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LAPLACE INTEGRAL ON RATIONAL NUMBERS

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1. Introduction.

In 1934 D. V. Widder proved [5, p. 325] that a function $\varphi: I \to \mathrm{R}$ defined on an open interval of real numbers is of the form $\varphi(t) = \int_{\mathrm{R}} e^{tx}\, d\mu(x)$, where μ is a positive Radon measure on R, if and only if φ is continuous and positive definite on I, which means that φ is continuous and satisfies the following condition: for each natural number $n > 0$ and each family of real numbers $c_1, \ldots, c_n, r_1, \ldots, r_n$, such that $r_i \in \frac{1}{2}I$, we have

$$\sum_{j,k=1}^{n} c_j c_k \varphi(r_j + r_k) \geqq 0.$$

We prove in Section 2 of this paper a similar theorem for a function defined on an open interval of rational numbers.

In [3] it is proved without using any integral representation that a completely monotone function on a commutative semigroup with neutral element is positive definite. We give in Section 3 of this work an analogous result for a function defined on the set $]a, \infty[\, \cap\, \mathrm{Q}$, where a is a real number.

We obtain in Section 4 a Levy-Khinchin type formula for negative definite functions defined on an open interval of rational numbers. This formula is of the same type as that obtained in [4] for negative definite functions defined on a commutative semigroup without zero.

In Section 5 are given integral representations which extend some results from [1] on positive and negative definite functions on Q_+.

2. Laplace integral on an open interval of rational numbers.

Let $a, b \in \mathrm{R} \cup \{-\infty, \infty\}$, such that $a < b$. We denote by Γ the set $]a, b[\, \cap\, \mathrm{Q}$. We shall say that a function $\varphi: \Gamma \to \mathrm{R}$ is positive definite on Γ if for each natural number $n > 0$, each family of real numbers $c_1, \ldots, c_n$ and each family of rational numbers $r_1, \ldots, r_n$ from the set $\frac{1}{2}\Gamma$, we have

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$$\sum_{j,k=1}^{n} c_j c_k \varphi(r_j + r_k) \geqq 0.$$

For each $r \in \Gamma$ let f_r be the real function defined on R by $f_r(x) = e^{rx}$. We shall say that a function $\varphi: \Gamma \to \mathrm{R}$ can be represented by a Laplace integral if there exists a positive Radon measure μ on R such that the functions $(f_r)_{r \in \Gamma}$ are μ integrable and we have

$$\varphi(r) = \int_{\mathrm{R}} e^{rx}\, d\mu(x), \quad r \in \Gamma.$$

We see as in [1], p. 210 that if such a measure exists, it is uniquely determined by φ.

THEOREM 1. *A function $\varphi: \Gamma \to \mathrm{R}$ can be represented by a Laplace integral if and only if it is positive definite on Γ.*

PROOF. It is clear that a function which can be represented by a Laplace integral is positive definite. Let $\mathscr{F}$ be the set of all families $(a_r)_{r \in \Gamma}$ such that $a_r \neq 0$ only for a finite number of r. We denote by V the real vector space generated by the family $(f_r)_{r \in \Gamma}$.

Let V_+ be the set

$$\{f \in V \mid f(x) \geqq 0, \quad x \in \mathrm{R}\}.$$

We can easily verify that V is an adapted space (cf. [2], p. 4–04 or [1], p. 42, 2.6). Let f be an element of V. Then there exists a family of real numbers $(a_r)_{r \in \Gamma}$ from $\mathscr{F}$ such that $f = \sum_{r \in \Gamma} a_r f_r$. We can define a linear mapping $L_\varphi: V \to \mathrm{R}$ by $L_\varphi(\sum_{r \in \Gamma} a_r f_r) = \sum_{r \in \Gamma} a_r \varphi(r)$. L_φ is well defined because the functions $(f_r)_{r \in \Gamma}$ are linearly independent. We shall prove that L_φ is positive on V_+. Let $f \in V_+, f \neq 0$. Then there exist a natural number $n > 0$ and rational numbers $r_1 = \dfrac{p_1}{q_1}, \ldots, r_n = \dfrac{p_n}{q_n}$, where $p_j, q_j \in \mathrm{Z}$, $q_j > 0$ and $r_1 < \ldots < r_n$, such that

$$f(x) = a_{r_1} e^{\frac{p_1}{q_1}x} + \ldots + a_{r_n} e^{\frac{p_n}{q_n}x}.$$

We denote by z the number $e^{\frac{x}{q_1 \cdots q_n}}$. The polynomial function $g: \mathrm{R} \to \mathrm{R}$ defined by $g(z) = a_{r_1} + \ldots + a_{r_n} z^{q_1 \cdots q_{n-1} p_n - p_1 q_2 \cdots q_n}$ is positive for z positive.

It follows that there exist four real polynomial functions $(P_j)_{1 \leqq j \leqq 4}$ such that

$$g(z) = (P_1(z))^2 + (P_2(z))^2 + z((P_3(z))^2 + (P_4(z))^2).$$

Let n_j be the degree of the polynomial P_j. We write $m = \max(n_1, n_2)$ and $l = \max(n_3, n_4)$. We have

$$0 \leqq \frac{m}{q_1 \cdots q_n}, \frac{l+\frac{1}{2}}{q_1 \cdots q_n} \leqq \frac{1}{2}\left(\frac{p_n}{q_n} - \frac{p_1}{q_1}\right).$$

The fact that φ is positive definite on Γ, yields that L_φ is positive on V_+.

We obtain from a result of Choquet about adapted cones ([2], p. 4–05, Proposition 2) or ([1], p. 43, 2.7) that there exists a positive Radon measure μ on R, such that the functions $(f_r)_{r\in\Gamma}$ are μ integrable for every $r \in \Gamma$, and we have

$$\varphi(r) = L_\varphi(f_r) = \int_{\mathsf{R}} e^{rx}\, d\mu(x).$$

REMARK 1. In the case when $\Gamma = \mathsf{Q}$, Theorem 1 becomes Proposition 5.10 from [1], p. 211.

REMARK 2. The Widder's result from ([5], p. 325) is an immediate consequence of Theorem 1.

THEOREM 2. *For a function* $\varphi: \Gamma \to \mathsf{R}$ *the following conditions are equivalent:*

(i) *φ is positive definite and there is a rational number $\alpha \in]0, b-a[$ such that the function $r \mapsto \varphi(r) - \varphi(\alpha + r)$ is positive definite on $]a, b-\alpha[\cap \mathsf{Q}$.*

(ii) *There exists a positive Radon measure μ on $]-\infty, 0]$, such that the functions $(x \mapsto e^{rx})_{r\in\Gamma}$ are μ integrable, which satisfies*

$$\varphi(r) = \int_{]-\infty,0]} e^{rx}\, d\mu(x), \quad r \in \Gamma.$$

PROOF. The implication (ii) $\Rightarrow$ (i) is immediate.

(i) $\Rightarrow$ (ii) Using Theorem 1 we find the positive Radon measures μ and σ_α on R such that

$$\varphi(r) = \int e^{rx}\, d\mu(x), \quad r \in \Gamma$$

and

$$\varphi(r) - \varphi(\alpha + r) = \int e^{rx}\, d\sigma_\alpha(x), \quad r \in]a, b-\alpha[.$$

Then also

$$\varphi(r) - \varphi(\alpha + r) = \int e^{rx}(1 - e^{\alpha x})\, d\mu(x),$$

so

$$(1 - e^{\alpha x})^+ \mu = (1 - e^{\alpha x})^- \mu + \sigma_\alpha$$

by unicity of the positive representing measure, hence

$$(1 - e^{\alpha x})\mu = \sigma_\alpha,$$

or $\alpha x \leqq 0$ for $x \in \operatorname{supp}(\mu)$. This shows that $\operatorname{supp}(\mu) \subseteq]-\infty, 0]$.

REMARK 3. In the case when $\Gamma =]0, \infty[\cap \mathbb{Q}$, the function $\varphi: \Gamma \to \mathbb{R}$ has the integral representation given by Theorem 2, (ii) iff it is positive definite and satisfies the condition:

$$\sup_{s \in [0, \infty[\cap \mathbb{Q}} \varphi(s + r) < \infty, \quad r \in \Gamma.$$

This is a consequence of [4, p. 295, theorem 1].

3. Finite differences and Laplace integral.

Let a be a real number and let Γ be the set $]a, \infty[\cap \mathbb{Q}$. For a rational number $r \geqq 0$ we denote by E_r: $\mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ the operator defined by $E_r \varphi(x) = \varphi(x + r)$.

THEOREM 3. *For a function $\varphi: \Gamma \to \mathbb{R}$ the following conditions are equivalent:*

(i) *For each natural number $p > 0$ and each family of strictly positive rational numbers $r_1, \ldots, r_p$ we have*

$$(E_0 - E_{r_1}) \ldots (E_0 - E_{r_p})(\varphi)(r) \geqq 0, \quad r \in \Gamma.$$

(ii) *φ is positive definite on Γ and there is a strictly positive rational number α such that the function $r \mapsto \varphi(r) - \varphi(\alpha + r)$ is positive definite on Γ.*

(iii) *There is a positive Radon measure μ on $]-\infty, 0]$, such that the functions $(x \mapsto e^{rx})_{r \in \Gamma}$ are μ integrable, which satisfies*

$$\varphi(r) = \int_{]-\infty, 0]} e^{rx}\, d\mu(x), \quad r \in \Gamma.$$

PROOF. (ii)$\Leftrightarrow$(iii) is a particular case of Theorem 2. (iii)$\Rightarrow$(i) is obvious.

(i)$\Rightarrow$(ii) It is enough to prove that φ is positive definite on Γ. Let $c_1, \ldots, c_n$ be a family of real numbers and $r_1, \ldots, r_n$ be a family of strictly positive rational numbers. We suppose that $c_1, \ldots, c_l > 0$ and $c_{l+1}, \ldots, c_n < 0$. We denote by M the number $c_1 + \ldots + c_l$ and by P the number $-(c_{l+1} + \ldots + c_n)$. We write

$$X = c_1 E_{r_1} + \ldots + c_l E_{r_l},$$

$$Y = -(c_{l+1} E_{r_{l+1}} + \ldots + c_n E_{r_n}),$$

$$ME_0 - X = U, \quad PE_0 - Y = V.$$

For each natural number $m \geqq 2$ we denote by Z_m the sum

$$\sum_{j,k=0}^{m} \left[\frac{j^2 M^2 + k^2 P^2}{m(m-1)} - 2\frac{jkMP}{m^2}\right] \binom{m}{j} \binom{m}{k} X^j U^{m-j} Y^k V^{m-k}.$$

We have $Z_m(\varphi)(r) \geqq 0$ for each $r \in \Gamma$.

We also can write

$$Z_m = \sum_{j,k=0}^{m} \left[\frac{j(j-1)M^2}{m(m-1)} + \frac{k(k-1)P^2}{m(m-1)} - 2\frac{jM}{m}\frac{kP}{m} + \frac{jM^2}{m(m-1)} + \frac{kP^2}{m(m-1)} \right] \binom{m}{j} \binom{m}{k} X^j U^{m-j} Y^k V^{m-k}$$

$$= (X-Y)^2 M^m P^m + \frac{X}{m-1} M^{m+1} P^m + \frac{Y}{m-1} M^m P^{m+1}$$

Dividing by $M^m P^m$ and letting m tend to ∞ we obain $(X - Y)^2(\varphi)(r) \geqq 0$ for each $r \in \Gamma$ which finishes our proof.

REMARK 4. It results from ([4], p. 303, Lemma 3 and p. 305, Theorem 6) that the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3 in the case when $a = 0$ is a consequence of ([4], p. 303, Theorem 4).

REMARK 5. For the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 3 we have not used the integral representation from (iii). An analogous proof, for completely monotone functions defined on a commutative semigroup with neutral element, is in ([3], p. 318, Theorem 3.1).

4. Negative definite functions on an open interval.

Let Γ and $\mathscr{F}$ be as in Section 2. We say that a function $\varphi: \Gamma \to \mathsf{R}$ is negative definite on Γ if for each natural number $n > 0$, each family of real numbers $c_1, \ldots, c_n$ such that $c_1 + \ldots + c_n = 0$ and each family of rational numbers $r_1, \ldots, r_n$ from the interval $\frac{1}{2}\Gamma$ we have

$$\sum_{j,k=1}^{n} c_j c_k \varphi(r_j + r_k) \leqq 0.$$

THEOREM 4. *For a function $\varphi: \Gamma \to \mathsf{R}$ the following conditions are equivalent:*

(i) *φ is negative definite on Γ.*

(ii) *There are real numbers A, B, C, α, β, such that $C \geqq 0$, $\beta > 0$, and α, $\alpha + 2\beta \in \Gamma$ and a positive Radon measure μ on $\mathsf{R}\backslash\{0\}$, such that the functions $\{x \mapsto \sum_{r\in\Gamma} a_r e^{rx} \mid (a_r)_{r\in\Gamma} \in \mathscr{F}, \sum_{r\in\Gamma} a_r = 0, \sum_{r\in\Gamma} a_r e^{rx} \geqq 0\}$ are μ integrable, which satisfy the relation*

$$\varphi(r) = A + Br - Cr^2 - \int_{\mathsf{R}\backslash\{0\}} \left(e^{rx} - e^{\alpha x} - \frac{r-\alpha}{\beta} e^{\alpha x}(e^{\beta x} - 1) \right) d\mu(x).$$

PROOF. (ii) $\Rightarrow$ (i) is immediate.

(i) $\Rightarrow$ (ii) Consider the set

$$V_+ = \left\{ f\colon \mathsf{R}\setminus\{0\} \to \mathsf{R} \mid f(x) = \sum_{r\in\Gamma} a_r e^{rx}, (a_r)_{r\in\Gamma} \in \mathscr{F}; \lim_{x\to 0} f(x) = 0; f \geqq 0 \right\}.$$

Define $L_\varphi\colon V_+ - V_+ \to \mathsf{R}$ by $L_\varphi(x \mapsto \sum_{r\in\Gamma} a_r e^{rx}) = -(\sum_{r\in\Gamma} a_r \varphi(r))$. Remark, as in Theorem 1, that L_φ is positive on V_+. We denote by $C^c(\mathsf{R}\setminus\{0\})$ the vector space $\{f\colon \mathsf{R}\setminus\{0\} \to \mathsf{R} \mid f$ continuous with compact support$\}$. Because $(e^{ux} - e^{vx})^2 > 0$ for $u \neq v$ and $x \neq 0$, it is possible to obtain as in ([1], p. 43, Theorem 2.7) a real vector space $\tilde{V} \supset V \cup C^c(\mathsf{R}\setminus\{0\})$ and a linear function $\tilde{L}_\varphi\colon \tilde{V} \to \mathsf{R}$, positive on $\{f \in \tilde{V} \mid f \geqq 0\}$, such that the restriction of $\tilde{L}_\varphi$ to V is L_φ and the restriction of $\tilde{L}_\varphi$ to $C^c(\mathsf{R}\setminus\{0\})$ is a positive Radon measure μ on $\mathsf{R}\setminus\{0\}$ for which the functions from V_+ are μ integrable and we have $L_\varphi(f) \geqq \int_{\mathsf{R}\setminus\{0\}} f\,d\mu$ for each $f \in V_+$. Let us choose rational numbers α and β such that $\alpha, \alpha + 2\beta \in \Gamma$ and $\beta > 0$. We take an arbitrary element r of Γ and find rational numbers γ and δ such that

$$a < \gamma < \alpha < \alpha + 2\beta < \delta < b \quad \text{and} \quad \gamma < r < \delta.$$

We denote by $c(r)$ the number $\dfrac{r-\alpha}{\beta}$ and by $d(r)$ the number $\frac{1}{2}\left(\left(\dfrac{r-\alpha}{\beta}\right)^2 - \dfrac{r-\alpha}{\beta}\right)$. We remark that the function $x \mapsto e^{rx} - e^{\alpha x} - c(r)e^{\alpha x}(e^{\beta x} - 1)$, defined on $\mathsf{R}\setminus\{0\}$, is in V_+ or in $-V_+$.

Let $g\colon \mathsf{R}\setminus\{0\} \to \mathsf{R}$ be the function defined by

$$x \mapsto e^{rx} - e^{\alpha x} - c(r)e^{\alpha x}(e^{\beta x} - 1) - d(r)e^{\alpha x}(e^{\beta x} - 1)^2,$$

and

$$h\colon \mathsf{R}\setminus\{0\} \to \mathsf{R}$$

the function defined by

$$x \mapsto (e^{\frac{\gamma}{2}x} - e^{\frac{\delta}{2}x})^2 + e^{\alpha x}(e^{\beta x} - 1)^2.$$

The function g is in $V_+ - V_+$ and the function h is in V_+. We have

$$\lim_{x\to y} \frac{|g(x)|}{h(x)} = 0 \quad \text{for} \quad y = 0, -\infty, \infty.$$

It results that for each natural number $n > 0$ there is a compact $K_n \subset \mathsf{R}\setminus\{0\}$ such that

$$|g(x)| \leqq \frac{1}{n} h(x), \quad x \in (\mathsf{R}\setminus\{0\})\setminus K_n.$$

We can choose the K_n such that $K_n \subset K_{n+1}$ and $\cup K_n = \mathsf{R}\setminus\{0\}$. Let $\varphi_n\colon \mathsf{R}\setminus\{0\} \to [0,1]$ be a function of $C^c(\mathsf{R}\setminus\{0\})$ such that $\varphi_n(x) = 1$ for $x \in K_n$. For $p \geqq n$ we have

$$-\frac{1}{n}h \leqq g - g\varphi_p \leqq \frac{1}{n}h.$$

From the fact that $\tilde{L}_\varphi$ is an increasing linear mapping on $\tilde{V}$ we obtain

$$L_\varphi(g) = \lim_{p\to\infty} \tilde{L}_\varphi(g\varphi_p) = \lim_{p\to\infty} \int_{\mathsf{R}\setminus\{0\}} g\varphi_p \, d\mu = \int_{\mathsf{R}\setminus\{0\}} g \, d\mu.$$

From this relation we have:

$$(1) \quad -(\varphi(r) - \varphi(\alpha) - c(r)(\varphi(\alpha + \beta) - \varphi(\alpha)) - d(r)(\varphi(\alpha + 2\beta) - 2\varphi(\alpha + \beta) + \varphi(\alpha)))$$
$$= \int_{\mathsf{R}\setminus\{0\}} (e^{rx} - e^{\alpha x} - c(r)e^{\alpha x}(e^{\beta x} - 1) - d(r)e^{\alpha x}(e^{\beta x} - 1)^2) \, d\mu(x).$$

This last relation and the inequality

$$(2) \qquad -(\varphi(\alpha + 2\beta) - 2\varphi(\alpha + \beta) + \varphi(\alpha)) \geqq \int_{\mathsf{R}\setminus\{0\}} e^{\alpha x}(e^{\beta x} - 1)^2 \, d\mu(x),$$

imply our result.

REMARK 6. Because we have

$$2\varphi(r + \beta) - \varphi(r) - \varphi(r + 2\beta) = 2C\beta^2 + \int_{\mathsf{R}\setminus\{0\}} e^{rx}(1 - e^{\beta x})^2 \, d\mu(x)$$

for $r \in]a, \alpha[\cap \mathsf{Q}$ we see, as in [1], p. 215, that A, B, C and μ are uniquely determined by α, β and φ.

5. Positive and negative definite functions on a half-open interval of rational numbers.

Let $a \in \mathsf{R}$ and $b \in \mathsf{R} \cup \{\infty\}$ be such that $a < b$. We denote by Γ the set $[a, b[\cap \mathsf{Q}$. We say that a function $\varphi: \Gamma \to \mathsf{R}$ is positive (resp. negative) definite on Γ if it satisfies the same condition as in Section 2 (resp. 4).

We define $\mathscr{F}$ as in Section 2.

Let $\chi_{\{a\}}: \Gamma \to \mathsf{R}$ denote the function defined by $\chi_{\{a\}}(a) = 1$ and $\chi_{\{a\}}(r) = 0$ for $r \in]a, b[$.

PROPOSITION 1. *For a function $\varphi: \Gamma \to \mathsf{R}$ the following conditions are equivalent:*

(i) *φ is positive definite on Γ.*

(ii) *There is a positive Radon measure μ on R, such that the functions $(x \mapsto e^{rx})_{r\in\Gamma}$ are μ integrable, and a positive real number A, which satisfy*

$$\varphi(r) = A\chi_{\{a\}}(r) + \int_{\mathsf{R}} e^{rx} \, d\mu(x), \quad r \in \Gamma.$$

PROOF. It is enough to prove (i) $\Rightarrow$ (ii). We find as in the proof of the Theorem 1 a positive Radon measure μ on R such that the functions $(x \mapsto e^{rx})_{r \in \Gamma}$ are integrable and we have

$$\varphi(r) = \int_{\mathsf{R}} e^{rx}\, d\mu(x), \quad r \in]a, b[$$

and

$$\varphi(a) \geqq \int_{\mathsf{R}} e^{ax}\, d\mu(x).$$

This yields the representation given in (ii).

REMARK 7. We remark, as in Section 2, that μ and A are uniquely determined by φ. When Γ is the set $\mathsf{Q}_+ = [0, \infty[\cap \mathsf{Q}$ we deduce from the Proposition 1 that the semigroup Q_+ is perfect; a result proved in ([1], p. 209, 5.6).

PROPOSITION 2. *For a function $\varphi\colon \Gamma \to \mathsf{R}$ the following conditions are equivalent:*

(i) *φ is negative definite on Γ.*

(ii) *There are real numbers A, B, C, D, β, such that $C, D \geqq 0$, $\beta > 0$ and $a + 2\beta \in \Gamma$, and a positive Radon measure μ on $\mathsf{R}\backslash\{0\}$, such that the functions $\{x \mapsto \sum_{r \in \Gamma} a_r e^{rx} \mid (a_r)_{r \in \Gamma} \in \mathscr{F},\ \sum_{r \in \Gamma} a_r = 0,\ \sum_{r \in \Gamma} a_r e^{rx} \geqq 0\}$ are μ integrable, which satisfy*

$$\varphi(r) = A + Br - Cr^2 - D\chi_{\{a\}}(r)$$
$$- \int_{\mathsf{R}\backslash\{0\}} \left(e^{rx} - e^{ax} - \frac{r-a}{\beta} e^{ax}(e^{\beta x} - 1)\right) d\mu(x).$$

PROOF. Because (ii) $\Rightarrow$ (i) is evident, we only have to prove (i) $\Rightarrow$ (ii).

Without loss of generality we may assume that $a = 0$.

Consider the set

$$V_+ = \{(\mathsf{R}\backslash\{0\}) \cup \{-\infty\} \to \mathsf{R} \mid f(x) = \sum_{r \in \Gamma} a_r e^{rx},$$
$$(a_r)_{r \in \Gamma} \in \mathscr{F};\ \lim_{x \to 0} f(x) = 0;\ f \geqq 0\},$$

where $e^{0(-\infty)} = 1$ and $e^{r(-\infty)} = 0$ for $r > 0$.

If $r \in \Gamma$ and $r > 0$ the function $x \mapsto (e^{rx} - 1)^2 \in V_+$ and is strictly positive on $(\mathsf{R}\backslash\{0\}) \cup \{-\infty\}$.

It results that we can construct a positive Radon measure μ on $(\mathsf{R}\backslash\{0\}) \cup \{-\infty\}$, as in the proof of Theorem 4, such that the analog of the relations (1) and (2) holds with $\alpha = 0$ and $0 < 2\beta < b$.

Consequently there are the real numbers A', B' and the positive real number C such that we have the relation

$$\varphi(r) = A' + B'r - Cr^2 - \int_{(\mathbb{R}\setminus\{0\})\cup\{-\infty\}} \left(e^{rx} - 1 - \frac{r}{\beta}(e^{\beta x} - 1)\right) d\mu(x), \quad r \in \Gamma$$

which is equivalent to the representation given by (ii).

REMARK 8. We see as in Remark 6 that A, B, C, D and μ are uniquely determined by β and φ. When $\Gamma = \mathbb{Q}_+$ we reobtain, with Proposition 2, the integral representation from ([1], p. 231, 5.13).

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