} \to H^{l,\mu}) \subset M(H^{m-l,\mu} \to H^{0,\mu})$ together with (8) implies that the space $M(H^{m,\mu} \to H^{l,\mu})$ is continuously embedded into $M(H^{m-j,\mu} \to H^{l-j,\mu})$. From this and Theorem 1 it follows that (7) is equivalent to

$$\text{(9)} \qquad \|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu} \to H^{l,\mu})}$$
$$\sim \sup_{\{e:\, \mathrm{diam}\,(e) \leqq 1\}} \left(\sum_{j=0}^{[l]} \frac{\int_e (\mathscr{D}_{l-j,\mu}\gamma(x))^2\, dx}{\mathrm{cap}_{m-j}(e)} + \sum_{j=0}^{[l]} \frac{\int_e (\mathscr{D}_{j,\mu}\gamma(x))^2\, dx}{\mathrm{cap}_{m-l+j}(e)} \right)^{1/2}.$$

For $m = l$ the term corresponding to $j = 0$ in the second sum should be replaced by $\mathrm{ess\,sup}_{x \in \mathsf{R}^n} \|\gamma(x, \cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}$. Clearly, for integer l both sums in (9) coincide. The restriction $\mathrm{diam}(e) \leqq 1$ can be omitted.

REMARK 3. Let Q be an arbitrary cube in R^n and let G_{2l} denote the kernel of the Bessel potential $J_{2l} = (1 - \Delta)^{-l}$, i.e., the function whose Fourier transform is equal to $(1 + |\xi|^2)^{-l}$. Theorem 1 together with the main result of the paper [8] leads to the following relation for the norm in $M(H^{m,\mu} \to H^{l,\mu})$, different from (7),

$$\text{(10)} \qquad \|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu} \to H^{l,\mu})}$$
$$\sim \sup_{\{Q\}} \left(\frac{\int_Q \int_Q G_{2m}(x-y)(\mathscr{D}_{l,\mu}\gamma(x))^2 (\mathscr{D}_{l,\mu}\gamma(y))^2\, dxdy}{\int_Q (\mathscr{D}_{l,\mu}\gamma(x))^2\, dx} \right)^{1/2}$$
$$+ \begin{cases} \sup_{x \in \mathsf{R}^n} \left(\int_{B_1^n(x)} \|\gamma(y, .); S^{n-1}\|^2_{\mathscr{H}_\mu}\, dy\right)^{1/2} & \text{for } m > l, \\ \mathrm{ess\,sup}_{x \in \mathsf{R}^n} \|\gamma(x, \cdot); S^{n-1}\|_{\mathscr{H}_\mu} & \text{for } m = l. \end{cases}$$

Another description of the space $M(H^m(\mathsf{R}^n) \to L_2(\mathsf{R}^n))$, obtained in [9], enables one to replace the first item on the right in (10) by the supremum of the function

$$\left(\frac{J_m((J_m(\mathscr{D}_{l,\mu}\gamma)^2)^2)}{J_m(\mathscr{D}_{l,\mu}\gamma)^2} \right)^{1/2}.$$

Duplicating the proof of Theorem 1.3.3 from [5], one arrives at the following assertion.

COROLLARY 1. *For $2m > n$*

$$\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu}\to H^{l,\mu})} \sim \sup_{x\in\mathsf{R}^n}\left(\int_{B_1^n(x)} (\mathscr{D}_{l,\mu}\gamma(y))^2\,dy + \int_{B_1^n(x)} \|\gamma(y,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}\,dy\right)^{1/2}. \tag{11}$$

One can verify directly that the right-hand side of (11) is equivalent to the norm $\|\gamma; B \times S^{n-1}\|_{H^{l,\mu}}$.

From Theorem 1 one can obtain upper estimates for the norm in $M(H^{m,\mu} \to H^{l,\mu})$ using well-known lower estimates for the capacity of a compact set in terms of its Lebesgue measure mes_n.

COROLLARY 2. *For* $2m < n$

$$c\,\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu}\to H^{l,\mu})} \leqq \sup_{\{e:\,\mathrm{diam}(e)\leqq 1\}} \frac{(\int_e (\mathscr{D}_{l,\mu}\gamma(x))^2\,dx)^{1/2}}{(\mathrm{mes}_n e)^{\frac{1}{2}-\frac{m}{n}}} + \sup_{x\in\mathsf{R}^n}\left(\int_{B_1^n(x)} \|\gamma(y,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}\,dy\right)^{1/2}. \tag{12}$$

For $2m = n$

$$c\,\|\gamma; \mathsf{R}^n \times S^{n-1}\|_{M(H^{m,\mu}\to H^{l,\mu})} \leqq \sup_{\{e:\,\mathrm{diam}(e)\leqq 1\}} \left(\log\frac{2^n}{\mathrm{mes}_n e}\right)^{1/2}\left(\int_e (\mathscr{D}_{l,\mu}\gamma(x))^2\,dx\right)^{1/2} + \sup_{x\in\mathsf{R}^n}\left(\int_{B_1^n(x)} \|\gamma(y,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}\,dy\right)^{1/2}. \tag{13}$$

In case $m = l$ *one should replace the second item in the right-hand sides of* (12), (13) *by* $\mathrm{ess\,sup}_{x\in\mathsf{R}^n} \|\gamma(x,\cdot); S^{n-1}\|_{\mathscr{H}_\mu}$.

One can derive various upper and (separately) lower estimates for the norm of a function in $M(H^{m,\mu} \to H^{l,\mu})$ using estimates for the constant C in (5) obtained in [5], [9], [10].

3. Continuity of singular integral operators in pairs of Sobolev spaces.

Let σ be a measurable function on R^n with values in $L_2(S^{n-1})$. For any $u \in C_0^\infty(\mathsf{R}^n)$ we define the singular integral operator S with the symbol σ by the equality

$$\mathscr{S}u(x) = \mathscr{F}^{-1}_{\xi\to x}[\sigma(x, \xi/|\xi|)(\mathscr{F}u)(\xi)], \tag{14}$$

where $\mathscr{F}$ is the Fourier transform in R^n and $\mathscr{F}^{-1}$ is its inverse.

In what follows we use the notation

$$\mathscr{K} = \left(\int_{\mathsf{R}^{n-1}} \frac{d\tau}{\mu(\tau)^2}\right)^{1/2}. \tag{15}$$

THEOREM 2. *Let* $\mathcal{K} < \infty$ *and let*

$$\sigma \in M(H^{m,\mu} \to H^{l,\mu}), \quad m \geqq l \geqq 0. \tag{16}$$

Then the operator (14) *maps* $H^m(\mathsf{R}^n)$ *continuously into* $H^l(\mathsf{R}^n)$. *Moreover, the estimate*

$$\|\mathcal{S}\|_{H^m \to H^l} \leqq c\mathcal{K}\ \|\sigma\|_{M(H^{m,\mu} \to H^{l,\mu})} \tag{17}$$

is valid.

PROOF. We use a device proposed in [11], where singular integral operators in $L_2(\mathsf{R}^n)$ are considered. Let $x, \xi \in \mathsf{R}^n$, $\theta = \xi/|\xi|$ and let u be an arbitrary function from $C_0^\infty(\mathsf{R}^n)$. We write the operator $\mathcal{S}$ as

$$\mathcal{S}u(x) = \int_0^\infty \int_{S^{n-1}} e^{2\pi i x\xi} \sigma(x, \theta) \mathcal{F}u(\xi)\, |\xi|^{n-1} d\,|\xi|\, d\theta$$

or, briefly,

$$\mathcal{S}u(x) = \int_{S^{n-1}} \sigma(x, \theta) v(x, \theta)\, d\theta, \tag{18}$$

where

$$v(x, \theta) = \int_0^\infty e^{2\pi i x\xi} \mathcal{F}u(\xi)\, |\xi|^{n-1} d\,|\xi|, \tag{19}$$

and

$$\mathcal{F}u(\xi) = \int_{\mathsf{R}^n} e^{-2\pi i y\xi} u(y)\, dy.$$

Using the above introduced structure of the class C^∞ on S^{n-1}, one has

$$\mathcal{S}u(x) = \sum_k \int_{\mathsf{R}^{n-1}} \nu_k(\varphi_k^{-1}(t)) \sigma(x, \varphi_k^{-1}(t)) v(x, \varphi_k^{-1}(t))\, |J_k(t)|\, dt,$$

where J_k is the Jacobian of the mapping φ_k^{-1}. Let $\eta_k \in C_0^\infty(U_k)$ be such that $\eta_k \nu_k = \nu_k$. We put

$$\sigma_k(x, t) = \nu_k(\varphi_k^{-1}(t)) \sigma(x, \varphi_k^{-1}(t)),$$
$$v_k(x, t) = \eta_k(\varphi_k^{-1}(t)) v(x, \varphi_k^{-1}(t))\, |J_k(t)|.$$

By Parseval's theorem,

$$\mathcal{S}u(x) = \sum_k \int_{\mathsf{R}^{n-1}} \sigma_k(x,t)v_k(x,t)\,dt \tag{20}$$
$$= \sum_k \int_{\mathsf{R}^{n-1}} F\sigma_k(x,\tau)\overline{F^{-1}\overline{v_k(x,\tau)}}\,d\tau.$$

Taking into account (19), one obtains

$$\overline{F^{-1}\overline{v_k(x,\tau)}} = \int_{\mathsf{R}^{n-1}} e^{-2\pi i\tau t}\eta_k(\varphi_k^{-1}(t))v(x,\varphi_k^{-1}(t))\,|J_k(t)|\,dt \tag{21}$$
$$= \int_{S^{n-1}} e^{-2\pi i\tau\varphi_k(\theta)}\eta_k(\theta)v(x,\theta)\,d\theta$$
$$= \int_{\mathsf{R}^n} e^{2\pi i x\xi}\eta_k(\theta)e^{-2\pi i\tau\varphi_k(\theta)}\mathcal{F}u(\xi)\,d\xi.$$

The last integral can be interpreted as a family of singular integral convolution operators $E_k(\tau)$, depending on a parameter $\tau \in \mathsf{R}^{n-1}$, with symbols

$$\eta_k(\theta)e^{-2\pi i\tau\varphi_k(\theta)}, \quad k = 1, 2, \ldots$$

Now, from (20) and (21) it follows that $\mathcal{S}$ can be represented in the form

$$\mathcal{S}u(x) = \sum_k \int_{\mathsf{R}^{n-1}} F\sigma_k(x,\tau)E_k(\tau)u(x)\,d\tau. \tag{22}$$

Let l be fractional and let

$$D_l w(x) = \left(\int_{\mathsf{R}^n} |\Delta_h \nabla_{[l]} w(x)|^2 \frac{dh}{|h|^{n+2\{l\}}}\,dh\right)^{1/2}.$$

We have

$$|D_l\mathcal{S}u(x)|^2$$
$$\leq c\sum_{j=0}^{[l]}\sum_k \Bigg\{\int_{\mathsf{R}^n}\left(\int_{\mathsf{R}^{n-1}} |F\nabla_{j,x}\sigma_k(x+h,\tau)|\,|\Delta_h\nabla_{[l]-j,x}E_k(\tau)u(x)|\,d\tau\right)^2 \frac{dh}{|h|^{n+2\{l\}}}$$
$$+ \int_{\mathsf{R}^n}\left(\int_{\mathsf{R}^{n-1}} |F\Delta_h\nabla_{[l]-j,x}\sigma_k(x,\tau)|\,|\nabla_{j,x}E_k(\tau)u(x)|\,d\tau\right)^2 \frac{dh}{|h|^{n+2\{l\}}}\Bigg\}.$$

The right-hand side does not exceed

$$c \sum_{j=0}^{[l]} \sum_k \left\{ \int_{\mathrm{R}^n} \int_{\mathrm{R}^{n-1}} |\mu(\tau) F \nabla_{j,x} \sigma_k(x+h,\tau)|^2 \, d\tau \right.$$
$$\times \int_{\mathrm{R}^{n-1}} |\Delta_h \nabla_{[l]-j,x} E_k(\lambda) u(x)|^2 \frac{d\lambda}{\mu(\lambda)^2} \frac{dh}{|h|^{n+2\{l\}}}$$
$$+ \int_{\mathrm{R}^n} \int_{\mathrm{R}^{n-1}} |\mu(\tau) F \Delta_h \nabla_{[l]-j,x} \sigma_k(x,\tau)|^2 \, d\tau$$
$$\left. \times \int_{\mathrm{R}^{n-1}} |\nabla_{j,x} E_k(\lambda) u(x)|^2 \frac{d\lambda}{\mu(\lambda)^2} \frac{dh}{|h|^{n+2\{l\}}} \right\}. \tag{23}$$

Consequently,

$$\|D_l \mathscr{S} u; \mathrm{R}^n\|_{L_2}^2$$
$$\leqq c \sum_{j=0}^{[l]} \sum_k \left\{ \int_{\mathrm{R}^n} \|\nabla_{j,x} \sigma_k(x,\cdot); \mathrm{R}^{n-1}\|_{\mathscr{H}_\mu}^2 \int_{\mathrm{R}^{n-1}} |(D_{l-j} E_k(\lambda) u)(x)|^2 \frac{d\lambda}{\mu(\lambda)^2} \, dx \right.$$
$$\left. + \int_{\mathrm{R}^n} \left(\int_{\mathrm{R}^n} \|\Delta_h \nabla_{[l]-j,x} \sigma_k(x,\cdot); \mathrm{R}^{n-1}\|_{\mathscr{H}_\mu}^2 \frac{dh}{|h|^{n+2\{l\}}} \right) \left(\int_{\mathrm{R}^{n-1}} |(\nabla_j E_k(\lambda) u)(x)|^2 \frac{d\lambda}{\mu(\lambda)^2} \right) dx \right\}.$$

This and Lemma 1 imply the estimate

$$\|D_l \mathscr{S} u; \mathrm{R}^n\|_{L_2}^2$$
$$\leqq c \sum_{j=0}^{[l]} \sum_k \left\{ \sup_e \frac{\int_e \|\nabla_{j,x} \sigma(x,\cdot); \mathrm{R}^{n-1}\|_{\mathscr{H}_\mu}^2 \, dx}{\mathrm{cap}_{m-l+j}(e)} \int_{\mathrm{R}^{n-1}} \|E_k(\lambda) u; \mathrm{R}^n\|_{H^m}^2 \frac{d\lambda}{\mu(\lambda)^2} \right.$$
$$\left. + \sup_e \frac{\int_e \|D_{l-j} \sigma_k(x,\cdot); \mathrm{R}^{n-1}\|_{\mathscr{H}_\mu}^2 \, dx}{\mathrm{cap}_{m-j}(e)} \int_{\mathrm{R}^{n-1}} \|E_k(\lambda) u; \mathrm{R}^n\|_{H^m}^2 \frac{d\lambda}{\mu(\lambda)^2} \right\}.$$

Since the operators $E_k(\lambda)$ are uniformly bounded in $H^m(\mathrm{R}^n)$, it follows that

$$\|D_l \mathscr{S} u; \mathrm{R}^n\|_{L_2}$$
$$\leqq c\mathscr{K} \sup_e \left(\sum_{j=0}^{[l]} \frac{\int_e (\mathscr{D}_{j,\mu} \gamma(x))^2 \, dx}{\mathrm{cap}_{m-l+j}(e)} + \sum_{j=0}^{[l]} \frac{\int_e (\mathscr{D}_{l-j,\mu} \gamma(x))^2 \, dx}{\mathrm{cap}_{m-j}(e)} \right)^{1/2} \|u; \mathrm{R}^n\|_{H^m},$$

which together with Remark 2 gives

$$\|D_l \mathscr{S} u; \mathrm{R}^n\|_{L_2} \leqq c\mathscr{K} \, \|\sigma\|_{M(H^{m,\mu} \to H^{l,\mu})} \, \|u; \mathrm{R}^n\|_{H^m}. \tag{24}$$

For integer l the proof is similar and somewhat easier. In particular, the counterpart of (23) is

$$c \sum_{j=0}^{l} \sum_k \int_{\mathrm{R}^{n-1}} |\mu(\tau) F \nabla_{j,x} \sigma_k(x,\tau)|^2 \, d\tau \int_{\mathrm{R}^{n-1}} |\nabla_{l-j,x} E_k(\lambda) u(x)|^2 \frac{d\lambda}{\mu(\lambda)^2}.$$

Duplicating the above arguments we arrive at the analogue of (24) with D_l replaced by ∇_l in the left hand-side. This together with the inequality

$$\|\mathscr{S}u; \mathsf{R}^n\|_{L_2} \leqq c\mathscr{K}\, \|\sigma\|_{M(H^{m,\mu}\to H^{0,\mu})}\| u; \mathsf{R}^n\|_{H^m},$$

corresponding to $l = 0$, completes the proof.

REMARK 4. We show that Theorem 2 is precise in a sense.

Let the symbol of $\mathscr{S}$ have the form $a(x)b(\theta)$, $x \in \mathsf{R}^n$, $\theta \in S^{n-1}$, and let $b \in \mathscr{H}_\mu(S^{n-1})$ and $|b(\theta)| \geqq \text{const} > 0$. Clearly, $\mathscr{S}$: $H^m(\mathsf{R}^n) \to H^l(\mathsf{R}^n)$ is continuous if and only if the operator of multiplication by a is a continuous operator from $H^m(\mathsf{R}^n)$ into $H^l(\mathsf{R}^n)$. In other words the condition (16) follows from the continuity of $\mathscr{S}$.

Now let $\mathscr{S}$ be an operator (16) with the symbol $b(\theta)$, $\theta \in S^{n-1}$. Its continuity from $H^m(\mathsf{R}^n)$ into $H^l(\mathsf{R}^n)$ is equivalent to the inequality

$$|b(\theta)|\,(1 + |\xi|^2)^{(l-m)/2} \leqq \text{const}$$

which gives the boundedness of b. Therefore, if for any $b \in \mathscr{H}_\mu(S^{n-1})$ the operator $\mathscr{S}$: $H^m(\mathsf{R}) \to H^l(\mathsf{R}^n)$ is continuous, then $\mathscr{H}_\mu(S^{n-1}) \subset L_\infty(S^{n-1})$, which implies (2).

4. Corollaries.

In this section we give sufficient conditions for the continuity of the operator $\mathscr{S}$: $H^m(\mathsf{R}^n) \to H^l(\mathsf{R}^n)$ which follow from Theorem 2 and from either necessary and sufficient or sufficient conditions for a function to belong to $M(H^{m,\mu} \to H^{l,\mu})$ (see Sec. 2).

The next assertion is a direct corollary of Theorems 1 and 2.

COROLLARY 4. *The estimate* (17) *is equivalent to*

$$\|\mathscr{S}\|_{H^m\to H^l} \leqq c\mathscr{K}\left[\sup_{\{e\subset \mathsf{R}^n:\,\mathrm{diam}(e)\leqq 1\}} \frac{\int_e (\mathscr{D}_{l,\mu}\sigma(x))^2\,dx}{\mathrm{cap}_m(e)} + \sup_{x\in\mathsf{R}^n}\int_{B_1^n(x)} \|\sigma(y,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}\,dy\right]^{1/2} \tag{25}$$

for $m > l \geqq 0$. For $m = l$ the second item on the right in (25) *must be replaced by* $\operatorname{ess\,sup}_{x\in\mathsf{R}^n} \|\sigma(x,\cdot); S^{n-1}\|^2_{\mathscr{H}_\mu}$.

Theorem 2 and Corollary 1 imply the following assertion.

COROLLARY 5. *Let $2m > n$. The inequality* (17) *is equivalent to*

$$\|\mathscr{S}\|_{H^m\to H^l} \leqq c\mathscr{K} \sup_{x\in\mathsf{R}^n} \|\sigma; B_1^n(x) \times S^{n-1}\|_{H^{l,\mu}}. \tag{26}$$

Combining Theorem 2 with Corollary 2 one can remove the capacity from inequality (25) as follows.

COROLLARY 6. *Let* $2m < n$. *Then*

$$\|\mathscr{S}\|_{H^m\to H^l} \tag{27}$$

$$\leq c\mathscr{K}\left[\sup_{\{e\subset \mathsf{R}^n:\,\mathrm{diam}(e)\leq 1\}}\frac{\int_e(\mathscr{D}_{l,\mu}\sigma(x))^2\,dx}{(\mathrm{mes}_n\, e)^{1-2m/n}}+\sup_{x\in\mathsf{R}^n}\int_{B_1^n(x)}\|\sigma(y,\cdot);S^{n-1}\|^2_{\mathscr{H}_\mu}\,dy\right]^{1/2}.$$

For $2m = n$ *the expression* $(\mathrm{mes}_n e)^{1-2m/n}$ *should be replaced by* $(\log(2^{n/}\mathrm{mes}_n e))^{-1}$. *In case* $m = l$ *the second term on the right in* (27) *should be changed by*

$$\operatorname*{ess\,sup}_{x\in\mathsf{R}^n}\|\sigma(x,\cdot);S^{n-1}\|^2_{\mathscr{H}_\mu}.$$

One can easily write inequalities, equivalent to (17) by combining Theorem 1 and Remark 3. A number of sufficient conditions for the continuity of the operator $\mathscr{S}$: $H^m(\mathsf{R}^n)\to H^l(\mathsf{R}^n)$ follow from Theorem 1 and upper estimates for the norm in $M(H^{m,\mu}\to H^{l,\mu})$ which can be obtained due to results in [5], [9], [10].

REMARK 5. For $m = l = 0$ Theorem 2 coincides with the result obtained in [11]. Corollaries 5 and 6 improve the following sufficient condition for the continuity of $\mathscr{S}$ in $H^l(\mathsf{R}^n)$, $\{l\} = 0$, due to Mikhlin [12]:

$$\sup_{x\in\mathsf{R}^n}\sum_{j=0}^{l}\|\nabla_j\sigma(x,\cdot);S^{n-1}\|_{H^\lambda}<\infty,$$

where $2\lambda > n - 1$.

REMARK 6. Theorem 2 and its corollaries can be directly extended to classical pseudo-differential operators with symbols of the form

$$\zeta(\xi)\sum_{k=1}^{N}\sigma_k(x,\xi/|\xi|)|\xi|^{r_k},$$

where $r_1 > \ldots > r_N$ and $\zeta\in C^\infty(\mathsf{R}^{n-1})$, $\zeta(\xi) = 1$ for $|\xi| > 2$, $\zeta(\xi) = 0$ for $|\xi| < 1$ (see [13]).

REFERENCES

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. 2, Berlin-New York, Springer-Verlag, 1983.
2. L. R. Volevich and B. P. Paneyah, *Some spaces of generalized functions and embedding theorems*, Uspehi Mat. Nauk 20 (1965), 3–74.
3. J. Peetre, *On the differentiability of the solutions of quasilinear partial differential equations*, Trans. Amer. Math. Soc. 104 (1962), 476–482.

4. V. G. Maz'ya, *Sobolev Spaces*, Berlin-New York, Springer-Verlag, 1985.
5. V. G. Maz'ya and T. O. Shaposhnikova, *Theory of Multipliers in Spaces of Differentiable Functions*, London, Pitman, 1985.
6. N. S. Landkof, *Foundations of Modern Potential Theory*, Berlin-New York, Springer-Verlag 1972.
7. H. Triebel, *Interpolation Theory. Function Spaces. Differential Operators*, Berlin, VEB Deutscher Verlag, Wissenschaften, 1978.
8. E. Sawyer, *Multipliers of Besov and power-weighted L_2 spaces*, Indiana Univ. Math. J. 33 (1984), 353–366.
9. V. G. Maz'ya and I. E. Verbitsky, *Capacitary inequalities for fractional integrals with applications to partial differential equations and Sobolev multipliers*, Preprint LiTHMAT-R-92-36 (to appear in Ark. Mat.).
10. C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. 9 (1983), 129–206.
11. V. G. Maz'ya and Ju. E. Haikin, *A note on continuity in L_2 of a singular integral operator*, Vestnik Leningrad. Univ. 19, Nr. 4 (1969), 156–159.
12. S. G. Mikhlin and S. Prössdorf, *Singuläre Integraloperatoren*, Berlin, Akademie-Verlag, 1980.
13. J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. 18 (1965), 269–305.

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