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SYMBOLIC CALCULUS FOR SUBSPACES OF $C_0(X)$

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Abstract.

If h is a function which operates on a uniformly closed point separating subspace B of continuous functions vanishing at infinity on a locally compact space X, and if either h is not odd and differentiable at 0 or, if h is increasing and $h'(0) = \infty$, then $B = C_0(X)$.

The Stone-Weierstrass theorem says that if X is a compact Hausdorff space and B is a uniformly closed subspace of $C(X)$, containing the constant functions and separating the points of X, and if the function $h(t) = t^2$ operates on B then $B = C(X)$. (A function h defined on an interval of the real line is said to operate on B if $h \circ b \in B$ for all $b \in B$ for which the composite function $h \circ b$ is defined.) In [4] K. de Leeuw and Y. Katznelson extended this result by showing that the function $h(t) = t^2$ can be replaced by any function defined and continuous on an interval which is not affine, i.e. not of the form $h(t) = \alpha t + \beta$.

It turns out that there is a somewhat similar result for subspaces of $C_0(X)$, where X is a locally compact Hausdorff space, but that the proof of the result of de Leeuw and Katznelson can not be applied to this situation, except in a special case. To see why, we give a brief sketch of the proof.

First we consider the case where the operating function is twice continuously differentiable. Differentiating the function

$$t \to h \circ (b + tc),$$

where $b, c \in B$, twice and then putting $t = 0$ we find that

$$c^2 \cdot h'' \circ b \in B. \tag{1}$$

Since h is not affine, it is possible to choose a constant function $b \in B$ such that $h'' \circ b \neq 0$ and then we conclude that

$$c^2 \in B \quad \text{for all} \quad c \in B.$$

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This method also works for subspaces of $C_0(X)$. From (1) we deduce that

$$c \cdot d \cdot h'' \circ b \in B \quad \text{for all} \quad c, d \in B.$$

Now,

$$A := \{f \in C_0(X) : fB \subseteq B\}$$

is clearly an algebra containing the functions $d \cdot h'' \circ b$. If h is not of the form $h(t) = \alpha t$ on any interval containing 0, then there is for any $x \in X$, a funtion $b \in B$ such that $h''(b(x)) \neq 0$, and hence, since the functions $d \cdot h'' \circ b$ belong to A, it follows that if these functions separate the points of X then $B = C_0(X)$.

If h is not a smooth function we pick a function $\phi \in C_0^\infty(R)$ and form the smooth function $h_\phi = h * \phi$. Approximating the integral below by Riemann sums, we find that if $b \in B$ and ϕ has support in a sufficiently small neighbourhood of 0 then

$$h_\phi \circ b = \int h \circ (b + s)\phi(s)\, ds \in B,$$

i.e. the function h_ϕ operates on B. If h is not affine, then it is possible to choose ϕ such that $h_\phi'' = h * \phi'' \neq 0$, and thus, h_ϕ is a smooth non-affine operating function.

This proof does not carry over to a subspace B of $C_0(X)$, because if s is a real number the function b_s, given by $b_s(x) = h(b(x) + s)$ where $b \in B$, does not have to belong to B. It even does not have to belong to $C_0(X)$. Thus a different approach is needed to deal with subspaces of $C_0(X)$. As we shall see, we will make use of the method invented by de Leeuw and Katznelson.

Let us begin by looking at two very simple examples, which indicate that one has to make some extra assumptions in the locally compact case.

EXAMPLE 1. Let $X = [-1, 1]\setminus\{0\}$ and let $B = \{f \in C(X) : f(t) = \alpha t \text{ if } t < 0 \text{ and } f(t) = \beta t \text{ if } t > 0\}$. Clearly, the function $h(t) = |t|$ operates on B.

EXAMPLE 2. Let $X = [-1, 1]\setminus\{0\}$ and let B be the space of all continuous odd functions on X. Then the function $h(t) = t^3$ operates on B, and also any continuous odd function on an interval centered at 0.

We are now ready to state the main result of this note.

THEOREM. *Let X be a locally compact Hausdorff space and let B be a uniformly closed subspace of $C_0(X)$ separating the points of X in the following strong sense: Firstly, for each pair x_1, x_2 of points in X there is a function $b \in B$ such that $|b(x_1)| \neq |b(x_2)|$, and secondly, the functions in B do not all vanish at some point in X. Let h be a function defined and continuous on an open interval containing zero, with $h(0) = 0$, which operates on B. If either*

i) *h is differentiable at* 0, *and not an odd function on any interval containing* 0 *or*

ii) *h is increasing and* $h'(0) = \infty$, i.e.

$$\lim_{t \to 0} \frac{h(t)}{t} = \infty,$$

then $B = C_0(X)$.

PROOF. For a function $b_0 \in B$ we let

$$B(b_0) = \{b \in B : |b| \leqq k\,|b_0| \text{ for some positive number } k > 0\}.$$

Further, let

$$A(b_0) = \{f \in C_0(X) : f \cdot \overline{B(b_0)} \subseteq \overline{B(b_0)}\}.$$

(Here, and elsewhere, "bar" means closure in the sup-norm topology). Clearly $A = A(b_0)$ is an algebra. If we can show that A separates points that are separated by b_0 it will follow that $b_0 \in A$ and hence that $b_0^2 \in \overline{B(b_0)} \subseteq B$. Since b_0 is an arbitrary function in B we will then be able to conclude that $B = C_0(X)$.

Let $c \in B(b_0)$. If r, s and t are sufficiently small real numbers then $h \circ (rb_0 + tc - sb_0) \in B$. Approximating the integral below by Riemann sums, we see that if $\phi \in C_0^\infty(R)$ has support in a sufficiently small neighbourhood of 0, then the function

$$c_t = \int h \circ (rb_0 + tc - sb_0)\phi(s)\,ds$$

belongs to B. Let us put

$$\Delta_\delta^2 c_t = \frac{1}{\delta^2}(c_{t+\delta} + c_{t-\delta} - 2c_t).$$

Then $\Delta_\delta^2 c_t \in B$. If $b_0(x) = 0$ then $\Delta_\delta^2 c_t(x) = 0$. If $b_0(x) \neq 0$ then, changing variables and using a mean value theorem for the second derivative, we find that

$$\Delta_\delta^2 c_t(x) = \frac{c^2(x)}{b_0^2(x)} \int h(rb_0(x) + (t + \lambda\delta)c(x) - sb_0(x))\phi''(s)\,ds$$

where $-1 < \lambda < 1$. Putting $t = 0$ and letting $\delta \to 0$ we get

$$\Delta^2 c(x) := \lim_{\delta \to 0} \Delta_\delta^2 c_0(x) = \frac{c^2(x)}{b_0^2(x)} \int h(rb_0(x) - sb_0(x))\phi''(s)\,ds$$

Let us put

(2) $$d = \int h \circ (rb_0 - sb_0)\phi''(s)\,ds$$

Then $d \in B$ and $d(x) = 0$ if $b_0(x) = 0$. Thus, if we put $\Delta^2 c(x) = 0$ whenever $b_0(x) = 0$ then $\Delta^2 c \in C_0(X)$, and since $\Delta_\delta^2 c_0 \in B$ and since $\Delta_\delta^2 c_0 \to \Delta^2 c$ boundedly as $\delta \to 0$, it follows that $\Delta^2 c \in B$, that is,

$$\frac{c^2 d}{b_0^2} \in B \quad \text{for all } c \in B(b_0)$$

and hence

$$\frac{c_1 c_2 d}{b_0^2} \in B \quad \text{for all} \quad c_1, c_2 \in B(b_0)$$

Let us now assume that h satisfies condition i). Replacing $h(t)$ by $h(t) - h'(0)t$ we may assume that $h'(0) = 0$ *i.e.* $h(t)/t \to 0$ as $t \to 0$, and then $cd/b_0^2 \in A$ for all $c \in B(b_0)$. We are going to show that if b_0 separates x_1 and x_2, then cd/b_0^2 separates x_1 and x_2, for a suitable choice of $c \in B(b_0)$.

So, let us suppose that $b_0(x_1) \neq b_0(x_2)$, and that $b_0(x_1) \neq 0$. Since h is not affine in any open interval containing 0, we can choose r and ϕ such that $d(x_1) \neq 0$, where d is defined in (2). If $b_0(x_2) = 0$ then $d/b_0 = b_0 d/b_0^2 \in A$ separates x_1 and x_2. If $d(x_1)/b_0(x_1) = d(x_2)/b_0(x_2)$, we put $c = h \circ (ub_0)$, where u is a real number, and try to choose u to get separation.

Supposing that $|b_0(x_1)| \leqq |b_0(x_2)|$ we put $\lambda = b_0(x_1)/b_0(x_2)$. If $c(x_1)/b_0(x_1) = c(x_2)/b_0(x_2)$ then $h(\lambda v) = \lambda h(v)$ where $v = ub_0(x_2)$. If this holds for all sufficiently small v then $h(\lambda^n v) = \lambda^n h(v)$ for all sufficiently small v. But, since h is not odd, $\lambda \neq -1$ and hence $|\lambda| < 1$. But then we have a contradiction, because $h'(0) = 0$ and h is not identically zero in any neighbourhood of 0. We conclude that there must be a number u such that $c(x_1)/b_0(x_1) \neq c(x_2)/b_0(x_2)$ and then $cd/b_0^2 \in A$ separates x_1 and x_2.

Let us now assume that h satisfies condition ii). Replacing $h(t)$ by $h(t) - h(-t)$ we may assume that h is odd on some interval containing 0.

We begin by showing that if K is a compact subset of X for which there is a function $b_0 \in B$ such that $|b_0| > 0$ on K then $B \mid K$ is dense in $C(K)$. To this end let $b, c \in B$. Then

$$\int h \circ (rb + tc - sb_0)\phi(s)\,ds \,|K \in \overline{B| K}$$

if r and t are sufficiently small real numbers and if $\phi \in C_0^\infty(R)$ has support in a sufficiently small neighbourhood of 0. Here, we write $f \mid K$ for the restriction to K of a function f defined on X. Differentiating twice w.r.t. the variable t and putting $t = 0$ we find that

$$\frac{c^2}{b_0^2}\int h\circ(rb-sb_0)\phi''(s)\,ds\,|K\in\overline{B|K}\quad\text{for all}\quad c\in B.$$

Let us put

$$d=\int h\circ(rb-sb_0)\phi''(s)\,ds \tag{3}$$

Then $c_1c_2d/b_0^2\,|K\in\overline{B|K}$ for all $c_1,c_2\in B$. Now, the set

$$A=\{f\in C(K):f\cdot B\,|K\subseteq\overline{B|K}\}$$

is a uniformly closed algebra containing the constant functions. If we can show that A separates the points of K, it will follow that $A=C(K)$, and hence that $B\,|\,K$ is dense in $C(K)$.

Let $x_1,x_2\in K$. Since functions of the form cd/b_0^2 belong to A, it suffices to show that a function of this type separates x_1 and x_2. Since h is not affine we can choose b and r such that if d is the function defined in (3) then $d(x_1)\neq 0$. If $d/b_0\,|\,K=b_0d/b_0^2\,|\,K$ does not separate x_1 and x_2, we try to find a function $c\in B$ such that $c/b_0\,|\,K$ separates x_1 and x_2. This is clearly possible if $|b_0(x_1)|=|b_0(x_2)|$, because by assumption, there is a function $c\in B$ such that $|c(x_1)|<|c(x_2)|$. If $|b_0(x_1)|<|b_0(x_2)|$ we put $c=h\circ(ub_0)$, where u is a real number. As in the first part a suitable choice of u will give a function c such that $c/b_0\,|\,K$ separates x_1 and x_2 and then $cd/b_0^2\,|\,K\in A$ separates x_1 and x_2. We have thus shown that $A\,|\,K$ is dense in $C(K)$.

In the next step we show that if E and F are disjoint subsets of X, for which there is a function $b\in B$ such that $b\,|\,E<-1$ and $b\,|\,F>1$, and if we put

$$L=L_{E,F}=\inf\{\|b\|:b\in B,\ b\,|E<-1\text{ and }b|\,F>1\},$$

then $L=1$.

We choose a sequence $\{b_n\}$ in B such that $b_n\,|E<-1$, $b_n|\,F>1$ and $\lim\|b_n\|=L$. For any sufficiently small positive value of λ we consider the functions $c_n=h\circ\left(\frac{\lambda}{\|b_n\|}b_n\right)$. Since h is an increasing odd function, $\|c_n\|=h(\lambda)$ and also, $c_n\,|\,E<-h\left(\frac{\lambda}{\|b_n\|}\right)$ and $c_n\,|\,F>h\left(\frac{\lambda}{\|b_n\|}\right)$. Thus by the definition of L,

$$h\left(\frac{\lambda}{\|b_n\|}\right)^{-1}h(\lambda)=\left\|h\left(\frac{\lambda}{\|b_n\|}\right)^{-1}\cdot c_n\right\|\geqq L.$$

Taking the limit we find that

$$h(\lambda)\geqq Lh\left(\frac{\lambda}{L}\right)$$

or, equivalently that

$$\frac{h(\lambda)}{\lambda} \geqq \frac{h\left(\dfrac{\lambda}{L}\right)}{\dfrac{\lambda}{L}},$$

for all sufficiently small positive numbers λ. Since $h'(0) = \infty$, and since $L \geqq 1$, we deduce that $L = 1$.

Let us finally show that $B = C_0(X)$. If not, then there is a non-zero measure $\mu \in B^{\perp}$, and hence the number

$$M = \sup\left\{\int |b|\, d\,|\mu| : b \in B \text{ and } \|b\| \leqq 1\right\}$$

is larger than zero.

For any $\varepsilon > 0$ we pick a function $b \in B$ with $\|b\| \leqq 1$ such that

$$\int |b|\, d\,|\mu| > M - \varepsilon.$$

We can then choose a compact set K, such that $|b| > 0$ on K, and such that

$$\int_K |b|\, d\,|\mu| > M - \varepsilon.$$

Let $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of μ, and let E and F be disjoint compact subsets of K, such that $\mu^+(E) = \mu^-(F) = 0$, and such that

$$\int_{E \cup F} |b|\, d\,|\mu| > M - \varepsilon.$$

As has already been shown, there is a function $b_1 \in B$ such that $b_1 \mid E < -1$, $b_1 \mid F > 1$, and such that $\|b_1\| < 1 + \varepsilon$. Then,

$$\int_{E \cup F} b_1\, d\mu \geqq \mu^-(E) + \mu^+(F) \geqq \int |b|\, d\,|\mu| > M - \varepsilon.$$

Also, since $\mu \in \mathrm{B}^{\perp}$,

$$\left|\int_{X \setminus (E \cup F)} b_1\, d\mu\right| = \left|\int_{E \cup F} b_1\, d\mu\right| > M - \varepsilon,$$

and hence

$$\int |b_1|\, d\,|\mu| > 2M - 2\varepsilon.$$

By the definition of M, this implies that $(1 + \varepsilon)M \geqq 2M - 2\varepsilon$. Since this holds for all $\varepsilon > 0$, it follows that $M \geqq 2M$, contradicting the fact that $M > 0$. Thus the only measure in $B^{\perp}$ is the zero measure, and hence $B = C_0(X)$.

REMARK. In the first part of the proof it suffices to assume that the points of X are separated by functions in B, and that the functions in B do not all vanish at some point in X.

EXAMPLE 3. Functions satisfying conditions i) or ii) are f.ex. functions h which are not odd, satisfying $|h(t)| \leqq k|t|^{1+\delta}$, for some positive numbers k and δ, and functions such as $h(t) = t^{1/p}$, where p is an odd natural number larger than 1. If $h(t) = |t|^{1/p}$, where p is an even natural number, operates B, then $B = C_0(X)$. This is because in that case, the function $k(t) = |\,|t|^{1/p} + t|^{1/p} - |t|^{t/p^2}$, operates on B and satisfies condition ii). The idea to construct the function k is due to A. Bernard [1].

REMARK. Clearly there might be weaker conditions on h which could still imply that $B = C_0(X)$. It would be interesting to know, whether one can drop the assumption that h is increasing in condition ii). We finally note, that condition ii) enters in the proof of Katznelsons square root theorem. (See [2] and [3]).

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