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A SPATIAL CHARACTERIZATION OF ω CONDITIONAL EXPECTATIONS ON VON NEUMANN ALGEBRAS

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Abstract.

Representations of the predual of von Neumann algebras as spaces of densely defined sesquilinar forms on the Hilbert space on which the von Neumann algebras itself operate are taken as a setting on which to extend ω-conditional expectations. This allows us to give a spatial characterization of ω-conditional expectations through a property which is a natural generalization of a characterising property of norm one projections on von Neumann algebras.

1. Introduction.

In [1] the ω-conditional expectation (ε_ω) from a von Neumann algebra to a von Neumann subalgebra M_0 with respect to a faithful normal state ω on M (with restriction ω_0 to M_0) has been first introduced as a generalization of ω preserving norm one projections. In [4], of which this paper can be seen as a sequel, it has been characterized as a dual map of a canonical state extension (cfr. [5] and [6]) perturbed with convenienient partial isometries in M. Our main result, stated now for convenience in the simple case of matrix algebras, says that a positive linear contraction ε from M to M_0 preserving ω coincides with ε_ω iff for all a in M and all normal states ϕ_0 on M_0 we have:

$$\varepsilon([\rho_\omega(\phi_0)/\omega]^+ a[\rho_\omega(\phi_0)/\omega]) = [\phi_0/\omega_0]^+ \varepsilon(u^+ au)[\phi_0/\omega_0],$$

where $\rho_\omega(\phi_0)$ is the canonical extension of ϕ_0 to M with respect to ω, u is the partial isometry involved in the extension and the notation $[\phi/\omega]$ denotes, for any pair of normal states ϕ, ω on a matrix algebra, the analytic extension of the Connes' cocycle for ϕ and ω in the point $i/2$ (see [2] and [7]). The above formula in the case in which $[\rho_\omega(\phi_0)/\omega] = [\phi_0/\omega_0]$ and u is the identity reduces to:

$$\varepsilon_\omega([\phi_0/\omega_0]^+ a[\phi_0/\omega_0]) = [\phi_0/\omega_0]^+ \varepsilon_\omega(a)[\phi_0/\omega_0].$$

The case in which ε_ω is a norm one projection occurs when the above formula

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holds for all normal states ϕ_0 on M_0. This can be seen by setting $a = 1$ and recalling that any positive element in M_0 is of the form $|[\phi_0/\omega_0]|^2$ for some normal state ϕ_0 on M_0.

For general von Neumann algebras the analytic extension of the Connes' cocycle for two normal states ϕ and ω (faithful) in the point $i/2$ does not always exist (at least as a bounded operator). So, in order to generalize the above stated characterization of ω-conditional expectations to the general situation, after giving the necessary preliminaries and establishing our notations in section 2, section 3 will be devoted to the task of enlarging a von Neumann algebra by embedding it in a representation of its predual as a linear space of sesquilinear forms defined on a dense linear subspace of the Hilbert space on which the von Neumann algebra itself acts. We shall also extend ω-conditional expectations to this representation. Finally section 4 will contain our main result.

2. Preliminaries and notations.

Let M be a von Neumann algebra acting on a separable Hilbert space H with commutant M', and ω be a faithful state on M. We shall denote, as in [7], by $D(H, \omega)$ the dense linear subspace of all ξ in H such that the functional $a \to \langle \xi, a\xi \rangle$ on M is majorized by some positive multiple of ω (or dominated by ω). The action of M' maps $D(H, \omega)$ into itself. Let now ω be faithful; we shall denote by π the left representation of M on a standard Hilbert space H with a cyclic and separating vector Ω such that $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$ and by J the isometrical involution associated to Ω. Let ϕ be in $(M*)_+$; $\pi(\phi)$ shall be defined by $\pi(\phi)(\pi(a)) = \phi(a)$, For a in M we set $\pi'(a) = J\pi(a)^+J$; so π' is the right representation for M. For each ξ in $D(H, \omega)$ (cfr. [6]) there is a unique bounded linear operator $R(\xi)$: $\mathsf{H} \to H$ such that $R(\xi)\pi(a)\Omega = a\xi$. For all ξ, η in $D(H, \omega) R(\xi)R(\eta)^+$ is in M', and (see [3], 3.2) $\mathrm{R}(\xi)^+\mathrm{R}(\eta)$ is in $\pi'(M)$. We shall set $\pi^{-1}(JR(\xi)^+R(\eta)J) = \theta(\xi, \eta)$. Let now ϕ be also in $(M*)_+$ and $(D\phi : \mathrm{D}\omega)_t$ the Connes' cocycle for ϕ and ω in M. If ϕ is dominated by ω the mapping $t \to (D\phi : \mathrm{D}\omega)_t$ admits a continuous extension to the strip S of the complexes z with $0 \leqq \mathrm{Im}\, z \leqq 1/2$, which is analytic in its interior (see [7]); we shall denote its value at $i/2$ by $[\phi/\omega]$ (see also [2]). In general, however, for all ξ in $D(H, \omega)$ the mapping $t \to (D\phi : D\omega)_t\xi$ admits a continuous extension to S analytic in its interior. We shall denote again its value by $[\phi/\omega]\xi$; so the mapping $[\phi/\omega]$ is always a linear operator on $D(H, \omega)$ commuting with M'. The selfdual positive cone in H containing Ω, which is pointwise invariant under J, is the closure of the set: $\{[\pi(\phi)/\pi(\omega)]\Omega$: ϕ normal state dominated by ω on $M\}$.

Let now M_0 be a von Neumann subalgebra of M, $\omega_0 = \omega \mid M_0$. Let H_0 be the subspace of H closure of $\{a_0\Omega$, a_0 in $M_0\}$, and E the orthogonal projection from H to H_0. Then $\pi(M_0)$ acts standardly on H_0, Ω is a cyclic and separating vector for $\pi(M_0)$ in H_0 and for all a_0 in M_0 we have $E\pi(a_0)E \mid \mathsf{H}_0 = \pi_0(a_0)$, the left representation of M_0 on H_0. We shall often identify H_0 with the Hilbert space on which the standard representation of M_0 acts. We shall endow with a subscript "0" all the above defined objects when referred to M_0. As in [1] the ω conditional expectation ε_ω from M to M_0 is the mapping satisfying $\pi_0(\varepsilon_\omega(a))\Omega = J_0 E J \pi(a)\Omega$ for all a in M. Let ϕ_0 be a normal state on M_0. We define (cfr. [4], [5] and [6]) the canonical state extension $\rho_\omega(\phi_0)$ of ϕ_0 to M with respect to ω by setting, for all a in M, $\rho_\omega(\phi_0(a)) = \omega([\phi_0/\omega_0]^+ a[\phi_0/\omega_0])$ if ϕ_0 is dominated by ω_0, and otherwise by continuity. We shall denote by $u(\phi_0, \omega)$ the (unique) partial isometry in M with initial projection the identity introduced in [5] which satisfies the equality $J\pi(u(\phi_0, \omega))J[\pi_0(\phi_0)/\pi_0(\omega_0)]\Omega = [\pi(\rho_\omega(\phi_0))/\pi(\omega)]\Omega$.

3. Extending ω-conditional expectations.

Let M be a von Neumann algebra acting on a separable Hilbert space H, and ω be a faithful state on M.

3.1. Definition. We call $M_2(H, \omega)$ the vector space of all linear (not necessarily continuous) operators T from $D(H, \omega)$ to H which commute with M' and which satisfy the following condition:

$$(*) \qquad \left| \sum_{n=1}^{N} \langle \xi_n, T\eta_n \rangle \right| \leqq \alpha \left\| \sum_{n=1}^{N} R(\eta_n)^+ \xi_n \right\|$$

for some $\alpha > 0$ and all ξ_n, η_n in $D(H, \omega)$.

The above condition (*) is nothing else than the continuity in the $\omega - L_2$ norm of the linear map which maps $\theta(\xi, \eta)$ into $\langle \xi, T\eta \rangle$ on the linear span of $\{\theta(\xi, \eta)\colon \xi, \eta$ in $D(H, \omega)\}$.

3.2. Proposition. *For all a in M $a \,|\, D(H, \omega)$ is in $M_2(H, \omega)$.*

Proof. For all ξ_n, η_n in $D(H, \omega)$ we have:

$$\left| \sum_{n=1}^{N} \langle \xi_n, a\eta_n \rangle \right| = \left| \sum_{n=1}^{N} \langle \xi_n, R(\eta_n)\pi(a)\Omega \rangle \right|$$

$$= \left| \left\langle \sum_{n=1}^{N} R(\eta_n)^+ \xi_n, \pi(a)\Omega \right\rangle \right| \leqq \|\pi(a)\Omega\| \left\| \sum_{n=1}^{N} R(\eta_n)^+ \xi_n \right\|.$$

3.3. Proposition. *Let T be in $M_2(H, \omega)$. There is a unique $\pi(T)$ in $(\pi(M))_2(\mathsf{H}, \pi(\omega))$ such that for all ξ, η in $D(H, \omega)$ we have $R(\eta)\pi(T)\Omega = T\eta$. The map $T \to \pi(T)$ is linear, for all a in M we have $\pi(a \,|\, D(H, \omega)) = \pi(a) \,|\, D(\mathsf{H}, \pi(\omega))$, and for all a in M and T in $M_2(H, \omega)$ we have $\pi(aT) = \pi(a)\pi(T)$. It is also invertible, in the sense that for all T in $(\pi(M))_2(\mathsf{H}, \pi(\omega))$ there is a T in $M_2(H, \omega)$ such that $\pi(T) = \mathsf{T}$.*

Proof. From [8] and [3] it follows that the linear space spanned by

$\{R_\omega(\xi)^+ R_\omega(\eta)\Omega\colon \xi, \eta \text{ in } D(H,\omega)\}$ is dense in H. By (*) there is therefore a unique vector Ω_T in H such that:

$$\left\langle \sum_{n=1}^{N} R(\eta_n)^+ R(\xi_n)\Omega, \Omega_T \right\rangle = \left\langle \sum_{n=1}^{N} R(\eta_n)^+ \xi_n, \Omega_T \right\rangle = \sum_{n=1}^{N} \langle \xi_n, T\eta_n \rangle$$

for all ξ_n, η_n in $D(H,\omega)$. Set now $\pi(T)\pi'(a)\Omega = \pi'(a)\Omega_T$ for all $\pi'(a)$ in $\pi'(M)(=\pi(M)')$. We note first that $\{\pi'(a)\Omega\colon a \text{ in } M\} = D(\mathsf{H}, \pi(\omega))$ and that $\pi(T)$ obviously commutes with $\pi(M)'$. Also, for all a_n, b_n in M we have:

$$\left| \sum_{n=1}^{N} \langle \pi'(a_n)\Omega, \pi(T)\pi'(b_n)\Omega \rangle \right| = \left| \sum_{n=1}^{N} \langle \pi'(b_n)^+ \pi'(a_n)\Omega, \pi(T)\Omega \rangle \right|$$

$$\leqq \left\| \sum_{n=1}^{N} \pi'(b_n)^+ \pi'(a_n)\Omega \right\| \left\| \pi(T)\Omega \right\|,$$

which is (*) for the standard representation. Our construction implies, for ξ, η in $D(H,\omega)$, $\langle \xi, R(\eta)\pi(T)\Omega \rangle = \langle \xi, T\eta \rangle$, and by the density of $D(H,\omega)$ in H the equality $R(\eta)\pi(T)\Omega = T\eta$ follows.

The linearity of π is obvious from our construction, as well as the equality $\pi(a \mid D(H,\omega)) = \pi(a) \mid D(\mathsf{H}, \pi(\omega))$ for all a in M.

For a in M we have, using our previous notations:

$$\left\langle \sum_{n=1}^{N} R(\eta_n)^+ R(\xi_n)\Omega, \pi(aT)\Omega \right\rangle = \sum_{n=1}^{N} \langle R(\xi_n)\Omega, aT\eta_n \rangle$$

$$= \sum_{n=1}^{N} \langle a^+ R(\xi_n)\Omega, T\eta_n \rangle = \sum_{n=1}^{N} \langle R(\xi_n)\pi(a)^+ \Omega, T\eta_n \rangle$$

$$= \left\langle \sum_{n=1}^{N} R(\eta_n)^+ R(\xi_n)\Omega, \pi(a)\pi(T)\Omega \right\rangle,$$

which by density implies $\pi(aT) = \pi(a)\pi(T)$.

Let now T be in $(\pi(M))_2\,(\mathsf{H}, \pi(\omega))$; clearly if we define T by setting $R(\eta)\mathsf{T}\Omega\mathsf{T} = \mathsf{T}\eta$ for all η in $D(H,\omega)$, T is in $M_2(H,\omega)$ and $\pi(T) = \mathsf{T}$.

3.4. Remark. Our proof of prop. 3.3 implies also that if H contains a separating vector Ψ for M then condition (*) is automatically satisfied for all linear operators from $D(H,\omega)$ to H which commute with M' (whose set therefore coincides with $M_2(H,\omega)$). In this case T is completely characterized by the vector $T\Psi$ (in other words Ψ is separating also for $M_2(H,\omega)$); moreover, in general T is completely characterized by $\pi(T)\Omega$. So we can establish a linear bijection between the vectors in H and $M_2(H,\omega)$, and remark that $M_2(H,\omega)$ is therefore nothing else than one of the possible concrete spatial realizations of $L^2(M,\omega)$. This allows us to look at $M_2(H,\omega)$ as a Banach space with $\|T\|_2 = \|\pi(T)\Omega\|$ for

T in $M_2(H,\omega)$ as a $L^2(M,\omega)$ norm. Prop. 3.3 implies that if there is an isomorphism λ from a von Neumann algebra M_1 acting on a Hilbert space H_1 to a von Neumann algebra M_2 acting on a Hilbert space H_2, and ω_i $(i = 1, 2)$ are normal faithful states on M_i such that $\omega_1(a) = \omega_2(\lambda(a))$, then λ can be extended to a linear mapping (which we shall also denote by λ) from $(M_1)_2(H_1, \omega_1)$ to $(M_2)_2(H_2, \omega_2)$ such that, for all a in M_1 and T in $(M_1)_2(H, \omega)$, $\lambda(aT) = \lambda(a)\lambda(T)$.

We also note that if ϕ is in $(M*)_+$ then $[\phi/\omega]$ is in $M_2(H,\omega)$ and $[\pi(\phi)/\pi(\omega)] = \pi([\phi/\omega])$; the selfdual positive cone in H containing Ω, is the set $\{\pi([\phi/\omega])\Omega$: ϕ normal state on $M\}$.

3.5. Definition. We shall denote by $M_1(H,\omega)$ the set of all complex valued sesquilinear mappings q on $D(H,\omega) \times D(H,\omega)$ of the form $q(\xi,\eta) = \langle T_1\xi, T_2\eta\rangle$ with T_1, T_2 in $M_2(H,\omega)$ for all ξ,η in $D(H,\omega)$.

We shall often denote the above defined mapping q by $[q(T_1, T_2)]$.

3.6. Lemma. *$M_1(H,\omega)$ is a linear space. Let ϕ in $M*$ be such that for a in $M\phi(a) = \langle\Phi_1, \pi(a)\Phi_2\rangle$ with Φ_1, Φ_2 in H, and take T_1, T_2 in $M_2(H,\omega)$ such that $\pi(T_1)\Omega = J\Phi_1$, $\pi(T_2)\Omega = J\Phi_2$. Then the mapping μ which maps ϕ into $\mu(\phi) = [q(T_1, T_2)]$ is a linear bijection from $M*$ to $M_1(H,\omega)$.*

Proof. We have: $\mu(\phi)(\xi, \eta) = \langle T_1\xi, T_2\eta\rangle = \langle T_1R(\xi)\Omega, T_2R(\eta)\Omega\rangle = \langle R(\xi)\pi(T_1)\Omega, R(\eta)\pi(T_2)\Omega\rangle = \langle R(\xi)J\Phi_1, R(\eta)J\Phi_2\rangle = \phi(\theta(\xi,\eta))$. It is now enough to note that $\{\theta(\xi,\eta){:}\xi,\eta$ in $D(H,\omega)\}$ is weakly dense in M and that all elements in $M*$ are of the form $a \to \langle\Phi_1, \pi(a)\Phi_2\rangle$ with Φ_1, Φ_2 in H to get our claim.

Lemma 3.6 tells us $M_1(H,\omega)$ is one of the many possible spatial realizations of $M*$; it becomes a Banach space with the norm $\|\mu(\phi)\|_1 = \|\phi\|$ for ϕ in $M*$. We shall denote the embedding of $M_2(H,\omega)$ into $M_1(H,\omega)$ by setting $q^T = [q(I, T)]$ (for a in M we shall make no distinction between a and $a \mid D(H,\omega)$).

3.7. Definition. For q in $M_1(H,\omega)$ as above we define $\pi(q)$ by setting, for a, b in M:

$$\pi(q)(\pi'(a)\Omega, \pi'(b)\Omega) = \langle\pi(T_1)\pi'(a)\Omega, \pi(T_2)\pi'(b)\Omega\rangle.$$

3.8. Lemma. *$\pi(q)$ is in $(\pi(M))_1(\mathsf{H}, \pi(\omega))$, and if T is in $M_2(H,\omega)$, then $q^{\pi(T)} = \pi(q^T)$.*

Proof. Immediate.

Let now M_0 be a von Neumann subalgebra of M, $\omega_0 = \omega \mid M_0$, and let us use our established notations and their natural generalization to M_0 and ω_0.

3.9. Lemma. *Let q be in $M_1(H,\omega)$, ξ_0, η_0 in $D(\mathsf{H}_0, \pi_0(\omega_0))$. We define Eq by setting:*

$$(\boldsymbol{E}q)(\xi_0,\eta_0) = \pi(q)(JJ_0\xi_0, JJ_0\eta_0).$$

Then $\boldsymbol{E}$ *is a linear mapping from* $M_1(H,\omega)$ *to* $(\pi_0(M_0))_1(\mathsf{H}_0, \pi_0(\omega_0))$, *and* $\boldsymbol{E}q^a = q_0\pi_0(\varepsilon_\omega(a))$ *for all a in M.*

PROOF. Since Ω is a cyclic and separating vector in $E\mathsf{H}$ for $E\pi(M_0)\,|\,E\mathsf{H} = \pi_0(M_0)$ we identify $E\mathsf{H}$ with H_0. There are a_0, b_0 in M_0 such that $\xi_0 = \pi_0'(a_0)\Omega$ and $\eta_0 = \pi_0'(b_0)\Omega$. So $J_0\xi_0 = \pi_0(a_0^+)\Omega = \pi(a_0^+)\Omega$, and $JJ_0\xi_0 = J\pi(a_0^+)\Omega = \pi'(a_0)\Omega$ (resp. $J_0\eta_0 = \pi(b_0^+)\Omega$ and $JJ_0\eta_0 = \pi'(b_0)\Omega$), which is in $D(\mathsf{H},\pi(\omega))$. So the right hand side of our equality is well defined. We also have (cfr. lemma 3.6.):

$$\pi(q)(JJ_0\xi_0, JJ_0\eta_0) = \mu^{-1}(q)(\theta(JJ_0\xi_0, JJ_0\eta_0)) = \mu^{-1}(q)(a_0^+b_0)$$
$$= \mu^{-1}((q))_0(a_0^+b_0).$$

On the other hand $\theta_0(\xi_0,\eta_0) = \pi_0(a_0^+b_0)$; so $\pi(q)(JJ_0\xi_0, JJ_0\eta_0) = \mu^{-1}(q)$ $(\pi_0^{-1}(\theta_0(\xi_0,\eta_0))) = \mu_0(\mu^{-1}(q))_0)(\theta_0(\xi_0,\eta_0))$. So $\boldsymbol{E}q$ is the element of $(\pi_0(M_0))_1(\mathsf{H}_0,\pi_0(\omega_0))$ satisfying $(\boldsymbol{E}q)(\xi_0,\eta_0) = \mu_0((\mu^{-1}(q))_0)(\xi_0,\eta_0)$ for all ξ_0,η_0 in $D(\mathsf{H}_0,\pi_0(\omega_0))$.

Let now a be in M and $q = q^a$. Then $(\boldsymbol{E}q)(\xi_0,\eta_0) = \langle JJ_0\xi_0, \pi(a)JJ_0\eta_0\rangle = \langle \xi_0, J_0EJ\pi(a)JJ_0\eta_0\rangle = \langle \xi_0, \pi_0(\varepsilon(a))\eta_0\rangle = q_0\pi_0(\varepsilon(a))(\xi_0,\eta_0) = q_0\pi_0(\varepsilon(a))(\xi_0,\eta_0) = q_0\pi(\varepsilon(a))(\xi_0,\eta_0)$.

Let π_0 denote the mapping from $(M_0)_1(H,\omega)$ extension of the (faithful) isomorphism from M_0 to $\pi_0(M_0)$ as in def. 3.7.

3.10. DEFINITION. We denote by $\boldsymbol{E}$ the mapping from $M_1(H,\omega)$ to $(M_0)_1(H,\omega_0)$ defined by $\boldsymbol{E}(q) = \pi_0^{-1}(\boldsymbol{E}(q))$ and call it the 1-ω-conditional expectation for M and M_1.

3,11. THEOREM. *The above defined mapping* $\boldsymbol{E}$ *is a linear contraction, which extends the* ω*-conditional expectation* ε_ω *from* M *to* M_0 *both on* $M_1(H,\omega)$ *and on* $M_2(H,\omega)$.

(Our last statement means, more precisely, that if T is in $M_2(H,\omega)$ then there is a T_0 in $(M_0)_2(H,\omega_0)$ such that $\boldsymbol{E}(q^T) = q^{T_0}$ and $\|T\|_2 \leqq \|T\|_2$).

PROOF. It is an immediate consequence of lemma 3.9. that $\boldsymbol{E}$ is a linear extension of ε_ω. As noted in lemma 3.9. for all q *in* $M_1(H,\omega)$ $\boldsymbol{E}q$ is the element of $(\pi_0(\mathrm{M}_0))_1(\mathsf{H}_0,\pi_0(\omega_0))$ satisfying $(\boldsymbol{E}q)(\xi_0,\eta_0) = \mu_0((\mu^{-1}(q))_0)(\xi_0,\eta_0)$ for all ξ_0,η_0 in $D(\mathsf{H}_0,\pi_0(\omega_0))$.

So, since $\{\theta_0(\xi_0,\eta_0)\colon \xi_0,\eta_0 \text{ in } D(\mathsf{H},\omega_0)\}$ is weak operator dense in M_0, we have $\|\boldsymbol{E}q\|_1 = \|\mu_0^{-1}(\boldsymbol{E}q)\| = \|\mu^{-1}(q))_0\| \leqq \|\mu^{-1}(q)\| = \|q\|_1$.

Let now T be in $M_2(H,\omega)$. Then $(\boldsymbol{E}q^T)(\xi_0,\eta_0) = \langle JJ_0\xi_0, \pi(T)JJ_0\eta_0\rangle = q_0{}^{J_0EJ\pi(T)JEJ_0}(\xi_0,\eta_0)$, so we can set $T_0 = \pi_0^{-1}(J_0EJ\pi(T)JEJ_0)$.

So $\|T_0\|_2 = \|J_0EJ\pi(T)JEJ_0\Omega\| \leqq \|\pi(T)\Omega\| = \|T\|_2$.

As already noted in [1] not only is ε_ω in general not a norm one projection, but also its range can fail to be the whole of M_0; moreover, given an operator a_0 in its range, in general there is no a in M such that $\varepsilon_\omega(a) = a_0$ and $\|a\| = \|a_0\|$ (with any reasonable norms for a_0 and a). The following proposition shows us that the correct spaces for the analogues of these properties to hold are $M_2(H,\omega)$ and $(M_0)_2(H,\omega_0)$.

3.12. PROPOSITION. *Let T_0 be in $(M_0)_2(H,\omega_0)$. There is then a T in $M_2(H,\omega)$ such that $E(q^T) = q_0 T_0$ and $\|T_0\|_2 = \|T\|_2$.*

PROOF. Using the linear bijection established in 3.4. between the vectors in H and $M_2(H,\omega)$, we let T be the operator in $M_2(H,\omega)$ such that $\pi(T)\Omega = JJ_0\pi_0(T_0)\Omega$. Clearly $\|T_0\|_2 = \|T\|_2$, and for ξ_0, η_0 in $D(\mathrm{H}_0,\pi_0(\omega_0))$, $\eta_0 = \pi_0'(\mathrm{b}_0)\Omega$ with b_0 in M_0, we have:

$$(Eq^T)(\xi_0,\eta_0) = \langle JJ_0\xi_0, \pi(T)JJ_0\pi_0'(b_0)\Omega\rangle$$
$$= \langle JJ_0\xi_0, \pi(T)J\pi(b_0^+)\Omega\rangle = \langle JJ_0\xi_0, J\pi(b_0^+)J\pi(T)\Omega\rangle$$
$$= \langle JJ_0\xi_0, J\pi(b_0^+)J_0\pi_0(T_0)\Omega\rangle = \langle \xi_0, J_0\pi(b_0^+)J_0\pi_0(T_0)\Omega\rangle$$
$$= \langle \xi_0, \pi_0(T_0)J_0\pi(b_0^+)J_0\Omega\rangle = \langle \xi_0, \pi_0(T_0)\eta_0\rangle = q_0\pi_0(T_0)(\xi_0,\eta_0),$$

which implies our claim.

4. A spatial characterization of ω conditional expectations.

4.1. THEOREM. *Let E be the 1-ω-conditional expectation from $M_1(H,\omega)$ to $(M_0)_1(H,\omega_0)$ preserving ω. Then, for all a in M and all normal states ϕ_0 on M_0*

$$E([q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])]) = [q_0([\phi_0/\omega_0], \varepsilon_\omega(u^+au)[\phi_0/\omega_0])]$$

with $u = u(\phi_0,\omega)$.

PROOF. Let us recall first that if $\xi_0(\eta_0)$ is in $D(\mathrm{H}_0,\pi_0(\omega_0))$ then there is some $b_0(c_0)$ in M_0 such that $\xi_0 = \pi_0'(b_0)\Omega$ ($\xi_0 = \pi_0'(b_0)\Omega$). Then clearly $JJ_0\xi_0 = JJ_0\pi_0'(b_0)\Omega = J\pi_0(b_0^+)\Omega = J\pi(b_0^+)\Omega = \pi'(b_0)\Omega$, and similarly $JJ_0\eta_0 = \pi'(c_0)\Omega$. We have therefore, for all a in M and ξ_0, η_0 in $D(\mathrm{H}_0, \pi_0(\omega_0))$:

$$E([q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])])(\xi_0,\eta_0)$$
$$= \pi([q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])])(JJ_0\xi_0, JJ_0\eta_0)$$
$$= \langle \pi([\rho_\omega(\phi_0)/\omega])JJ_0\xi_0, \pi(a[\rho_\omega(\phi_0)/\omega])JJ_0\eta_0\rangle$$
$$= \langle \pi([\rho_\omega(\phi_0)/\omega])\pi'(b_0)\Omega, \pi(a[\rho_\omega(\phi_0)/\omega])\pi'(c_0)\Omega\rangle$$
$$= \langle \pi'(b_0)\pi([\rho_\omega(\phi_0)/\omega])\Omega, \pi'(c_0)\pi(a)\pi([\rho_\omega(\phi_0)/\omega])\Omega\rangle$$

$$= \langle \pi'(b_0)J\pi([\rho_\omega(\phi_0)/\omega])\Omega, \pi'(c_0)\pi(a)J\pi([\rho_\omega(\phi_0)/\omega])\Omega\rangle$$

$$= \langle \pi'(b_0)\pi(u)J\pi_0([\phi_0/\omega_0])\Omega, \pi'(c_0)\pi(a)\pi(u)J\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle \pi'(b_0)J\pi_0([\phi_0/\omega_0])\Omega, \pi'(c_0)\pi(u^+au)J\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle J\pi'(c_0)\pi(u^+au)JJ_0\pi_0([\phi_0/\omega_0])\Omega, J\pi'(b_0)JJ_0\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle \pi(c_0^+)J\pi(u^+au)JJ_0\pi_0([\phi_0/\omega_0])\Omega, \pi(b_0^+)J_0\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle J\pi(u^+au)JJ_0\pi_0([\phi_0/\omega_0])\Omega, \pi_0(c_0)\pi_0(b_0^+)J_0\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle J_0\pi_0(c_0)\pi_0(b_0^+)J_0\pi_0([\phi_0/\omega_0])\Omega, J_0EJ\pi(u^+au)JJ_0\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle J_0\pi_0(c_0)\pi_0(b_0^+)J_0\pi_0([\phi_0/\omega_0])\Omega, \pi_0(\varepsilon_\omega(u^+au))\pi_0([\phi_0/\omega_0])\Omega$$

$$= \langle \pi_0'(b_0)\pi_0([\phi_0/\omega_0])\Omega, \pi_0'(c_0)\pi_0(\varepsilon_\omega(u^+au))\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle \pi_0([\phi_0/\omega_0])\pi_0'(b_0)\Omega, \pi_0(\varepsilon_\omega(u^+au))\pi_0([\phi_0/\omega_0])\pi_0'(c_0)\Omega\rangle$$

$$= \langle \pi_0([\phi_0/\omega_0])\xi_0, \pi_0(\varepsilon_\omega(u^+au))\pi_0([\phi_0/\omega_0])\eta_0\rangle$$

$$= [q_0([\phi_0/\omega_0], \varepsilon_\omega(u^+au)[\phi_0/\omega_0])](\xi_0, \eta_0).$$

Our claim now follows immediately.

4.2. COROLLARY. *Let $\rho_\omega(\phi_0)$ be dominated by ω. Then for all a in M* $\varepsilon_\omega([\rho_\omega(\phi_0)/\omega]^+a[\rho_\omega(\phi_0)/\omega]) = [\phi_0/\omega_0]^+\varepsilon_\omega(u^+au)[\phi_0/\omega_0]$.

PROOF. Immediate.

4.3. THEOREM. *Let σ be a linear weakly continuous contraction from to M_0 and Σ a contradiction from $M_1(H,\omega)$ to $(M_0)_1(H,\omega_0)$ such that for all a in M and all normal states ϕ_0 on M_0 we have (setting $u = u(\phi_0,\omega)$):*

$$\Sigma([q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])]) = [q_0([\phi_0/\omega_0], \sigma(u^+au)[\phi_0/\omega_0])],$$

Then σ is the ω preserving ω-conditional expectation ε_ω from M_1 to M_0 and Σ the corresponding $1-\omega$-conditional expectation.

PROOF. We have, for all ξ_0, η_0 in $D(H,\omega_0)$, a_0 in M_0:

$$\langle J_0\pi_0(\theta_0(\eta_0,\xi_0))J_0\pi_0([\phi_0/\omega_0])\Omega, \pi_0(a_0)\pi_0([\phi_0/\omega_0])\Omega\rangle$$

$$= \langle R_0(\eta_0)^+R_0(\xi_0)\pi_0([\phi_0/\omega_0])\Omega, \pi_0(a_0)\pi_0([\phi_0/\omega_0])\Omega$$

$$= \langle \pi_0([\phi_0/\omega_0])R_0(\xi_0)\Omega, \pi_0(a_0)\pi_0([\phi_0/\omega_0])R_0(\eta_0)\Omega\rangle$$

$$= \langle [\phi_0/\omega_0]\xi_0, a_0[\phi_0/\omega_0]\eta_0\rangle = \langle [\phi_0/\omega_0]\xi_0, a_0[\phi_0/\omega_0]\eta_0\rangle$$

$$= [q_0([\phi_0/\omega_0], a_0[\phi_0/\omega_0])](\xi_0, \eta_0).$$

So if we let $\theta_0(\eta_0^\alpha, \xi_0^\alpha) \to 1$ weakly,

$$[q_0([\phi_0/\omega_0], a_0[\phi_0/\omega_0])](\eta_0^\alpha, \xi_0^\alpha) \to \phi_0(a_0).$$

Similarly, for all in $D(H,\omega)$, a in M:

$$[q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])](\xi, \eta)$$
$$= \langle J\theta(\eta, \xi)J\pi([\rho_\omega(\phi_0)/\omega])\Omega, \pi(a)\pi([\rho_\omega(\phi_0)/\omega])\Omega\rangle,$$

and if $\theta_0(\eta^\alpha, \xi^\alpha)$ goes weakly to the identity then

$$[q([\rho_\omega(\phi_0)/\omega], a[\rho_\omega(\phi_0)/\omega])](\eta^\alpha, \xi^\alpha) \text{ goes to } \rho_\omega(\phi_0)(a).$$

So the mapping σ is the dual mapping of the mapping $\phi_0 \to \rho_\omega(\phi_0)(u^+u)$ and therefore coincides with ε_ω as proved in [4].

If we set $\phi_0 = \omega_0$ we get $\Sigma(q^a) = q_0\varepsilon_\omega(a)$ for a in M and the continuity of Σ completes our proof.

4.4. COROLLARY. *Let M be a matrix algebra. Then the ω conditional expectation from M to M_0 is the (unique) linear contraction ε from M to M_0 such that for all normal states ϕ_0 on M_0 we have:*

$$\varepsilon([\rho_\omega(\phi_0)/\omega]^+ a[\rho_\omega(\phi_0)/\omega]) = [\phi_0/\omega_0]^+\varepsilon(u^+au)[\phi_0/\omega_0].$$

PROOF. It is enough to recall that for matrix algebras $\rho_\omega(\phi_0)$ is always dominated by ω and $M_1(H,\omega)$ coincides with M.

4.5. REMARK. The formula in cor. 4.4. recalls the defining property of Haagerup's operator valued weights ([9] and [10]), giving a hint on a possible way to pursue in order to generalize them in the direction of ω-conditional expectations. An approach to this problem can be found in [11].

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