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INVARIANT SUBSPACES IN CERTAIN FUNCTION SPACES ON EUCLIDEAN SPACE

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§1. Introduction and formulation of the main results.

Let G be a transitive group of transformations of the set M, $\mathscr{F}$ be some locally convex space (LCS) consisting of functions on M,

$$\pi(g): f(x) \mapsto f(g^{-1}x)$$

be the quasiregular representation of G on the LCS $\mathscr{F}$. A linear subspace $H \subseteq \mathscr{F}$ we call an invariant subspace (ISS) if H is closed and invariant with respect to the quasiregular representation π. We shall also assume that an ISS H not coinciding with the whole space $\mathscr{F}$. One of the main problems in harmonic analisis on group G is the problem of describing the invariant subspaces of some concrete function spaces $\mathscr{F}$. In particular we have the problem of describing the irreductible and indecomposable invariant subspaces, where an ISS H is said to be irreductible if there does not exist an invariant subspace $H_1 \subset H$ other then H itself and $\{0\}$, and H is said to be indecomposable if $H \neq H_1 + H_2$ for ISS's H_1, H_2 such that $H_1 \neq \{0\}$, $H_2 \neq \{0\}$ and $H_1 \cap H_2 = \{0\}$ (here $H_1 + H_2$ is the closure of the algebraic sum of H_1 and H_2).

In this paper we study the case when M is the n-dimensional Euclidean space R^n, G is the group of all orientation-preserving isometries, $\mathscr{F}$ is one of the spaces C^d, L^p_*, C^d_*, where C^d is the space of all C^d-class functions on R^n with the usual topology ($d = 0, 1, \ldots, \infty$; in particular $C^0 = C$ is the space of all continuous functions, $C^\infty = \mathscr{E}$ is the space of all infinitely differentiable functions), the spaces C^d_* and L^p_* will be defined below. All functions will be complex-valued unless otherwise stated. Let $x = (x_1, \ldots x_n) \in \mathsf{R}^n$, $O = (0, \ldots 0)$, $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

Denote by C_k the space of continuous functions $f(x)$ on R^n such that

$$|f(x)|\, e^{-k|x|} \to 0$$

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as $|x| \to \infty$; C_k is a Banach space (BS) with the norm

$$n_k(f) = \sup_{x \in \mathbf{R}^n} |f(x)|\, e^{-k|x|}. \tag{1.1}$$

The space

$$C_* = \bigcup_{k>0} C_k$$

is equipped with the topology of the inductive limit of the BS's C_k.

Let $\mathbf{Z}_+$ be the set of nonnegative integers. If $r = (r_1, \dots r_n) \in \mathbf{Z}_+^n$, then we put $|r| = r_1 + \dots + r_n$, $\partial^r f = \partial_{x_1}^{r_1} \dots \partial_{x_n}^{r_n} f$.

For $d \in \mathbf{Z}_+$ denote by C_k^d the space of all functions $f(x)$ such that

$$\partial^r f \in C_k \quad \forall r \in \mathbf{Z}_+^n, \quad |r| \leqq d.$$

C_k^d is a BS with the norm

$$n_{k,d}(f) = \sum_{|r| \leqq d} n_k(\partial^r f).$$

The space

$$C_*^d = \bigcup_{k>0} C_k^d$$

is equipped with the topology of the inductive limit of the BS's C_k^d.

For $d = \infty$ let

$$C_k^\infty = \mathscr{E}_k = \bigcap_{d=0}^{\infty} C_k^d.$$

The topology in ε_k is given by the family of seminorms (even norms) $n_{k,d}$, $d \in \mathbf{Z}_+$. The space

$$\mathscr{E}_* = C_*^\infty = \bigcup_{k>0} \mathscr{E}_k$$

is equipped with the topology of the inductive limit of the locally convex spaces (LCS) $\mathscr{E}_k$.

Let the space L_k^p consist of all measurable functions $f(x)$ on $\mathbf{R}^n$ such that

$$N_{p,k}(f) = \left(\int |f(x)|^p e^{-k|x|}\, dx \right)^{1/p} < \infty, \tag{1.2}$$

where dx is the element of the Lebesgue measure, the integral is taken over the whole space $\mathbf{R}^n$, functions are taken to within values on a set measure zero. With respect to the norm $N_{p,k}$ the space L_k^p is the BS. The space

$$L^p_* = \bigcup_{k>0} L^p_k$$

is equipped with the topology of the inductive limit of BS's L^p_k.

If $\mathscr{F}$ is a function spaces on the set M, E is a finite-dimensional normed space over $\mathbf{C}$, then the vector space $\mathscr{F} \otimes E$ is naturally indentified with the function space of E-valued functions on M and is equipped with the topology of the tensor products of LCS's [2].

Let $K = \{g \in G : gO = O\}$ be the isotopy subgroup of the point O. The group K is isomorphic to the group $\mathrm{SO}(n)$. An arbitrary irreductible representation of $\mathrm{SO}(n)$ is determined by its highest weight, which can be identified with an integer tuple $\lambda = (\lambda_1, \dots \lambda_m)$ $(m = [n/2]$ is the integer part of $n/2)$ satisfying the conditions

$$\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_m \tag{1.3}$$

for $n = 2m + 1$ and

$$\lambda_1 \geqq \lambda_2 \geqq \dots \lambda_{m-1} \geqq |\lambda_m| \tag{1.3'}$$

for $n = 2m$.

Let Λ be the set of highest weights of the group K. Denote by λ_0 the set of highest weights of K of the form $(l, 0, \dots 0)$, where $l \in \mathbf{Z}$ for $n = 2$ and $l \in \mathbf{Z}_+$ for $n \geqq 3$. Let T^l be the irreductible representation of K with highest weight $(l, 0, \dots 0)$, E^l be the space of the representation T^l, and fix in E^l a K-invariant Hermitian form $\langle \xi, \eta \rangle$ $(\xi, \eta \in E^l)$.

Let $\mathscr{F}$ be a complete LCS consisting of functions on $\mathbf{R}^n$. Suppose $\mathscr{F}$ is invariant with respect to the quasiregular representation π and the mapping $g \mapsto \pi(g)f$ from G to $\mathscr{F}$ is continuous $(f \in \mathscr{F})$. Let $\mathscr{F}^{(l)}$ be the set of all functions $F(x) \in \mathscr{F} \otimes E^l$ such that

$$F(ux) = T^l(u)F(x) \quad \forall u \in K. \tag{1.4}$$

This space is equipped with the topology induced from $\mathscr{F} \otimes E^l$. In particular we have the spaces $\mathscr{E}^{(l)}$ $C^{d(l)}$, $\mathscr{E}^{(l)}_*$, $C^{d(l)}_*$.

For every invariant subspace $H \subset \mathscr{F}$ denote by $H^{(l)}$ the set of all functions $F(x) \in \mathscr{F}^{(l)}$ such that the functions $\varphi_\xi = \langle F(x), \xi \rangle \in H$ for all $\xi \in E^l$. The subspace H can be uniquely recovered from all subspaces $H^{(l)}$, namely, H coincides with closure of the linear span of all functions $\langle F(x), \xi \rangle$ for $F \in H^{(l)}, \xi \in E^l, l \in \mathbf{Z}_+$ (or $l \in \mathbf{Z}$ for $n = 2$). The subspaces $H^{(l)}$ will be called the cells of the invariant subspace H, or simply the invariant cells. To describe an invariant subspace it suffices to describe all its cells.

Below $\mathscr{F}$ is one of the spaces L^p_* $(p \geqq 1)$, C^d, C^d_* $(d \in \mathbf{Z}_+ \cup \{\infty\}$, in particular $C^0_* = C_*, C^\infty_* = \mathscr{E}_*)$. The spaces L^p_* and C^d_* will be called the spaces of type 1, the spaces C^d will be called the spaces of type 2.

Let μ be a complex number, and r a positive integer. Denote $V_{\mu,r}^{(l)}$ the linear subspace consisting of all functions $F(x) \in \mathscr{E}^{(l)}$ such that $(\Delta + \mu^2)^r F = 0$, where $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$ is the Laplace operator on $\mathbf{R}^n$. If we denote by

(1.5) $$\mathbf{C}_+ = \{z \in \mathbf{C} : \operatorname{Re} z \geqq 0, \text{ and } \operatorname{Im} z \geqq 0 \text{ for } \operatorname{Re} z = 0\}$$

then without loss of generality we can assume that $\mu \in \mathbf{C}_+$.

It will be shown later that $\dim V_{\mu,r}^{(l)} = r$ and that $V_{\mu,r}^{(l)}$ has a Jordan basis, i.e. a basis $F_1, \ldots F_r$ such that $\Delta F_1 = \mu^2 F_1$ and $\Delta F_k = -\mu^2 F_k + F_{k-1}$ for $k \geqq 2$. Besides we have that $V_{\mu,r}^{(l)} \subset \mathscr{E}_*^{(l)}$.

The subspace $V_{\mu,r}^{(l)}$ is the simplest invariant cell. A general invariant cell can be described by the next theorem.

THEOREM 1. *For every invariant cell $H^{(l)}$ of $\mathscr{F}^{(l)}$ there exist the unique finite or countable set of complex numbers $\{\mu_j\}$ (μ_j can occur with a finite multiplicity r_j; $\mu_j \in \mathbf{C}_+$) such that $H^{(l)}$ is the closure in $\mathscr{F}^{(l)}$ of the linear span of the subspaces $V_{\mu,r}^{(l)}$, where μ runs through the set $\{\mu_j\}$ and r is the multiplicity of μ in this set.*

The set $\{\mu_j\}$ will be called the spectrum of the invariant cell $H^{(l)}$. There is the description of spectrums of invariant cells.

Let $\mathscr{F}$ be a space of type 1. The set $\{\mu_j\}$ ($\mu_j = a_j + ib_j \in \mathbf{C}_+$) is a spectrum of some invariant cell of $\mathscr{F}^{(l)}$ if and only if the following condition hold:

(A) *For each $t > 0$ the numbers $\mu_j = a_j + ib_j$ with $|b_j| < t$, after renumbering in order to increasing a_j ($0 \leqq a_1 \leqq a_2 \leqq \ldots$), either are such that*

$$a_n / \ln n \to \infty$$

as $n \to \infty$, or form a finite set.

If $\mathscr{F}$ is a space of type 2 then the set $\{\mu_j\}$ is a spectrum of some invariant cell of $\mathscr{F}^{(l)}$ if and only if the following conditions hold:

(B) *There exists an entire nonzero function $\Phi(\lambda)$ such that each number μ_j is the root of $\Phi(\lambda)$ with the multiplicity μ_j, and*

$$|\Phi(\lambda)| \leqq A e^{B|\operatorname{Im} \lambda|}(1 + |\lambda|)^C$$

for any $A, B, C > 0$ (such function is the Fourier transform of distribution with the compact supports).

Suppose that in each space $\mathscr{F}^{(l)}$ we fix a cell $H^{(l)}$ of some invariant subspace, depending on l in general, and let $\sigma(l)$ be the spectrum of $H^{(l)}$.

THEOREM 2. *The cells $H^{(l)}$ are the cells of a single invariant subspace if and only if the following conditions hold:*

1) *The spectrums $\sigma(l)$ differ only by the multiplicity $r_0^{(l)}$ of the number 0 in $\sigma(l)$ for various l.*

2) *For $l \geqq 0$ the multiplicity $r_0^{(l+1)}$ is equal to $r_0^{(l)}$ or $r_0^{(l)} - 1$.*

3) *For* $l \leqq 0$ *the multiplicity* $r_0^{(l-1)}$ *is equal to* $r_0^{(l)}$ *or* $r_0^{(l)} - 1$.
Only conditions (1) *and* (2) *remain for* $n \geqq 3$.

Combination of Theorems 1 and 2 gives a complete description of invariant subspaces of $\mathscr{F}$. The description of irreducible and indecomposable subspaces can be easily obtained from this theorems.

There are two variants for the spectrums of an irreductible subspace: (1) all spectrums consist of the unique number $\mu \neq 0$ with multiplicity 1; (2) the spectrum $\sigma(0)$ consist of the number 0 with multiplicity 1, the others spectrums are the empty sets.

In the second case the corresponding irreductible ISS of $\mathscr{F}$ consists of all constants. In the first case the corresponding irreductible ISS of $\mathscr{E}$ consists of all functions $f \in \mathscr{E}$ such that

$$(\Delta + \bar{\mu})f = 0, \tag{1.6}$$

where $\bar{\mu}$ is the complex conjugate number to μ. Denote this subspace by $\mathscr{E}(\bar{\mu})$. *If* $\mathscr{F}$ is a space of type 2 (i.e. $\mathscr{F} = C^d$), then the corresponding irreductible subspace is the closure $[\mathscr{E}(\bar{\mu})]$ in $\mathscr{F}$. But if $f \in [\mathscr{E}(\bar{\mu})]$, then f is a weak solution of the equation (1.6) and, by the regularity theorem, $f \in \mathscr{E}$ since Δ is an elliptic operator. Therefore $[\mathscr{E}(\bar{\mu})] = \mathscr{E}(\bar{\mu})$ and $\mathscr{E}(\bar{\mu})$ is an irreductible ISS of $\mathscr{F}$. If $\mathscr{F}$ is a space of type 1 (i.e. $\mathscr{F} = C_*^d$ or $\mathscr{F} = L_*^p$) then the corresponding irreductible ISS of $\mathscr{F}$ is

$$\mathscr{E}_*(\bar{\mu}) = \mathscr{E}_* \cap \mathscr{E}(\bar{\mu})$$

(it is easy to see that $\mathscr{E}_*(\bar{\mu})$ is closed in $\mathscr{F}$ since Δ is an elliptic operator).

The subspace H is indecomposable if and only if every spectrum $\sigma(l)$ consists of the unique number μ with some multiplicity. For $\mu \neq 0$ the multiplicities of μ must be equal for all $\sigma(l)$; for $\mu = 0$ the multiplicities can be changed such that the conditions of Theorem 2 hold.

If every spectrum $\sigma(l)$ consists of the number μ with the multiplicity r then the corresponding indecomposable ISS of $\mathscr{E}$ (and of any space $\mathscr{F}$ of type 2) consists of all functions $f \in \mathscr{E}$ such that

$$(\Delta + \bar{\mu})^r f = 0$$

Denote this subspace by $\mathscr{E}(\bar{\mu}, r)$. The corresponding subspace of $\mathscr{E}_*$ (and of any space $\mathscr{F}$ of type 1) is $\mathscr{E}(\bar{\mu}, r) \cap \mathscr{E}_*$. We shall say that the indecomposable ISS's $\mathscr{E}(\mu, r)$ and $\mathscr{E}_*(\mu, r)$ are general, the other indecomposable subspaces are exceptional.

If H is an exceptional subspace then every spectrum $\sigma(l)$ consists of the number 0 with some multiplicity d_l. Hence an exceptional subspace can be described by a sequence $d_l \in \mathbf{Z}_+$ such that the conditions (2) and (3) of Theorem 2 are true. Further we shall obtain the more clear description of such subspaces..

Let $n \geqq 3$, then $l \in \mathbf{Z}_+$. Note that the subspace corresponding to the sequence

$d_j = k - j$ for $0 \leqq j \leqq k$ and $d_j = 0$ for $j > k$ is the minimal invariant subspace containing the cell $V^{(0)}_{0,k}$, and the cell $V^{(0)}_{0,k}$ is the linear span of the functions

$$1, |x|^2, |x|^4, \ldots |x|^{2(k-1)}.$$

Consequently this ISS is spaned by the functions $\partial^r |x|^{2(k-1)}$ with $r = (r_1, \ldots r_n) \in \mathbf{Z}^n_+$, $|r| \leqq 2(k-1)$. We denote this ISS by H_k.

If $m \leqq k$ then the subspace $H_{k,m} = H_k \cap \mathscr{E}(0, m)$ is determined by the sequence $d_j = m$ for $0 \leqq j \leqq k - m$, $d_j = k - j$ for $k - m \leqq j \leqq k$, $d_j = 0$ for $j > k$.

Every exceptional indecomposable subspace is the finite union of the subspaces $H_{k,m}$ and of the subspace $\mathscr{E}(0, d)$ (or $\mathscr{E}_*(0, d)$), where $d = \lim d_j$ as $j \to \infty$. For example, the sequence $d_0 = d_1 = 5$, $d_2 = d_3 = 4$, $d_4 = 3$, $d_j = 2$ for $j \geqq 5$ corresponds to the exceptional indecomposable subspace $H = H_{6,5} \cup H_{7,4} \cup \mathscr{E}(0, 2)$ of any type 2 space $\mathscr{F}$.

Now let $n = 2$. Let $z = x_1 + ix_2 \in \mathbf{C}$, $\bar{z} = x_1 - ix_2$, $\partial_z = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, then $\Delta = 4\partial_z\partial_{\bar{z}}$. It is easy to see that the cel $V^{(l)}_{0,k}$ is spanned by the functions $z^{l+t}\bar{z}^t$ for $l > 0$ and $z^+\bar{z}^{l+t}$ for $l \leqq 0$, where $t = 0, 1, \ldots k - 1$.

Let

$$\mathscr{E}^+(0, k) = \{f \in \mathscr{E} : \partial^k_{\bar{z}} f = 0\}.$$

Then $\mathscr{E}^+(0, k)$ is an invariant indecomposable subspace and $\mathscr{E}^+(0, k)$ corresponds to the sequence d_j with $d_j = k$ for $j \geqq 0$, $d_j = k + j$ for $(-k) \leqq j \leqq 0$ and $d_j = 0$ for $j < (-k)$. The corresponding subspace of $\mathscr{E}_*$ is $\mathscr{E}^+_*(0, k) = \mathscr{E}^+(0, k) \cap \mathscr{E}_*$. By analogy let

$$\mathscr{E}^-(0, k) = \{f \in \mathscr{E} : \partial^k_z f = 0\},$$

$\mathscr{E}^-_*(0, k) = \mathscr{E}^-(0, k) \cap \mathscr{E}_*$. Then the subspaces $\mathscr{E}^-(0, k)$ and $\mathscr{E}^-_*(0, k)$ are determined by the sequence d_j with $d_j = k$ for $j \leqq 0$, $d_j = k - j$ for $0 \leqq j \leqq k$, $d_j = 0$ for $j > k$. The subspace H_k is defined as above. Let $H^+_{k,m} = H_k \cap \mathscr{E}^+(0, m)$, $H^-_{k,m} = H_k \cap \mathscr{E}^-(0, m)$. Every exceptional invariant subspace is the finite union of the subspaces $H^{\pm}_{k,m}$ and of the subspaces $\mathscr{E}^+(0, d_+)$ and $\mathscr{E}^-(0, d_-)$ (or $\mathscr{E}^+_*(0, d_+)$ and $\mathscr{E}^-_*(0, d_-)$), where $d_+ = \lim\limits_{j \to +\infty} d_j$, $d_- = \lim\limits_{j \to -\infty} d_j$.

It follows from Theorems 1 and 2 that every ISS of $\mathscr{F}$ is the closure of the direct sum of countable number of indecomposable subspaces.

The main purpose of this paper is te proof of Theorems 1 and 2. The methods of this paper are similar to those of [4–8]. In §2 we study the problem of describing the submodules of Harish Chandra modules. The results of §2 can be used not only for describing the invariant subspaces, but also for others problems in harmonic analisis on Lie groups. We note that the irreductibility of the subspace $\mathscr{E}(\mu)$ for $\mu \neq 0$ was established by Helgason [3].

§2. On submodules of the Harish-Chandra modules.

Let G be a Lie group, K a compact connected subgroup of G, $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of G and K respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexifications of $\mathfrak{g}_0$ and $\mathfrak{k}_0$.

Let $g \to \mathrm{Ad}(g)$ be the adjoint representation of G on $\mathfrak{g}$. Denote by Λ the set of equivalence classes of irreductible finite-dimensional representations of K. For $\lambda \in \Lambda$ denote by E^λ the corresponding $\mathfrak{k}$-module, and by $T^\lambda(u)$ the corresponding representations of K.

For every $\mathfrak{g}$-module V and for $\lambda \in \Lambda$ let V^λ be the sum of all the $\mathfrak{k}$-submodules isomorphic to E^λ. We call a $\mathfrak{g}$-module V a Harish-Chandra module if $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ (direct sum of vector spaces). This definition differs slightly from the definition of a Harish-Chandra module in [15], where Λ is taken to be all irreductible finite dimensional representations of the algebra $\mathfrak{k}$.

Let $\mathrm{Hom}(E^\lambda, V)$ be the set of linear mappings from E^λ to V, and $V^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, V)$ be the set of $\mathfrak{k}$-module homomorphisms from E^λ to V. Any element $x \in V^\lambda$ can be represented as a sum of certain elements following the form $\Psi(y)$ for certain $y \in E^\lambda$ and $\Psi \in V^{(\lambda)}$. Therefore, we can assume that the Harish-Chandra module V can be uniquely recovered from all possible $V^{(\lambda)}$. If H is a Harish-Chandra submodule of V, i.e., $H = \bigoplus_{\lambda \in \Lambda} (H \cap V^\lambda)$, then we have the set of subspaces

$$H^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, H) \subseteq V^{(\lambda)}.$$

It turns out that in certain cases it is convenient to describe Harish-Chandra submodules H of V by specifying the corresponding subspaces $H^{(\lambda)} \subseteq V^{(\lambda)}$. The following problem arises naturally: given some subspace $H^{(\lambda)}$ in each $V^{(\lambda)}$, find conditions on the $H^{(\lambda)}$ under which V contains a Harish-Chandra submodule H such that

$$H^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, H) \subseteq V^{(\lambda)}.$$

This problem will be solved later.

The action of $\mathfrak{k}$ on V is extended to the representation $T(u)$ of the compact group K. This is possible as V is a Harish-Chandra module.

Fix in each E^λ a K -invariant Hermitian form $\langle \xi, \eta \rangle$ and let e_j^λ $(1 \leqq j \leqq n_\lambda)$ be an orthonormal basis in E^λ. We note that for $\Psi \in V^{(\lambda)}$, $y \in E^\lambda$, $u \in K$

$$\Psi(T^\lambda(u)y) = T(u)\Psi(y). \tag{2.1}$$

Since the representation $\mathrm{Ad}(u)$ of the group K on $\mathfrak{g}_0$ complete irreductible, there exist an invariant complement $\mathfrak{p}_0$ of $\mathfrak{k}_0$, that is $\mathfrak{g}_0 = \mathfrak{p}_0 + \mathfrak{k}_0$ and $\mathrm{Ad}(u)\mathfrak{p}_0 \subseteq \mathfrak{p}_0$ for $u \in K$.

Let $\mathfrak{p}$ be the complexification of $\mathfrak{p}_0$. $\mathfrak{p}$ becomes a $\mathfrak{l}$-module if we define

$kp = [k, p]$ for $k \in \mathfrak{k}$, $p \in \mathfrak{p}$. Let $p_1, \dots p_m$ be a basis in $\mathfrak{p}$. Denote by $\mathfrak{p}^*$ the dual $\mathfrak{k}$-module to $\mathfrak{p}$, let $p_1^*, \dots p_m^*$ be the dual basis in $\mathfrak{p}^*$.

For $\lambda, \mu \in \Lambda$ let $\mathrm{Hom}(E^\mu, E^\lambda)$ be the set of linear mappings from E^μ to E^μ. As usually $\mathrm{Hom}(E^\mu, E^\lambda)$ is a $\mathfrak{k}$-module (that is $(kA)\xi = k(A\xi) - A(k\xi)$ for $\xi \in E^\mu$). Let

$$\varphi\colon \mathfrak{p}^* \mapsto \mathrm{Hom}(E^\mu, E^\lambda) \tag{2.2}$$

be a $\mathfrak{l}$-module homomorphism from $\mathfrak{p}^*$ to $\mathrm{Hom}(E^\mu, E^\lambda)$. Let

$$\alpha_j = \varphi(p_j^*) \in \mathrm{Hom}(E^\mu, E^\lambda).$$

For $\Psi \in \mathrm{Hom}(E^\lambda, V)$ we define $L(\varphi)\Psi \in \mathrm{Hom}(E^\mu, V)$ by

$$(L(\varphi))(z) = \sum_{j=1}^{m} p_j \Psi(\alpha_j(z)), \quad \forall z \in E^\mu. \tag{2.3}$$

Denote by $s(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda))$ the set of $\mathfrak{k}$-module homomorphisms (intertwining operators) from $\mathfrak{p}^*$ to $\mathrm{Hom}(E^\mu, E^\lambda)$.

Lemma 2.1. *The operator $L(\varphi)$ maps $V^{(\lambda)}$ into $V^{(\mu)}$.*

Proof. Let $\Psi \in V^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, V)$, $\Phi = L(\varphi)\Psi$, $k \in \mathfrak{l}$. If

$$[k, p_j] = \sum_r \tau_{rj}(k) p_r, \tag{2.4}$$

then

$$kp_j^* = -\sum_r \tau_{jr}(k) p_r^*,$$

and so

$$k\alpha_j(z) - \alpha_j(kz) = -\sum_r \tau_{jr}(k)\alpha_r(z). \tag{2.5}$$

We get from (2.4) and (2.5) that

$$k\Phi(z) - \Phi(kz) = \sum_j ([k, p_j]\Psi(\alpha_j(z)) + p_j \Psi(k(\alpha_j(z)) - \alpha_j(kz))) = 0,$$

consequently

$$\Phi \in \mathrm{Hom}_{\mathfrak{k}}(E^\mu, V) = V^{(\mu)}.$$

Denote by $P_0(\lambda, \mu)$ the set of linear operators of the form $L(\varphi)$.

Proposition 2.2. *Let V be a Harish-Chandra module. In each $V^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, V)$ a subspace $H^{(\lambda)}$ is singled out so that $L(\varphi)(H^{(\lambda)}) \subseteq H^{(\mu)}$ for every $L(\varphi) \in P_0(\lambda, \mu)$. Let H be the linear subspace of V generated by all vectors $\Psi(y)$ for $\Psi \in H^{(\lambda)}$, $y \in E^\lambda$, $\lambda \in \Lambda$. Then H is a Harish-Chandra submodule of V and $H^{(\lambda)} = \mathrm{Hom}_{\mathfrak{k}}(E^\lambda, H)$ for all $\lambda \in \Lambda$.*

PROOF. A) Let us consider the tensor product $\mathfrak{p} \otimes \mathrm{Hom}(E^\mu, E^\lambda)$. It is a $\mathfrak{k}$-module as the tensor product of $\mathfrak{k}$-modules, that is

$$k(p \otimes A) = [k, p] \otimes A + p \otimes (kA), \quad p \in \mathfrak{p}, \quad A \in \mathrm{Hom}(E^\mu, E^\lambda). \tag{2.6}$$

An arbitrary element $I \in \mathfrak{p} \otimes \mathrm{Hom}(E^\mu, E^\lambda)$ can be represented as $I = p_1 \otimes A_1 + \ldots + p_m \otimes A_m$ where $A_j \in \mathrm{Hom}(E^\mu, E^\lambda)$. It is easy to see that the element I is an $\mathfrak{k}$-invariant (that is $kI = 0 \forall k \in \mathfrak{k}$) if and only if $A_j = \varphi(p_j^*)$ for some homomorphism $\varphi \in \mathfrak{s}(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda))$. Consequently, there exist a one-to-one correspondence between the invariants and the homorphisms.

B) Let $\rho(u)$ be the representation of the Lie group K on $\mathfrak{p} \otimes \mathrm{Hom}(E^\mu, E^\lambda)$ induced by the action (2.6) of the Lie algebra $\mathfrak{k}$. The explicit form of this representations is

$$\rho(u)(p \otimes A) = \mathrm{Ad}(u)p \otimes T^\lambda(u) A T^\mu(u^{-1}),$$

where $A \in \mathrm{Hom}(E^\mu, E^\lambda)$. An element $I \in \mathfrak{p} \otimes \mathrm{Hom}(E^\mu, E^\lambda)$ is a $\mathfrak{k}$-invariant if and only if $\rho(u)I = I \forall u \in K$.

For $p \in \mathfrak{p}$, $A \in \mathrm{Hom}(E^\mu, E^\lambda)$ let

$$I_{p,A} = \int \mathrm{Ad}(u)p \otimes T^\lambda(u) A T^\mu(u^{-1})\, du \tag{2.7}$$

where the integral is taken over the group K, du is the element of the Haar measure on K. Since the Haar measure is an invariant measure, the element $I_{p,A}$ is a $\mathfrak{k}$-invariant and let φ be the corresponding homomorphism in $\mathfrak{s}(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda))$. It is easy to see

$$(L(\varphi)\Psi)(z) = \int T(u)(p\Psi(A T^\mu(u^{-1})z)\, du \tag{2.8}$$

where $z \in E^\mu$, $\Psi \in V^{(\lambda)}$. In particular let

$$A(x) = \langle e_r^\mu, x \rangle e_j^\lambda, \quad x \in E^\mu.$$

If $\tau_{st}^{(\mu)}(u)$ are the matrix elements of the operator $T^\mu(u)$ then

$$A T^\mu(u^{-1}) e_t^\mu = A\left(\sum_s \tau_{st}^{(\mu)}(u^{-1}) e_s^\mu\right) = \tau_{rt}^{(\mu)}(u^{-1}) e_j^\lambda = \bar{\tau}_{tr}^{(\mu)}(u) e_j^\lambda.$$

Thus (2.8) gives

$$(L(\varphi))(e_t^\mu) = \int \bar{\tau}_{tr}^{(\mu)}(u) T(u)(p\Psi(e_j^\lambda))\, du. \tag{2.9}$$

C) Recall that H is the linear subspace of V spanned by all vectors $\Psi(y)$ for $\Psi \in H^{(\lambda)}$, $y \in E^\lambda$, $\lambda \in A$. Let H be the linear subspace spanned by the vectors $\Psi(y)$

for $\Psi \in H^{(\lambda)}, y \in E^{\lambda}$ with fixed λ. Then H^{λ} is a $\mathfrak{k}$-submodule of V and $H = \oplus_{\lambda \in \Lambda} H^{\lambda}$. Hence H is a $\mathfrak{k}$-module.

Let us verify that $\mathrm{Hom}_{\mathfrak{k}}(E^{\lambda}, H) = H^{(\lambda)}$. Let $\Psi \in \mathrm{Hom}_{\mathfrak{k}}(E^{\lambda}, H)$. Then $\Psi(e_1^{\lambda}) \in H$ ($e_1^{\lambda}, e_2^{\lambda}, \ldots$ is the orthonormal basis in E^{λ}), hence

$$\Psi(e_1^{\lambda}) = \sum_{s,t} c_{s,t} \Psi_s(e_t^{\lambda}), \tag{2.10}$$

where $\Psi_s \in H^{(\lambda)}$, $c_{s,t} \in \mathbf{C}$. Let $u \in K$, then (2.10) gives us that

$$\Psi(T^{\lambda}(u)e_1^{\lambda}) = \sum_r \tau_{r1}^{(\lambda)}(u)\Psi(e_r^{\lambda}) = \sum_{s,t} c_{s,t}\Psi_s(T^{\lambda}(u)e_t^{\lambda}) = \sum_{s,t,j} c_{s,t}\tau_{jt}^{(\lambda)}(u)\Psi_s(e_j^{\lambda}). \tag{2.11}$$

Multiplying (2.11) by $\bar{\tau}_{11}^{(\lambda)}(u)$, integrating with respect to u and using the orthogonality relations for the matrix elements of irreductible representations we get that

$$\Psi(e_1^{\lambda}) = \sum_s c_{s,1} \Psi_s(e_1^{\lambda}). \tag{2.12}$$

Since Ψ and Ψ_s are homomorphisms from E^{λ} to V we get from (2.12) that

$$\Psi = \sum_s c_{s,1} \Psi_s \in H^{(\lambda)}.$$

D) We verify that H is a $\mathfrak{g}$-submodule of V. Let $p \in \mathfrak{p}$. Since each $v \in H$ is a linear combination of the vectors $v_j^{\lambda} = \Psi(e_j^{\lambda})$ with $\Psi \in H^{(\lambda)}$, it suffices to show that $pv_j^{\lambda} \in H$.

Since V is a Harish-Chandra module we get that

$$pv_j^{\lambda} = \sum_{\mu \in \Lambda} (pv_j^{\lambda})_{\mu}, \tag{2.13}$$

where $(pv_j^{\lambda})_{\mu} \in V^{\mu}$. The orthogonality relations for the characters of irreductible representations give us that

$$(pv_j^{\mu})_{\mu} = \int \bar{\chi}^{\mu}(u) T(u)(pv_j^{\lambda})\, du, \tag{2.14}$$

where $\chi^{\mu}(u) = \sum_r \tau_{rr}^{(\mu)}(u)$ is the character of the representation T^{μ}.

If $L(\varphi) \in P_0(\lambda, \mu)$ then $L(\varphi)\Psi \in H^{(\mu)}$ and (2.8) gives us that

$$(L(\varphi)\Psi)(e_r^{\mu}) = \int \tau_{rr}^{(\mu)}(u) T(u)(p\Psi(e_j^{\lambda}))\, du \in H.$$

Hence (2.14) gives us that $(pv_j^{\lambda})_{\mu} \in H$ and $pv_j^{\lambda} \in H$.

We note that the results of this paragraph are the generalisation and the simplification of the results §1 of [5].

§3. Transition to the spaces $\mathscr{F}^{(l)}$.

As in §1 $\mathscr{F}$ is one of the spaces C^d, C^d_*, L^p_* ($p \geqq 1, d \in \mathsf{Z}_+ \cup \{\infty\}$). We denote the spaces C^∞ and C^∞_* by $\mathscr{E}$ and $\mathscr{E}_*$. The spaces C^d_* and L^p_* are the spaces of type 1, the spaces C^d are the spaces of type 2.

If φ and f are functions on R^n, then convolution $\varphi * f$ is defined by

$$\varphi * f(x) = \int_{\mathsf{R}^n} \varphi(y) f(x - y)\, dy.$$

PROPOSITION 3.1. *Let the spaces $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same type and $\mathscr{F}_1 \subseteq \mathscr{F}_2$. There is a one-to-one correspondence between the* ISS *of $\mathscr{F}_1$ and $\mathscr{F}_2$ obtained by assigning to the* ISS *$H \subset \mathscr{F}_1$ its closure $[H]$ in F_2. The same correspondence is obtained by assigning to the* ISS *$W \subset \mathscr{F}_2$ the subspace $W \cap \mathscr{F}_2 \subset \mathscr{F}_1$.*

PROOF. Assume that $\mathscr{F}_1 = C^d_*, \mathscr{F}_2 = L^p_*$. The other cases of $\mathscr{F}_1$ and $\mathscr{F}_2$ can be treated analogously.

Denote by C^∞_c the set of infinitely differentiable function on R^n with compact support. Let $\varphi \in C^\infty_c$. It is easy to verify that if $f \in L^p_*$ then $\varphi * f \in C^d_*$ and the mapping $f \to \varphi * f$ from L^p_* to C^d_* is continuous.

Recall that a sequence of functions $\varphi_u \in C^\infty_c$ is said to be an approximating unit if the following conditions hold:

1) $\varphi_n \geqq 0$;

2) $\int_{\mathsf{R}^n} \varphi_n(x)\, dx = 1$ for every n,

3) every neighborhood of zero contains the supports $\operatorname{supp} \varphi_n$ for sufficiently large n.

A standard argument shows that if $f \in L^p_*$ or C^d_*, then $\varphi_n * f \to f$ in L^p_* or C^d_*, respectively.

Let H be the ISS of C^d_*. We shall show that $[H] \cap C^d_* = H$. If $f \in [H] \cap C^d_*$ then for some net we have that $f_\alpha \to f$ in L^p_* and $f_\alpha \in H$. Let φ_n be an approximating unit. For every k the net $\varphi_k * f_\alpha$ in C^d_* and, since $\varphi_k * f_\alpha \in H$, it follows that $\varphi_k * f \in H$. The sequence $\varphi_k * f \to f$ converges to $\varphi_R * f$ in C^d_* whenever $k \to \infty$therefore $f \in H$.

If $H_1 \neq H_2$ are ISS's of C^d_* then $[H_1] \neq [H_2]$ since $[H_1] \cap C^d_* = H_1 \neq [H_2] \cap C^d_* = H_2$, it follows that the mapping $H \to [H]$ is an injection.

If W is an ISS of L_* then $W_0 = W \cap C^d_*$ is an ISS of C^d_*. For every $f \in W$ the functions $\varphi_k * f \in W_0$ and $\varphi * f \to f$ in L^p_*. Therefore $[W_0] = W$ and the mapping $H \to [H]$ is a surjection.

Let π be the representation of the group G on the complete locally convex space $\mathscr{F}$ (here G can be an arbitrary Lie group, K be a compact subgroup; $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of G and K, $\mathfrak{g}$ and $\mathfrak{k}$ be the complexifications of $\mathfrak{g}_0$ and $\mathfrak{k}_0$). The vector $\xi \in \mathscr{F}$ is said to be smooth (or analytic) if the mapping $g \to \pi(g)\xi$ of G into $\mathscr{F}$ is infinitely differentiable (respectively analytic). The vector $\xi \in \mathscr{F}$ is said to be

K-finite if the linear span of all vectors $\pi(u)\xi$ with $u \in K$ is finite-dimensional. Let $\mathscr{F}_\sigma$ be the set of all smooth K-finite vectors of $\mathscr{F}$, $\mathscr{F}_\#$ be the set of all analytic K-finite vectors of $\mathscr{F}$. $\mathscr{F}_\sigma$ and $\mathscr{F}_\#$ are Harish-Chandra modules with respect to the action of $\mathfrak{g}$ induced by the representation π.

If H is a closed π-invariant subspace, then $H_\sigma = H \cap \mathscr{F}_\sigma$ and $H_\# = H \cap \mathscr{F}_\#$ are Harish-Chandra submodules of $\mathscr{F}_\sigma$ and $\mathscr{F}_\#$ respectively. The subspace H_σ is dense in H. If $\mathscr{F}$ is a Banach space then $H_\#$ is dense in H also [1], but if $\mathscr{F}$ is a complete locally convex space then $H_\#$ can be a nondence subspace. We note that, if $W_\#$ is a Harish-Chandra submodule of $\mathscr{F}_\#$, then its closure $W = [W_\#]$ is a π-invariant subspace of $\mathscr{F}$.

Now let G be the group of isometries of R^n, π be the quasiregular representation of G on some complete locally convex function space. It follows from §2 that to describe the Harish-Chandra submodule H_σ of $\mathscr{F}_\sigma$ it suffices to describe the subspaces

$$H_\sigma^{(\lambda)} = \mathrm{Hom}_\mathfrak{k}(E^\lambda, H_\sigma) \subseteq \mathrm{Hom}_\mathfrak{k}(E^\lambda, \mathscr{F}_\sigma) = \mathscr{F}_\sigma^{(\lambda)}.$$

From a homomorphism $\Psi \in \mathscr{F}_\sigma^{(\lambda)}$ we construct a function $F(x)$ *on* R^n taking values in E^λ. By definition

$$\langle \xi, F(x)\rangle = [\Psi(\xi)](x) \quad \forall \xi \in E^\lambda, \tag{3.1}$$

where $\langle , \rangle$ is an invariant Hermitian form in E^λ.

For an E^λ-valued function $F(x)$ *on* R^n to correspond to some homomorphism $\Psi \in \mathscr{F}_\sigma^{(\lambda)}$, it is necessary and sufficient that the following two conditions hold:

$$F(ux) = T^\lambda(u)F(x) \quad \forall u \in K; \tag{3.2}$$

$$\langle \xi, F(x)\rangle \in \mathscr{F}_\sigma \quad \forall \xi \in E^\lambda. \tag{3.3}$$

The homomorphism Ψ will be identified with the corresponding function $F(x)$ in what follows and the space $\mathscr{F}_\sigma^{(\lambda)}$ is identified with the set of all E^λ-valued functions satisfying the conditions (3.2) and (3.3). Respectively the space $\mathscr{F}_\#^{(\lambda)}$ is identified with the set of all E^λ-valued functions $F(x)$ satisfying the conditions (3.2) and

$$\langle \xi, F(x)\rangle \in \mathscr{F}_\# \quad \forall \xi \in E^\lambda.$$

The irreductible representation of $K = \mathrm{SO}\,(n)$ is determined by the highest weight $\lambda = (\lambda_1, \dots \lambda_m)$, where $m = [n/2]$, the numbers $\lambda_j \in \mathsf{Z}$ and the conditions (1.3), (1.3′) hold.

LEMMA 3.2. *Let $F(x)$ be a nonzero E^λ-valued function on* R^n *satisfying the condition* (3.2). *Then $\lambda \in \Lambda_0$ (that is $\lambda_2 = \dots = \lambda_m = 0$).*

PROOF. Let $\alpha(t) \in G$ be the translation of R^n: $\alpha(t)(x) = x + te_n$, where $e_n = (0, \dots 0, 1)$, $t \in \mathsf{R}^n$. Every point $x \in \mathsf{R}^n$ can be represented in the form

$x = u\alpha(t)O$, where $u \in K$. Then $F(x) = T^\lambda(u)F(\alpha(t)O)$, hence the function $F(\alpha(t)O)$ is nonzero and for some $t \in \mathrm{R}^n$ the vector $\xi = F(\alpha(t)O) \neq 0$.

Let $K_1 = \{u \in K : ue_n = e_n\}$. The subgroup K_1 is isomorphic to $\mathrm{SO}(n-1)$. If $u \in K_1$ then

$$T^\lambda(u)\xi = T^\lambda(u)F(\alpha(t)O) = F(u\alpha(t)O) = F(\alpha(t)O) = \xi.$$

The one-dimensional subspace of E^λ spanned by the vector ξ is a K_1-invariant subspace and the representation of $K_1 = \mathrm{SO}(n)$ in this space has the highest weight $(0, \dots 0)$. On the other hand it is well known [1] that the restriction of T^λ to the subgroup K_1 is a direct sum of nonequivalent representations of K_1 and the representation with highest weight $(0, \dots 0)$ is contained in this sum if and only if

$$\lambda_1 \geqq 0 \geqq \lambda_2 \geqq 0 \geqq \dots \geqq \lambda_{m-1} \geqq 0 \geqq \lambda_m \geqq 0,$$

hence $\lambda = (l, 0, \dots 0)$ where $l = \lambda_1$.

Let

$$E_0^\lambda = \{\xi \in E^\lambda : T^\lambda(u)\xi = \xi \quad \forall u \in K_1\}.$$

It follows from the proof of Lemma 3.2 that $\dim E_0^\lambda = 1$ for $\lambda \in \Lambda_0$ and $\dim E_0^\lambda = 0$ for $\lambda \notin \Lambda_0$.

COROLLARY 3.3. *Let $F(x)$ be an E^λ-valued function satisfying the condition* (3.2), *$\lambda \in \Lambda_0$, ξ_0 is a nonzero vector of E_0^λ. Then there exists a complex-valued function $f(t)$, $t \in \mathrm{R}$, such that $F(\alpha(t)O) = f(t)\xi_0$.*

It follows from Lemma 3.2 that $\mathscr{F}_\sigma^{(\lambda)} = \{0\}$ for $\lambda \notin \Lambda_0$. It will be assumed below that $\lambda = (l, 0, \dots 0) \in \Lambda_0$ and we will write $\mathscr{F}_\sigma^{(l)}$, $\mathscr{F}_\#^{(l)}$ and so on. If H is an ISS of $\mathscr{F}$ then H_σ is a Harish-Chandra submodule of $\mathscr{F}_\sigma$, hence H_σ is the linear span of the functions $\langle \xi, F(x) \rangle$ for $F \in H_\sigma^{(l)}$, $\xi \in E^l$, $l \in \mathrm{Z}$ for $n = 2$ and $l \in \mathrm{Z}_+$ for $n \geqq 3$.

The subspaces $\mathscr{F}^{(l)}$ and the invariant cells $H^{(l)}$ of H were defined in § 1. We note that $H_\sigma^{(l)} \subseteq H^{(l)}$, hence H is the closure of the linear span of the functions $\langle \xi, F(x) \rangle$ for $F \in H_\sigma^{(l)}$, $\xi \in E^l$, $l \in \mathrm{Z}_+$ for $n \geqq 3$ or $l \in \mathrm{Z}$ for $n = 2$. In particular H can be uniquely recovered from all the cells $H^{(l)}$ of H.

Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of the Lie groups G and K respectively, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition. The Lie algebra $\mathfrak{g}_0$ is identified with the set of $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} & & & z_1 \\ & A & & \vdots \\ & & & z_n \\ 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.4}$$

where $A \in \mathrm{so}(n, \mathrm{R})$, $z = (z_1, \ldots z_n) \in \mathrm{R}^n$. Then $\mathfrak{k}_0$ and $\mathfrak{p}_0$ consist of the matrices

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} & A & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{p}_0 = \left\{ \begin{pmatrix} & 0 & & \begin{matrix} z_1 \\ \vdots \\ z_n \end{matrix} \\ 0 & \ldots & 0 & 0 \end{pmatrix} \right\}.$$

Let p_j be the matrix (3.4) with $z_j = 1$ and 0 elswere. The matrices $p_1, \ldots p_n$ form a basis in $\mathfrak{p}_0$.

Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$. Note that $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is an irreductible $\mathfrak{k}$-module with highest weight $(1, 0, \ldots 0)$. The dual $\mathfrak{k}$-module $\mathfrak{p}^*$ is isomorphic to $\mathfrak{p}$ and the basis $p_1, \ldots p_n$ is equal to its dual basis.

Using the method from §2 we must study the homomorphisms from $s(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda))$. If $\varphi \in s(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda))$ then the operator $L(\varphi)\colon \mathscr{F}_\sigma^{(\lambda)} \to \mathscr{F}_\sigma^{(\mu)}$ was defined in (2.3) and the action of this operator has the form

$$(L(\varphi)F)(x) = \sum_{j=1}^{m} \alpha_j^*[(p_j F)(x)], \tag{3.5}$$

where $p_j F$ is the action of an element of the Lie algebra on the function induced by the representation π, $\alpha_j = \varphi(p_j) \in \mathrm{Hom}(E^\mu, E^\lambda)$, $\alpha_j^* \in \mathrm{Hom}(E^\lambda, E^\mu)$ is the operator adjoint to α_j, that is

$$\langle \alpha_j^* \xi, \eta \rangle = \langle \xi, \alpha_j \eta \rangle \quad \forall \xi \in E^\lambda, \eta \in E^\mu.$$

Let E_1, E_2 and E_3 be arbitrary $\mathfrak{k}$-modules. Then the following sets of interwining operators are isomorphic as vector spaces:

$$s(E_3, \mathrm{Hom}(E_1, E_2)) \simeq s(E_1, \mathrm{Hom}(E_3, E_2)).$$

The isomorphism is obtained by assigning to the intertwining operator $\varphi\colon E_3 \to \mathrm{Hom}(E_1, E_2)$ the intertwining operator $\psi\colon E_1 \to \mathrm{Hom}(E_3, E_2)$ defined by

$$\psi(v_1)v_3 = \varphi(v_3)v_1 \quad \forall v_1 \in E_1 \; v_3 \in E_3.$$

We note also that the $\mathfrak{k}$-module $\mathrm{Hom}(E_1, E_2)$ is isomorphic to $E_1^* \otimes E_2$. Then

$$s(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda)) \simeq s(E^\mu, \mathrm{Hom}(\mathfrak{p}^*, E^\lambda)) \simeq s(E^\mu, \mathfrak{p} \otimes E^\lambda).$$

Let $\lambda = (\lambda_1, \ldots \lambda_n)$, ω_j be the row with 1 in j-th place and 0 elswere. It is known [9] that $\mathfrak{p} \otimes E^\mu$ is isomorphic to the direct sum

$$\bigoplus_{j=1}^{k} (E^{\lambda + \omega_j} \oplus E^{\lambda - \omega_j}) \quad \text{or} \quad E^\lambda \oplus \left[\bigoplus_{j=1}^{k} (E^{\lambda + \omega_j} \oplus E^{\lambda - \omega_j}) \right]$$

for even or odd n, respectively (if one of the rows $\lambda + \omega_j$ or $\lambda - \omega_j$ does not satisfy the conditions (1.3) and (1.3′) then the corresponding term is replaced by zero). Hence $s(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda)) \neq \{0\}$ if and only if $\mu = \lambda + \omega_j$ or $\mu = \lambda - \omega_j$ for some j, or $\mu = \lambda$ and n is odd. Under these conditions $\dim(\mathfrak{p}^*, \mathrm{Hom}(E^\mu, E^\lambda)) = 1$.

Let $\lambda, \mu \in \Lambda_0$ and $\lambda = (l, 0, \ldots 0)$. For $P_0(\lambda, \mu) \neq \{0\}$ it is necessary (but not sufficient) that $\mu = \lambda + \omega_1$ or $\mu = \lambda$ and n is odd. Under these conditions $\dim P_0(\lambda, \mu) \leqq 1$. We choose some operators (nonzero if $P_0(\lambda, \mu) \neq \{0\}$):

$$X_+^{(l)} \in P_0(\lambda, \lambda + \omega_1), \quad X_-^{(l)} \in P_0(\lambda, \lambda - \omega_1), \quad X_0^{(l)} \in P_0(\lambda, \lambda).$$

It will be shown in the next section that $P_0(\lambda, \lambda) = \{0\}$; therefore, the operator $X_0^{(l)}$ does not need to be considered. If $H^{(l)}$ are the cells of a single invariant subspace $H \subseteq \mathscr{F}$ it follows from Lemma 2.1 that

$$X_+^{(l)}(H_\sigma^{(l)}) \subseteq H_\sigma^{(l+1)} \quad \text{and} \quad X_-^{(l)}(H_\sigma^{(l)}) \subseteq H_\sigma^{(l-1)}.$$

PROPOSITION 3.4. *Suppose that in each space $\mathscr{F}_\#^{(l)}$ a linear subspace $H_\#^{(l)}$ is singled out such that*

$$X_+^{(l)}(H_\#^{(l)}) \subseteq H_\#^{(l+1)}, \quad X_-^{(l)}(H_\#^{(l)}) \subseteq H_\#^{(l-1)}.$$

Let H be the closure in $\mathscr{F}$ of the linear span of all functions $\langle \xi, F(x) \rangle$ for $F \in H_\#^{(l)}$, $\xi \in E^l$, $l \in \mathbf{Z}_+$ or $l \in \mathbf{Z}$ for $n \geqq 3$ and $n = 2$ respectively.

Then H is an invariant subspace and its cell $H^{(l)}$ is a closure of $H_\#^{(l)}$ in $\mathscr{F}^{(l)}$.

PROOF. Let $H_\#$ be the linear span of the functions $\langle \xi, F(x) \rangle$ for $F \in H_\#^{(l)}$, $\xi \in E^l$, $l \in \mathbf{Z}_+$ (or $l \in \mathbf{Z}$ for $n = 2$). By Proposition 2.2, $H_\#$ is a Harish-Chandra submodule and $H_\#^{(l)}$ coincides with the set of all functions $F(x) \in \mathscr{F}^{(l)}$ such that $\langle \xi, F(x) \rangle \in H_\# \, \forall \xi \in E^l$. Since $H_\#$ is a Harish-Chandra submodule of $\mathscr{F}_\#$, then H is an invariant subspace of $\mathscr{F}$.

Let $e_j (1 \leqq l \leqq n_l)$ be an orthonormal basis in E^l, $\tau_{jr}(u)$ be the matrix elements of the representation $T^l(u)$ in this basis. We construct the vector-valued function $F(x) = \sum \bar{F}^j(x) e_j$, where

$$F^j(x) = n_l^{-1/2} \int \bar{\tau}_{jr}(u^{-1}) f(ux) \, dx,$$

the integral is taken over the group K, r is a fixed number. It is easy to see that $F(x)$ satisfies condition (3.2) and $F^j \in \mathscr{F}$, consequently $F(x) \in \mathscr{F}^{(l)}$. The continuous mapping $f \to F$ from $\mathscr{F}$ to $\mathscr{F}^{(l)}$ arises, and we denote it by Γ_r.

We show that $\Gamma_r(H) = H^{(l)}$ for every ISS $H \subseteq \mathscr{F}$. If $f \in H$ then $\Gamma_r(f) \in H^{(l)}$. Conversely, let $F \in H^{(l)}$, $F(x) = \sum \bar{F}^j(x) e_j$. Then $F^j(x) = \langle e_j, F(x) \rangle$ and hence $F^j \in H$. It remains to observe that $\Gamma_r(F^r) = F$, what is easily obtained by using the orthogonality relations for the matrix elements of irreductible representations.

It is obvious that $\Gamma_r(H_\#) = H_\#^{(l)}$. Since $H_\#$ is dense in H, then $H_\#^{(l)} = \Gamma(H_\#)$ is dense in $H^{(l)} = \Gamma_r(H)$.

LEMMA 3.5. *Let the spaces $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same type and $\mathscr{F}_1 \subseteq \mathscr{F}_2$. There is a one-to-one correspondence between the invariant cells of $\mathscr{F}_1^{(l)}$ and $\mathscr{F}_2^{(l)}$ obtained by assigning to the cell $H^{(l)} \subseteq \mathscr{F}_1^{(l)}$ its closure $[H]$ in $\mathscr{F}_2$. The same correspondence is obtained by assigning to cell $W^{(l)} \subseteq \mathscr{F}_2^{(l)}$ the cell $W^{(l)} \cap \mathscr{F}_1^{(l)} \subseteq \mathscr{F}_1^{(l)}$.*

PROOF. We know (by Proposition 3.1) that the correspondence $H \to [H] = W$ is a bijection between the ISS's of $\mathscr{F}_1$ and $\mathscr{F}_2$ and in addition $W \cap \mathscr{F}_1 = H$. Let $H^{(l)}$ be a cell of the ISS $H \subseteq \mathscr{F}_1$, $W = [H] \subseteq \mathscr{F}_2$. Then $W^{(l)} = \Gamma_r([H]) = [H^{(l)}]$. It is clear that $W^{(l)} \cap \mathscr{F}_1 = H^{(l)}$, hence, the mapping $H^{(l)} \to [H^{(l)}]$ is an injection.

If $W^{(l)} \subseteq \mathscr{F}_2^{(l)}$ is a cell of the cell $W \subseteq \mathscr{F}_2$, then $H = W \cap \mathscr{F}_1$ is a dense subspace of W and $H^{(l)} = \Gamma_r(H)$ is dense in $W^{(l)}$, hence the mapping $H^{(l)} \to [H^{(l)}]$ is a surjection.

COROLLARY 3.6. *To prove Theorems* 1 *and* 2 *is suffices to prove these theorems for some space of type* 1 *and for some spaces of type* 2.

§4. Proof of Theorems 1 and 2.

Fix notation as in §3. Let $F(x) \in \mathscr{F}^{(l)}$. By Corollary 3.3 $F(\alpha(t)O) = f(t)\xi_0$, where $f(t)$ is a complex-valued function, $t \in \mathrm{R}$, $\xi_0 \in E_0^l$ and $\|\xi_0\| = 1$. It follows from (3.2) that $F(x)$ can be uniquely recovered from $f(t)$. We introduce the mapping D^l: $F(x) \to f(t)$. The action of the operators $X_\pm^{(l)}$, $X_0^{(l)}$ and the Laplace operator Δ on $F(x)$ can be expressed in terms of $f(t)$, that is we find the operators $D^l\Delta(D^l)^{-1}$, $D^{l\pm 1}X_\pm^{(l)}(D^l)^{-1}$, $D^lX_0^{(l)}(D^l)^{-1}$ (for brevity we denote them simply by Δ, $X_\pm^{(l)}$, $X_0^{(l)}$). We find an explicit form for this operators.

Let $x = u\alpha(t)O$, $u \in \mathrm{SO}(n)$. Then $F(x) = T^l(u)f(t)\xi_0$. The Laplace operator in polar coordinates has the form

$$(\Delta h)(x) = \partial_r^2 h + (n-1)r^{-1}\partial_r h + r^{-2}(Lh),$$

where L is the Laplace operator on the unit sphere S^{n-1}. If $x = u\alpha(t)O$ then $r = |t|$, therefore

$$(\Delta F)(x) = T^l(u)[\partial_t^2 f(t) + (n-1)t^{-1}\partial_t f(t)]\xi_0 + f(t)L(T^l(u)\xi_0).$$

Let e_j $(1 \leqq j \leqq n_l)$ be the orthonormal basis in E^l and let $e_1 = \xi_0$. Then $T^l(u) = \sum \tau_{j1}^{(l)}(u)e_j$. It is known [10, Ch. IX, §5] that

$$L(\tau_{j1}^{(l)}(u)) = -l(l+n-2)\tau_{j1}^{(l)}(u),$$

hence

$$L(T^l(u)\xi_0) = -l(l+n-2)T^l(u)\xi_0.$$

We get that

(4.1) $$(\Delta f)(t) = \partial_t^2 f + (n-1)t^{-1}\partial_t f - l(l+n-2)t^{-2} f.$$

Then computations of the operators $X_{\pm}^{(l)}, X_0^{(l)}$ is analogous to the computations of the corresponding operators in [6, §5]. Let $R(\varphi)$ be the rotation through the angle φ in the plane (x_n, x_{n-1}) of R^n. The product $\alpha(t)R(\varphi)\alpha(s)$ can be represented in the form $R(\psi)\alpha(t')R(\varphi')$, where ψ, t' and φ' are functions of the parameters t, φ and s. If s assumed to be small, then to within small quantities of first order

(4.2) $$\alpha(t)R(\varphi)\alpha(s) = R(\psi_1 s)\alpha(t + t_1 s)R(\varphi + \varphi_1 s),$$

where $t_1 = \cos\varphi$, $\psi_1 = -t^{-1}\sin\varphi$, $\varphi_1 = t^{-1}\sin\varphi$. This decomposition replaces the decomposition (5.7) in [6]. The other computations are obtained by repeating word-for-word the computations in [6]. Finally, we get that $X_0^{(t)} = 0$ and

(4.3) $$(X_+^{(l)} f)(t) = \partial_t f(t) - l t^{-1} f(t),$$

(4.4) $$(X_-^{(l)} f)(t) = \partial_t f(t) + (l+n-2)t^{-1} f(t).$$

It follows from the explicit form of $X_+^{(l)}$, $X_-^{(l)}$ and Δ that they are connected by the following relations:

$$X_-^{(l+1)} X_+^{(l)} f = \Delta f; \quad X_+^{(l-1)} X_-^{(l)} = \Delta f.$$

LEMMA 4.1. *Suppose that $H_{\#}^{(l_0)}$ is a linear subspace of $\mathscr{F}_{\#}^{(l_0)}$ such that $\Delta(H_{\#}^{(l_0)}) \subseteq H_{\#}^{(l_0)}$ and let $H^{(l_0)}$ be the closure of $H_{\#}^{(l_0)}$ in $\mathscr{F}^{(l_0)}$. Then $H^{(l_0)}$ is an invariant cell.*

PROOF. We introduce the collection of subspaces $H_{\#}^{(l)} \subseteq \mathscr{F}_{\#}^{(l)}$ for all l ($l \in \mathsf{Z}_+$ for $n \geqq 3$ or $l \in \mathsf{Z}$ for $n = 2$). We have already the subspace $H_{\#}^{(l_0)}$, for $l > l_0$ let $H_{\#}^{(l)} = X_+^{(l-1)} X_+^{(l-2)} \ldots X_+^{(l_0)} H_{\#}^{(l_0)}$, and for $l < l_0$ let $H_{\#}^{(l)} = X_-^{(l+1)} X_-^{(l+2)} \ldots X_-^{(l_0)} H_{\#}^{(l_0)}$. Then (4.5), and the fact that Δ commutes with operators $X_{\pm}^{(l)}$ give us that $X_{\pm}^{(l)}(H_{\#}^{(l)}) \subseteq H_{\#}^{(l\pm 1)}$. Then, by Proposition 3.4, $H^{(l)}$ is an invariant cell.

Let $\mathscr{L}_k$ denote the Banach space consisting of measurable complex-valued odd functions $h(t)$, $t \in \mathsf{R}$, such that the norm

$$\eta_{2,k}(h) = \left(\int_0^\infty |h(t)|^2 e^{-kt}\, dt\right)^{1/2}$$

is finite. The space $\mathscr{L}_* = \bigcup_{k>0} \mathscr{L}_k$ is equipped with the topology of the inductive limit of the BS's $\mathscr{L}_k$.

Let S_k denote the space consisting of continuous even complex-valued functions $h(t)$, $t \in \mathsf{R}$, such that $|h(t)|\, e^{-kt} \to 0$ as $t \to \infty$. The space S_k is a BS with the norm

$$\eta_k(h) = \sup_{t \geqq 0} |h(t)|\, e^{-kt}.$$

The space $S_* = \bigcup_{k>0} S_k$ is equipped with the topology of the inductive limit of the BS's S_k.

Let $C(\mathsf{R})$ be the space of all complex continuous functions on R and let $\mathscr{E}(\mathsf{R})$ be the space of all complex infinitely differentiable functions on R with the usual topologys, $C_0(\mathsf{R})$ and $\mathscr{E}_0(\mathsf{R})$ (or $C_1(\mathsf{R})$ and $\mathscr{E}_1(\mathsf{R})$) are the subspaces of all even (respectively, all odd) functions.

The proof of Theorem 1 consists of two steps: the first step is the proof of Theorem 1 for $l = 0$ and the second step is the proof for arbitrary l. The problem of describing the invariant cells of $\mathscr{F}^{(0)}$ will be reduced to the problem of describing the linear subspaces of $\mathscr{L}_*$ and $\mathscr{E}_1(\mathsf{R})$ that are closed and invariant with respect to the transformations

$$h(t) \to \tfrac{1}{2}(h(t+s) + h(t-s)) \quad \forall s \in \mathsf{R}. \tag{4.6}$$

This subspaces will be called the generalized invariant subspaces (GISS's), it will always be assumed that every GISS does not concide with the whole space $\mathscr{L}_*$ or $\mathscr{E}_1(\mathsf{R})$.

The problem of describing the GISS's of $\mathscr{L}_*$ was solved by Rashevskiĭ [4]. The GISS's of $\mathscr{L}_*$ are in a one-to-one correspondence with the sets σ of complex numbers satisfying the condition (A) in § 1. Corresponding to a set σ is the GISS that is the closure in $\mathscr{L}_*$ of the linear span of the functions

$$\sin \mu x, \quad x \cos \mu x = \partial_\mu \sin \mu x, \quad \partial_\mu^{r-1} \sin \mu x, \tag{4.7}$$

where μ runs through the set σ, r is the multiplicity of μ in σ. For $\mu = 0$ the functions (4.7) must be replaced by

$$x, \quad x^3, \quad \dots \quad x^{2r-1} \tag{4.8}$$

PROPOSITION 4.2. *The* GISS'*s of* $\mathscr{E}_1(\mathsf{R})$ *are in a one-to-one correspondence with the sets* $\sigma \subseteq \mathsf{R}_+$ *satisfying the condition* (B) *in* § 1. *Corresponding to a set* σ *is the* GISS *that is the closure in* $\mathscr{E}_1(\mathsf{R})$ *of the linear span of the functions* (4.7) *for* $\mu \neq 0$ *and* (4.8) *for* $\mu = 0$.

PROOF. A linear subspace $W \subseteq \mathscr{E}(\mathsf{R})$ will be called an invariant subspace (ISS) if W is closed and invariant under the transformations $h(t) \to h(t+s)$ $\forall s \in \mathsf{R}$. A linear subspace W is said to be symmetric if follows from $f(t) \in W$ that $f(-t) \in W$. We introduce the mapping P: $h(t) \to \frac{1}{2}(h(t) - h(-t))$. If W is a symmetric ISS of $\mathscr{E}(\mathsf{R})$ then $H = P(W)$ is the GISS of $\mathscr{E}_1(\mathsf{R})$. Conversely, if H is an GISS of $\mathscr{E}_1(\mathsf{R})$ then let W be the closure of the linear span of the functions $h(t+s)$ for $h \in H$, $s \in \mathsf{R}$. W is a symmetric ISS of $\mathscr{E}(\mathsf{R})$ and, since $P(h(t)) = \frac{1}{2}(h(t+s) + h(t-s)) \in H$, $P(W) = H$.

Finally, the proof of Prop. 4.2 follows from the Schwartz's results [11] that the

ISS of $\mathscr{E}(\mathsf{R})$ are in a one-to-one correspondence with the sets $\sigma \subset \mathsf{R}$ satisfying the condition (B) in §1 and corresponding to a set σ is the ISS that is closure of the linear span of the functions

$$e^{\mu t}, \quad te^{\mu t} = \partial_\mu e^{\mu t}, \quad \dots \quad t^{r-1}e^{\mu t} = \partial_\mu^{r-1} e^{\mu t}.$$

PROOF OF THEOREM 1 (THE CASE $l = 0$).

(a) By Corollary 3.6 it is sufficient to prove Theorem 1 for the spaces $C_*^{(0)}$ and $\mathscr{E}^{(0)}$. The spaces $C_*^{(0)}$ and $\mathscr{E}^{(0)}$ consists of complex-valued functions $F(x)$ satisfying the condition

$$F(ux) = F(x) \quad \forall u \in K. \tag{4.9}$$

The mapping

$$D^0\colon F(x) \mapsto f(t) = F(\alpha(t)O) \tag{4.10}$$

is an isomorphism of the topological vector space $C_*^{(0)}$ onto S and of $\mathscr{E}^{(0)}$ onto $\mathscr{E}_0(\mathsf{R})$. For the space $C_*^{(0)}$ it is obvious and for the space $\mathscr{E}^{(0)}$ it is the special case of Corollary Chapter in [12].

(b) We show that the linear subspace $H^{(0)} \subseteq C_*^{(0)}$ (or $H^{(0)} \subseteq \mathscr{E}^{(0)}$) is an invariant cell if and only if $H^{(0)}$ is closed and invariant under the transformations

$$F(x) \mapsto \int F(gux)\, du \quad \forall g \in G, \tag{4.11}$$

where du is the element of the Haar measure on K, the integral is taken over the group K.

Indeed, if $H^{(0)}$ is the cell of the ISS H then the functions $F(gux) \in H$, hence $H^{(0)}$ is invariant under (4.11). Conversely, let W be a closed subspace of $C_*^{(0)}$ (or $\mathscr{E}^{(0)}$) that is invariant under (4.11). Let H be the closure of the linear span of the functions $F(gx)$ with $F \in W$, $g \in G$. Then H is an ISS and the mapping

$$\Gamma\colon F(x) \mapsto \int F(ux)\, du$$

is a projection of H onto $H^{(0)}$. It is obvious that

$$\Gamma(F(gx)) = \int F(gux)\, du \in W.$$

Then $H^{(0)} = W$.

Since every $g \in G$ can be represented in the form $g = u_1 \alpha(t) u_2$, where $u_1, u_2 \in K$, then $H^{(0)}$ is invariant under the transformations (4.11) if and only if $H^{(0)}$ is invariant under (4.11) for $g = \alpha(s)\ \forall s \in \mathsf{R}$.

(c) Let T_x be the translation by the vector $x \in \mathrm{R}^n$ (that is $T_x(y) = y + x$). For $F(x) \in C_*^{(0)}$ (or $F(x) \in \mathscr{E}^{(0)}$) define

$$v(x, y) = \int F(T_y u T_x O)\, du.$$

It is obvious that

$$v(ux, y) = v(x, uy) = v(x, y) \quad \forall u \in K,$$

in particular

$$v(-x, y) = v(x, -y) = v(x, y).$$

If F is a C^2-class function then $v(x, y)$ satisfies the Darboux equation

$$\Delta_x v(x, y) = \Delta_y v(x, y),$$

where Δ is the Laplace operator. It is obvious that $v(x, O) = F(x)$. Let $u(t, s) = v(\alpha(t)O, \alpha(s)O)$, $t, s \in \mathrm{R}$. Then $u(t, s)$ satisfies the differential equation

$$D_t y(t, s) = D_s u(t, s), \tag{4.12}$$

where $D_t = \partial_t^2 + (n - 1)t^{-1}\partial_t$ is the Bessel differential operator, with the initial conditions

$$u(t, 0) = f(\alpha(t)O) = h(t); \tag{4.13}$$

$$\partial_s u(t, 0) = 0. \tag{4.14}$$

Let $(\tau^s h)(t) = u(t, s)$. It follows from (4.12)–(4.14) that τ^s is the Delsart-Levitan operator of generalized translation corresponding to Bessel operator [14]. The closed subspace $\mathscr{H} \subseteq S_*$ has the form $\mathscr{H} = D^0(H^{(0)})$ for some invariant cell $H^{(0)} \subset C^{(0)}$ if and only if $\mathscr{H}$ is invariant with respect to the transformations τ^s $\forall s \in \mathrm{R}$.

In [7] we have described the closed subspaces $\mathscr{H} \subset S_*$ that are invariant with respect to operators of generalized translation corresponding to the Bessel operator. It turns out that such subspaces are in one-to-one correspondence with the sets σ of complex numbers satisfying the condition (A) in § 1. Corresponding to a set σ is the closure in S_* of linear span of he functions

$$j_{n-1}(\mu t), \quad t j'_{n-1}(\mu t), \quad \ldots \quad t^{r-1} j_{n-1}^{(r-1)}(\mu t), \tag{4.15}$$

where μ runs over σ, r is the multiplicity of μ in σ, $j_{n-1}(\mu t)$ is the even eigenfunction of the operator D_t with the eigenvalue $(-\mu^2)$ that is normalized by the condition $j_{n-1}(0) = 1$ (let us observe that

$$j_{n-1}(\mu t) = \frac{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)}{(\mu t)^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(\mu t),$$

where $J_r(t)$ is the Bessel function). For $\mu = 0$ functions (4.15) must be replaced by

$$1, \quad t^2, \quad t^4, \quad \ldots \quad t^{2r-2}. \tag{4.16}$$

The proof of this results is obtained by reduction to the problem on describing the GISS's of $\mathscr{L}_*$.

Arguing as in [7], we get the description of the closed τ^s-invariant subspaces of $\mathscr{E}_0(\mathrm{R})$, only in this case the problem of describing the GISS's of $\mathscr{L}_*$ must be replaced by the problem of describing the GISS's of $\mathscr{E}_1(\mathrm{R})$. We get that the closed τ^s-invariant subspaces of $\mathscr{E}_0(\mathrm{R})$ are in a one-to-one correspondence with the sets $\sigma \subset \mathrm{C}_+$ satisfying the condition (B) in § 1. Corresponding to a set σ is the subspace $\mathscr{H}$ that is the closure in $\mathscr{E}_0(\mathrm{R})$ of the linear span of the functions (4.15) (or (4.16) for $\mu = 0$).

It remains to observe that the functions (4.15) or (4.16) are the basis of the solution space of the differential equation $(\Delta + \mu^2)^r h = 0$, where $\Delta = D_t$ is the Laplace operator in $\mathscr{E}^{(0)}$ that is the basis of $V^{(0)}_{\mu,r}$. Since $V^{(0)}_{\mu,r}$ contains only one (to within multiplication by a number) eigenfunctions of the operator Δ (which is $j_{n-1}(\mu t)$), it follows that $V^{(0)}_{\mu,r}$ has a Jordan basis. This completes the proof of Theorem 1 for the spaces $C^{(0)}_*$ and $\mathscr{E}^{(0)}_*$.

Let us pass to the general case $l \in \mathrm{Z}_+$ (or $l \in \mathrm{Z}$ for $n = 2$). Let $F(x) \in C^{(l)}$, $f(t) = D^l(F)$. The action of the operators $X^{(l)}_{\mp}$ has the form (3.5), hence the operators $X^{(l)}_{\pm}$ can be considered as continuous mappings of $C^{d(l)}$ onto $C^{(d-1)(l\pm 1)}$ (or of C^d_* onto $C^{(d-1)(l\pm 1)}_*$) for every $d \geqq 1$. The action of the operators $X^{(l)}_{\pm}$ in terms of $f(t)$ has the form (4.3) and (4.4). We remark that, if $n = 2$, then the operators $X^{(l)}_{\pm}$ are defined for every $l \in \mathrm{Z}$, and if $n \leqq 3$, then the operator $X^{(l)}_{+}$ is defined for $l \geqq 0$ and $X^{(l)}_{-}$ is defined for $l \geqq 1$.

Lemma 4.3. *The following properties are true (if the corresponding operators are defined):*

(1) dim $V^{(l)}_{\mu,r} = r$ *and* $V^{(l)}_{\mu,r} \subseteq C^{(l)}_{*\#}$;

(2) Ker $X^{(l)}_{-} = \{0\}$ *for* $l > 0$, Ker $X^{(l)}_{-} = V^{(l)}_{0,1}$ *for* $l \leqq 0$;

(3) Ker $X^{(l)}_{+} = V^{(l)}_{0,1}$ *for* $l \geqq 0$, Ker $X^{(l)}_{+} = \{0\}$ *for* $l < 0$;

(4) $X^{(l)}_{-}(V^{(l)}_{\mu,r}) = V^{(l-1)}_{\mu,r_1}$, *where* $r_1 = r$ *for* $\mu \neq 0$ *or for* $\mu = 0$ *and* $l > 0$, $r_1 = r - 1$ *for* $\mu = 0$ *and* $l \leqq 0$;

(5) $X^{(l)}_{+}(V^{(l)}_{\mu,r}) = V^{(l+1)}_{\mu,r_1}$, *where* $r_1 = r$ *for* $\mu \neq 0$ *or for* $\mu = 0$ *and* $l < 0$, $r_1 = r - 1$ *for* $\mu = 0$ *and* $l \geqq 0$.

Proof. Let $X^{(l)}_{-} f = 0$, then it follows from (4.4) that $f(t) = Ct^{-(l+n-2)}$ (C is a constant). For $l \geqq 0$ the function f has a discontinuity at $t = 0$ and hence does not belong to $C^{(l)}$. Therefore, $\operatorname{Ker} X_+^{(l)} = \{0\}$. Arguing similarly, we get $\operatorname{Ker} X_+^{(l)} = \{0\}$ for $l < 0$. Then, for $l \leqq 0$, $X_-^{(l)}F = 0$ if and only if $X_+^{(l-1)}X_-^{(l)}F = 0$ or, by (4.5), $\Delta F = 0$. Therefore, $\operatorname{Ker} X_-^{(l)} = V_{0,1}^{(l)}$ for $l \leqq 0$. Similarly, $\operatorname{Ker} X_+^{(l)} = V_{0,1}^{(l)}$ for $l \geqq 0$. This proves (2) and (3).

Since the operators $X_\pm^{(l)}$ commute with Δ then

$$X_\pm^{(l)}(V_{\mu,r}^{(l)}) \subseteq V_{\mu,r}^{(l\pm 1)}. \tag{4.17}$$

Proof of statement (1), (4), (5) is carried by induction in $|l|$. Assume that $n \geqq 3$, hence $l \geqq 0$. The case when $n = 2$ and $l < 0$ can be treated analogously.

Let $l = 0$. We know that $\dim V_{\mu,r}^{(0)} = r$. If $F \in V_{\mu,r}^{(0)}$ then $F \in C_k$ for some $k > 0$ and F is a solution of the elliptic equation $(\Delta + \mu^2)^r F = 0$, hence (see [13], Appendix 5) F is an analytic vector of the representation π. Thus $V_{\mu,r}^{(0)} \subseteq C_{*\#}^{(0)}$ and (1) holds true. The operator $X_-^{(0)}$ is not defined and (4) is meaningless.

Let $\mu \neq 0$, then it follows from (4.17) that $X_+^{(0)}(V_{\mu,r}^{(0)}) \subseteq V_{\mu,r}^{(1)}$ and $\operatorname{Ker} X_+^{(0)} \cap V_{\mu,r}^{(0)} = \{0\}$. Hence $\dim V_{\mu,r}^{(0)} \leqq \dim V_{\mu,r}^{(1)}$. On the other hand it follows from $X_-^{(1)}(V_{\mu,r}^{(1)}) \subseteq V_{\mu,r}^{(0)}$ that $\dim V_{\mu,r}^{(1)} \leqq \dim V_{\mu,r}^{(0)}$ (it is true and for $\mu = 0$). Hence $X_+^{(0)}(V_{\mu,r}^{(0)}) = V_{\mu,r}^{(1)}$ and $\dim V_{\mu,r}^{(1)} = r$.

Let $\mu = 0$, $F \in V_{0,r}^{(0)}$. Then by (4.5) $X_-^{(1)}\Delta^{r-1}X_+^{(0)}F = \Delta^r F = 0$ and it follows from (2) that $\Delta^{r-1}(X_+^{(0)}F) = 0$ and $X_+^{(0)}F \in V_{0,r-1}^{(1)}$. Hence $X_+^{(0)}(V_{0,r}^{(0)}) \subseteq V_{0,r-1}^{(1)}$ and, since $\dim \operatorname{Ker} X_+^{(0)} = 1$ and $\dim V_{0,r-1}^{(1)} \leqq r - 1$, $X_+^{(0)}(V_{\mu,r}^{(0)}) = V_{0,r-1}^{(1)}$.

Suppose $l > 0$ and Lemma 4.3 is true for $l - 1$. Thus by (5) $X_+^{(l-1)}(V_{\mu,r}^{(l-1)}) = V_{\mu,r}^{(l)}$ for $\mu \neq 0$ and $X_+^{(l-1)}(V_{0,r+1}^{(l-1)}) = V_{0,r}^{(l)}$. Since $\operatorname{Ker} X_+^{(l-1)} = V_{0,1}^{(l-1)}$ then $\dim V_{\mu,r}^{(l)} = r$ for every $\mu \in \mathbf{C}$. Since the operator $X_+^{(l-1)}$ maps $C_{*\#}^{(l-1)}$ onto $C_{*\#}^{(l)}$ then $V_{\mu,r}^{(l)} \subseteq C_{*\#}^{(l)}$. Statements (4) and (5) can be treated as (5) for $l = 0$.

Proof of Theoorem 1 (general case).

(a) By Corollary 3.6 it is sufficient to prove Theorem 1 for the spaces C_* and C. Proof is carried by induction in $|l|$. Assume that $l \geqq 0$, the case $l < 0$ is quite similar. We note that $V_{\mu,r}^{(l)} \subseteq C_{*\#}^{(l)}$ (see Lemma 4.3). Since $C_{*\#} \subset C_\#$ then $V_{\mu,r}^{(l)} \subset C_\#^{(l)}$. For the spaces C_* and C Theorem 1 are proved quite similar and we consider the space C_*.

Let $(X_-^{(l)}f)(t) = h(t)$. It follows from (4.4) that

$$f(t) = t^{-(l+n-2)} \int_0^t h(s) s^{l+n-2}\, ds. \tag{4.18}$$

We introduce the mapping $A\colon C_*^{(l-1)} \to C_*^{(l)}$ by (4.18). Thus $A(X_-^{(l)}F) = F$ for $F \in C_*^{d(l)}$, $d \geq 1$. We fix an integer $d \geqq 1$.

(b) Suppose now that Theorem 1 holds true for $l - 1$. Let $H^{d(l)}$ be the closure of $X_-^{(l)}(H^{d(l)})$ in $C_*^{(l)}$. Let $W^{(l-1)}$ be the closure of $X_-^{(l)}(H^{d(l)})$ in $C_*^{(l-1)}$. We verify that $W^{(l)}$ is an invariant cell.

Let H be an ISS of C_* such that $H^{(l)}$ is the cell of H and let $H^{(l-1)} \subset C_*^{(l-1)}$ be the other cell of H. By assumption $H^{(l-1)}$ can be describeed by its spectrum $\sigma \subset \mathbf{C}$ and $H^{(l-1)}$ is the closure of the linear span of the subspaces $V_{\mu,r}^{(l)}$, where $\mu \in \sigma$, r is the multiplicity of μ in σ. Assume that the number 0 belongs to σ and k is the multiplicity of 0. Let the set σ' be obtained from σ replacing the multiplicity k of 0 by $(k-1)$ (if $k=0$ than we put $\sigma' = \sigma$). Denote by $H_1^{(l-1)}$ the invariant cell of $C_*^{(l-1)}$ corresponding to σ'. Let $F_1, \ldots F_k$ be a Jordan basis of $V_{0,k}^{(l-1)}$. Let $H_0^{(l-1)}$ be the linear subspace spanned by the vector F_k (if $k=0$ then we put $H_0^{(l-1)} = \{0\}$). Thus $H^{(l-1)} = H_1^{(l-1)} \oplus H_0^{(l-1)}$ (the direct sum of linear spaces).

Let $\mu \neq 0$, $\mu \in \sigma$. It follows from $V_{\mu,r}^{(l-1)} \subseteq H^{(l-1)}$ that $V_{\mu,r}^{(l)} = X_+^{(l-1)}(V_{\mu,r}^{(l-1)}) \subseteq H^{d(l)}$ and $X_-^{(l)}(V_{\mu,r}^{(l)}) = V_{\mu,r}^{(l-1)} \subseteq W^{(l-1)}$. For $\mu = 0$ it follows from $V_{0,k}^{(l-1)} \subseteq H^{(l-1)}$ that $X_+^{(l-1)}(V_{0,k}^{(l-1)}) = V_{0,k-1}^{(l)} \subseteq H^{d(l)}$ and $X_-^{(l)}(V_{0,k-1}^{(l)}) = V_{0,k-1}^{(l-1)} \subseteq W^{(l-1)}$. Hence $H_1^{(l-1)} \subseteq W^{(l-1)}$.

On the other hand $X_-^{(l)}(H^{d(l)}) \subseteq H^{(l-1)}$ and $W^{(l-1)} \subseteq H^{(l-1)}$. Thus we obtain that $H_1^{(l-1)} \subseteq W^{(l-1)} \subseteq H^{(l-1)} = H_1^{(l-1)} \oplus H_0^{(l-1)}$ and there are only two possibilities for $W^{(l-1)}$ namely $W^{(l-1)} = H_1^{(l-1)}$ or $W^{(l-1)} = H^{(l-1)}$, hence $W^{(l-1)}$ is an invariant cell.

(c) Let $H_1^{d(l)} \neq H_2^{d(l)}$ be invariant cells of $C_*^{d(l)}$, $W_\nu^{(l)}$ $(\nu = 1,2)$ be the closure of $X_-^{(l)}(H_\nu^{d(l)})$ in $C_*^{(l-1)}$. We verify that $W_1^{(l-1)} \neq W_2^{(l-1)}$. Note that $H_\nu^{d(l)} \subseteq A(W_\nu^{(l-1)}) \subseteq H_\nu^{(l)}$ hence the closure of $A(W_\nu^{(l-1)})$ in $C_*^{(l)}$ is equal to $H_\nu^{(l)}$. If $W_\nu^{(l-1)} = W_2^{(l-1)}$ then $H_1^{(l)} = H_2^{(l)}$ in contradiction to Lemma 3.5.

(d) Let $H^{d(l)}$ be an invariant cell in $C_*^{d(l)}$ and let $W^{(l-1)} \subseteq C_*^{(l-1)}$ be constructed as in (b). Let $W^{(l-1)}$ be describe by the set σ. Denote by $H_1^{d(l)}$ the subspace of $C_*^{d(l)}$ that is closure of linear span of the subspaces $W_{\mu,r}^{(l)}$, where $\mu \in \sigma, r$ is the multiplicity of μ. Since $V_{\mu,r}^{(l)} \subseteq C_{*\#}^{(l)}$ and $\Delta(V_{\mu,r}^{(l)}) \subseteq V_{\mu,r}^{(l)}$ then, due to Lemma 4.1, $H_1^{d(l)}$ is invariant cell. It is obvious that the closure of $X_-^{(l)}(H_1^{d(l)})$ in $C_*^{(l-1)}$ coincides with $W^{(l-1)}$, hence $H_1^{d(l)} = H_2^{d(l)}$ that completes the proof of Theorem 1.

PROOF OF THEOREM 2. By Corollary 3.6 it is sufficient to prove Theorem 2 for the space C_* and C. Let us consider the space C_*, the space C can be considered quite similar.

Let $H^{(l)}$ be the invariant cells of a single ISS $H \subset C_*$ and $H^{(l)}$ can be described by a set $\sigma(l)$. If $V_{\mu,r}^{(l)} \subseteq H^{(l)}$ and $V_{\mu,s}^{(l+1)} \subseteq H^{(l+1)}$ then $X_+^{(l)}(V_{\mu,r}^{(l)}) \subseteq H^{(l+1)}$ and $X_-^{(l+1)}(V_{\mu,s}^{(l+1)}) \subseteq H^{(t)}$, but due to Lemma 4.3

$$X_+^{(l)}(V_{\mu,r}^{(l)}) = V_{\mu,r_1}^{(l+1)}, \quad X_-^{(l+1)}(V_{\mu,s}^{(l+1)}) = V_{\mu,s_1}^{(l)}, \tag{4.19}$$

where $r_1 = r$, or $r_1 = r-1$ and $s_1 = s$ or $s_1 = s-1$ depending on μ and l. It follows from (4.12) that the multiplicities r and s of μ in the sets $\sigma(l)$ and $\sigma(l+1)$ must be equal for $\mu \neq 0$, s must be equal to r or $r-1$ for $\mu = 0$ and $l \geqq 0$, r must be equal to s or $s-1$ for $\mu = 0$ and $l \leqq 0$. Hence the sets $\sigma(l)$ must satisfy conditions of Theorem 2.

Conversely, suppose that in each space $C_*^{(l)}$ we fix an invariant cell $H^{(l)}$, let $\sigma(l)$ be the spectrum of $H^{(l)}$, and assume conditions (1)–(3) of Theorem 2. Let $H_\#^{(l)}$ be the linear span of the subspaces $V_{\mu,r}^{(l)}$ ($\mu \in \sigma(l)$, r is the multiplicity of μ in $\sigma(l)$). Then it follows from Lemma 4.3 that $X_\pm^{(l)}(H_\#^{(l)}) \subseteq H_\#^{(l\pm 1)}$. Let H be the closure in C_* of the linear span of the functions $\langle \xi, F(x)\rangle$ for all $F \in H_\#^{(l)}$, $\xi \in E^l$, $l \in \mathbf{Z}_+$ (or $l \in \mathbf{Z}$ for $n = 2$). It follows from Proposition 3.4 that H is an invariant subspace and $H^{(l)}$ are the cells of this subspace.

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