MATH. SCAND 76 (1995), 179–193

ON FLATNESS AND PROJECTIVITY OF A RING AS A MODULE OVER A FIXED SUBRING

JUAN JOSÉ GARCÍA and ANGEL DEL RÍO*

Introduction and Preliminaries.

Let R be a ring, G a group of automorphisms of R and R^G the fixed subring of G on R. In [8], Jøndrup studied when R is projective as a right R^G-module. The first aim of this paper is to continue Jøndrup's work. On the other hand in [10], Lorenz gives a theorem relating the left global dimension of R and R^G under some assumptions. One of these assumptions is R to be flat as a right R^G-module. Lorenz theorem motivates the study of when R is flat as a right R^G-module (see [10, Remark 2.6]). This is our second aim.

While we are not aware of any result concerning the flatness of R as R^G-module, there are many results ensuring projectivity. For example, if G is finite then R is projective as right R^G-module if any of the following conditions hold:

- R is simple artinian and G is outer (Montgomery [11, Theorem 2.7]).
- The order of G is invertible in R and R is either a finite product of simple rings or biregular right selfinjective (Handelman and Renault [7]).
- The order of G is invertible in R, R is right hereditary and one of the following conditions hold: R is semiprime and right noetherian; R is a PI ring which is finitely generated as a module over a subring of its center; R is an artinian algebra; R is reduced, von Neumann regular PI ring and the order of G is a power of 2; R is reduced von Neumann regular and finitely generated over its center; R is commutative von Neumann regular; R is reduced von Neumann regular and G is solvable (Jøndrup [8]).
- The skew group ring $RG = R\pi R + J$, where

  $$\pi = \sum_{g\in G} g \text{ and } J = \{\sum_{g\in G} a_g g \in RG \mid \sum_{g\in G} (ra_g)^g = 0 \text{ for all } r \in R\}.$$

  In particular this happens if RG is biregular and R has an element of trace 1 (Kitamura [9]).

* The second author has been partially supported by DGICYT (PB90-0300-Co2-02).
Received March 10, 1994.

Our study starts with some necessary and sufficient conditions for R to be projective (resp. flat) as right R^G-module (under the assumption of G being finitely generated when we consider flatness) by means of properties of some right RG-modules. (See Theorems 3 and 5). The main tool to prove those theorems is the basic fact that R has a canonical structure of right RG-module such that $\text{End}_{RG}(R)$ is isomorphic to R^G.

If we assume that the module R_{RG} is quasi-projective part of the equivalent conditions to projectivity or flatness of R_{RG} automatically hold. This fact is used to improve the general theorems when R_{RG} is quasi-projective (Corollaries 7 and 8).

By applying the previous results, we are able to give the main results of the paper which consists in some sufficient conditions for R_{RG} to be projective or flat easier to check than the given in the necessary and sufficient theorems. For instance we prove that R_{RG} is projective if any of the following conditions holds for G a finite group of automorphisms of R.

- $R\pi R$ is projective as RG-module, where π is the element $\sum_{g\in G} g$ of RG.
- RG is right hereditary, in particular this last happens if R is right hereditary and has a central element of trace 1. (This extends Jøndrup's results mentioned above).
- RG is semihereditary (in particular, if R is right semihereditary and has a central element of trace 1) and R_{RG} is finitely generated as a right R^G-module.
- RG is biregular (This should be compared with Handelman-Renault and Kitamura's results).

Further we prove that R_{RG} is flat when one of the following conditions hold:

- $R\hat{\pi}R$ is flat as right RG-module (see below for the definition of $\hat{\pi}$).
- G is finite and RG has weak dimension at most 1 (in particular, if R has weak dimension at most 1 and has a central element of trace 1).

We finish with some examples showing that some of this sufficient conditions can not be aweakened. For instance, we give an example of an infinite group G of automorphisms of a ring R such that RG is von Neumann regular and R_{RG} is not flat.

In this paper "ring" means associative ring with unit. If R is a ring, mod-R will denote the category of right R-modules. The notation M_R will be used to emphasize that M is a right R-module.

All over this paper R will stand for a ring and G for a group of automorphisms of R. The action of $g\in G$ on $r\in R$ will by denoted by r^g.

The fixed ring is the subring $R^G = \{r\in R \mid r^g = r \text{ for all } g\in G\}$ of R.

The skew group ring is the ring RG defined as follows: As an abelian group RG coincides with the free left R-module with basis G. Thus every element of RG has a unique expression $\sum_{g\in G} r_g g$, with $r_g = 0$ for almost all $g \in G$. The multiplication in RG is defined by the rule $(rg)(sh) = rs^{g^{-1}}gh$, for all $r, s \in R$ and $g, h \in G$.

R can be considered as a subring of RG by identifying $r \in R$ with re, where e denotes the identity of G. Then RG is free with basis G both as a left and right R-module.

The ring R has a natural structure of R^G-RG-bimodule, given by: $ar(sg) = a(rs)^g$ for all $a \in R^G$, $r, s \in R$ and $g \in G$. Furthermore, the canonical isomorphism $\mathrm{End}(R_R) \simeq R$ restricts to an isomorphism $\mathrm{End}(R_{RG}) \simeq R^G$.

A subset X of R is said to be G-invariant if $x^g \in X$, for all $x \in X$ and $g \in G$.

Necessary and Sufficient Conditions.

For every $M \in$ mod-RG, we denote $M^G = \{m \in M \mid mg = m \text{ for all } g \in G\}$. Clearly, $M^G R^G \subseteq M^G$, thus M^G is a submodule of M_{R^G}. Furthermore, if $f \in \mathrm{Hom}_{RG}(M, N)$, then $f(M^G) \subseteq N^G$. (Note that this notation is consistent with the given for the fixed ring, when R is considered as a right RG-module, because $rg = r^g$, for all $r \in R$ and $g \in G$).

Let $(-)^G$: mod-$RG \to$ mod-R^G be the functor associating $M \in$ mod-RG to M^G and $f \in \mathrm{Hom}_{RG}(M, N)$ to the restriction f^G of f from M^G to N^G.

For every $M \in$ mod-RG, let $c_M: M^G \otimes_{R^G} R \to M$ be the map given by $c_M(m \otimes r) = mr$ $(m \in M, r \in R)$.

For any $N \in$ mod-R^G, we define $u_N: N \to (N \otimes_{R^G} R)^G$ by $u_N(n) = n \otimes 1$.

LEMMA 1. 1. *The functor* $(-)^G$ *is naturally isomorphic to* $\mathrm{Hom}_{RG}(R, -)$: mod-$RG \to$ mod-R^G. *Therefore* $(-)^G$ *is right adjoint to* $T = -\otimes_{R^G} R$: mod-$R^G \to$ mod-RG.

2. u: $1_{\text{mod-}R^G} \to (-)^G \circ T$ *and* c: $T \circ (-)^G \to 1_{\text{mod-}RG}$ *are respectively the unit and counit of the adjunction pair* $((-)^G, T)$.

3. $c_{R^{(I)}}$ *is an isomorphism for every set* I.

4. $u_{(R^G)^{(I)}}$ *is an isomorphism for every set* I.

PROOF. (1) Is well known [3] and (2) follows by straightforward computations. To show (3) it is enough to realize that $(R^{(I)})^G = (R^G)^{(I)}$ and $c_{R^{(I)}}$: $(R^G)^{(I)} \otimes_{R^G} R \to R^{(I)}$ is the canonical isomorphism. Finally, by (2), $(c_{R^{(I)}})^G \circ u_{(R^{(I)})^G} = 1_{(R^{(I)})^G}$ and hence, $u_{(R^{(I)})^G} = u_{(R^G)^{(I)}}$ is an isomorphism.

Let $\widehat{RG} = \prod_{g\in G} Rg$ where, for any $g \in G$, Rg is the additive subgroup of RG formed by the elements of the form rg $(r \in R)$. The element of $\widehat{RG}$ which has r_g at the g-th entry $(g \in G)$, will be denote by $\widehat{\sum_{g\in G}} r_g g$. We will use this notation with a wide meaning, for instance, $\widehat{\sum_{g\in G}} r_g gh$ is the element of $\widehat{RG}$ which has r_g at the gh-entry $(g \in G)$.

We endow $\widehat{RG}$ with an structure of RG-bimodule by setting:

$$rg \cdot \widehat{\sum_{h \in G}} s_h h = \widehat{\sum_{g \in G}} r s_h^{g^{-1}} \quad \text{and} \quad \widehat{\sum_{h \in G}} s_h h \cdot rg = \widehat{\sum_{g \in G}} s_h r^{h^{-1}} hg$$

LEMMA 2. *$\widehat{RG}^G \simeq R$ as RG-R^G-bimodule and u_R is injective.*

PROOF. To prove the first claim just check that the map $\phi\colon R \to \widehat{RG}$ given by $\phi(r) = \sum_{g \in G} rg$ is a monomorphism of RG-R^G-bimodules and that $\mathrm{Im}(\phi) = \widehat{RG}^G$. By last lemma $(c_{\widehat{RG}})^G \circ u_R = 1_R$. Thus u_R is injective.

THEOREM 3. *Let R be a ring and G a group of automorphisms of R. The following conditions are equivalent:*

1. *R is projective as a right R^G-module.*
2. *$R \otimes_{R^G} R$ is isomorphic to a direct summand of a direct sum of copies of R_{RG} and u_R is surjective (bijective).*
3. *There is a direct summand M of a direct sum of copies of R_{RG} such that $M^G \simeq R$ as right R^G-module.*

PROOF. (1) $\Rightarrow$ (2) Let $p\colon (R^G)^{(I)} \to R$ be a split epimorphism in mod-R^G. Then $p \otimes 1\colon (R^G)^{(I)} \otimes_{R^G} R \to R \otimes_{R^G} R$ is a split epimorphism in mod-RG and $(R^G)^{(I)} \otimes_{R^G} R \simeq R^{(I)}$ as a right RG-module. Furthermore, $u_R \circ p = (p \otimes 1)^G \circ u_{(R^G)^{(I)}}$ and, by Lemma 1, $u_{(R^G)^{(I)}}$ is an isomorphism. Thus u_R is an epimorphism.

(2) $\Rightarrow$ (3) If $u_R\colon R \to (R \otimes_{R^G} R)^G$ is an epimorphism and M is a direct summand of a direct sum of copies of R_{RG} isomorphic to $R \otimes_{R^G} R$, then M satisfies the required condition.

(3) $\Rightarrow$ (1) Let M be a direct summand of a direct sum of copies of R_{RG}, such that $M^G \simeq R$ as right R^G-module. Then, there exists a split epimorphism $f\colon R^{(I)} \to M$ in mod-RG. Thus $f^G\colon (R^{(I)})^G \to M^G \simeq R$ is an split epimorphism in mod-R^G and $(R^{(I)})^G \simeq (R^G)^{(I)}$.

To give necessary and sufficient conditions for R_{R^G} to be flat we need the following Lemma.

LEMMA 4. *R_{RG} is finitely presented if and only if G is finitely generated.*

PROOF. Let $\Phi\colon RG \to R$ be the map given by $\Phi(\sum_{g \in G} gr_g) = \sum_{g \in G} r_g$ $(\sum_{g \in G} r_g g \in RG)$. Φ is a homomorphism of right RG-modules and $\mathrm{Ker}(\Phi)$ is generated by $\{x - 1 \mid x \in X\}$, if X is a set of generators of G (cf. [1, Lemma 2.2]). Thus, the necessary condition is obvious.

For each subgroup H of G, we define $I_H = \left\{ \sum_{g \in G} gr_g \,\middle|\, \sum_{h \in H} r_{hg} = 0 \text{ for all } g \in G \right\}$. Note that if $x \in H$, then $1 - x \in I_H$.

Clearly $I_H + I_H \subseteq I_H$ and $I_H R \subseteq I_H$. Let $\alpha = \sum_{g\in G} g r_g \in I_H$ and $\sigma \in G$. Then $\alpha\sigma = \sum_{g\in G} g r^{\sigma}_{g\sigma^{-1}} = \sum_{g\in G} g s_g$ where $s_g = r^{\sigma}_{g\sigma^{-1}}$. If $g \in G$, then $\sum_{h\in H} s_{hg} = \sum_{h\in H} r^{\sigma}_{hg\sigma^{-1}} = \left(\sum_{h\in H} r_{hg\sigma^{-1}}\right)^{\sigma} = 0$. In other words, I_H is a right ideal of RG.

Furthermore $I_H \subseteq I_G = \mathrm{Ker}(\Phi)$ for all subgroup H of G, because if $\alpha = \sum_{g\in G} g r_g \in I_H$ and $\{a_x \mid x \in G/H\}$ is a set of representatives of right isomorphic classes of G/H then $\sum_{g\in G} r_g = \sum_{x\in G/H}\left(\sum_{h\in H} r_{ha_x}\right) = 0$.

If R_{RG} is finitely presented, $\mathrm{Ker}(\Phi) = \sum_{x\in G}(1-x)RG$ is finitely generated and hence there exists a finite subset $F \subseteq G$ such that $\mathrm{Ker}(\Phi) = \sum_{x\in F}(1-x)RG$. If H is the subgroup generated by F, then $\mathrm{Ker}(\Phi) = I_H$. If $x \in G - H$, then $1 - x \in \mathrm{Ker}(\Phi)$ but $1 - x \notin I_H$. Thus $G = H$ is finitely generated.

THEOREM 5. *Let R be a ring and G a finitely generated group of automorphisms of R. The following conditions are equivalent:*

1. *R is flat as a right R^G-module.*
2. *$R \otimes_{R^G} R \simeq \varinjlim M_i$ where each $M_i \in$ mod-RG is a finite direct sum of copies of R_{RG} and u_R is surjective (bijective).*
3. *$R \otimes_{R^G} R \simeq \varinjlim M_i$ where each $M_i \in$ mod-RG is a direct summand of a direct sum of copies of R_{RG} and u_R is surjective (bijective).*
4. *There is an $M = \varinjlim M_i$ in mod-RG, such that each M_i is isomorphic to a finite direct sum of copies of R and $M^G \simeq R_{R^G}$.*
5. *There is an $M = \varinjlim M_i$ in mod-RG, such that each M_i is isomorphic to a direct summand of a direct sum of copies of R and $M^G \simeq R_{R^G}$.*

PROOF. (1) $\Rightarrow$ (2) Assume that R_{R^G} is flat, then $R \simeq \varinjlim N_i$ where N_i is isomorphic to $(R^G)^{n_i}$ for some positive integer n_i. Let $g\colon \varinjlim (N_i \otimes_{R^G} R) \to (\varinjlim N_i) \otimes_{R^G} R$ be the canonical isomorphism. Then $R \otimes_{R^G} R \simeq \varinjlim (N_i \otimes_{R^G} R)$ and $N_i \otimes_{R^G} R \simeq R^{n_i}$. On the other hand, by Lemma 1 u_{N_i} is an isomorphism for every i. Consider the following commutative diagram:

$$\begin{array}{ccc}
\varinjlim N_i & \xrightarrow{\ \lim u_{N_i}\ } & \varinjlim (N_i \otimes_{R^G} R)^G \\
\downarrow u_{\lim N_i} & & \downarrow f \\
((\varinjlim N_i) \otimes_{R^G} R)^G & \xleftarrow[\ g^G\ ]{} & (\varinjlim (N_i \otimes_{R^G} R))^G
\end{array}$$

Where $f\colon \varinjlim (N_i \otimes_{R^G} R)^G \to (\varinjlim (N_i \otimes_{R^G} R))^G$ is the canonical homomorphisms. $\lim u_{N_i}$ is an isomorphisms since, by lemma 1, so is u_{N_i} and f is an isomorphism because R_{RG} is finitely presented (Lemma 4), $(-)^G \simeq \mathrm{Hom}_{RG}(R, -)$

(Lemma 1) and [15, Proposition V.3.4]. Thus $u_{\lim N_i}$ is an isomorphism and hence u_R is an isomorphism.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious and if (2) (resp. (3)) holds, then $M = R \otimes_{R^G} R$ satisfies (4) (resp. (5)).

(5) $\Rightarrow$ (1) Let $M = \varinjlim M_i$ in mod-RG, such that M_i is isomorphic to a direct summand of a direct sum P_i of copies of R. Set $P_i = M_i \oplus N_i$ and $0 \to N_i \to P_i \to M_i \to 0$ a split exact sequence. Then, one has a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & N_i^G \otimes_{R^G} R & \to & P_i^G \otimes_{R^G} R & \to & M_i^G \otimes_{R^G} R & \to & 0 \\ & & {\scriptstyle c_{N_i}}\downarrow & & {\scriptstyle c_{P_i}}\downarrow & & {\scriptstyle c_{M_i}}\downarrow & & \\ 0 & \to & N_i & \to & P_i & \to & M_i & \to & 0 \end{array}$$

By Lemma 1, c_{P_i} is an isomorphism for every i. Thus, c_{N_i} is a monomorphism and c_{M_i} is an epimorphism, for every i. By symmetry both are isomorphisms.

Consider the following commutative diagram with canonical homomorphisms:

$$\begin{array}{ccc} (\varinjlim M_i)^G \otimes_{R^G} R & \xleftarrow{f \otimes 1} & \varinjlim M_i^G \otimes_{R^G} R \\ {\scriptstyle c_{\lim M_i}}\downarrow & & \uparrow{\scriptstyle h} \\ \varinjlim M_i & \xleftarrow[\lim c_{M_i}]{} & \varinjlim (M_i^G \otimes_{R^G} R) \end{array}$$

Since $- \otimes_{R^G} R$ preserves direct limits, h is an isomorphism. By the previous paragraph, c_{M_i} is an isomorphism for every i and hence $\lim(c_{M_i})$ is an isomorphism. Since R_{R^G} is finitely presented (Lemma 4), f is an isomorphism (Lemma 1 and [15, Proposition V.3.4]) and hence $c_{\lim(M_i)}$ is an isomorphism. Therefore, c_M is an isomorphism and since $(c_M)^G \circ u_{M^G} = 1_{M^G}$, u_{M^G} is an isomorphism and hence u_R is an isomorphism.

Finally, $R \overset{u_R}{\simeq} (R \otimes_{R^G} R)^G \simeq (M^G \otimes_{R^G} R)^G \overset{(c_M)^G}{\simeq} M^G \overset{f}{\simeq} \lim(M_i^G)$ and M_i^G is projective for every i. Thus R_{R^G} is flat.

REMARK 6. At the end of this paper we give an example of a ring R and a group of automorphisms G of R, such that RG is von Neumann regular and R_{R^G} is not flat. This shows that the assumption of G to be finitely generated can not be dropped in Theorem 5. Indeed, let $\hat{\pi} = \sum_{x \in G} g \in \widehat{RG}$ and $M = R\hat{\pi}R$. M is generated by R_{RG}, that is, there is an epimorphism $f \colon R^{(I)} \to M$ in mod-RG. Moreover, in this case, M_{RG} is flat and hence f is a pure epimorphism. Thus, M_{RG} is isomorphic to a direct limit of modules which are direct sums of copies of R [17, Theorem 7.34.2]. In other words, M_{RG} satisfies condition (5) of Theorem 5.

A particular case.

Recall that a module M_R is said to be quasi-projective if for every submodule N of M and every homomorphism $f\colon M \to M/N$ there is an endomorphism $g\ M \to M$ such that $p \circ g = f$ where $p\colon M \to M/N$ is the canonical epimorphism.

If R_{RG} is quasi-projective then it satisfies the conditions of [14, Theorem 2.1]. This implies that u_N is an isomorphism for every $N \in \text{mod-}R^G$. From Theorem 3 and 5 we obtain the following two corollaries.

COROLLARY 7. *Let R be a ring and G a group of automorphisms of R such that R_{RG} is quasi-projective. Then R_{RG} is projective if and only if $R \otimes_{R^G} R$ is isomorphic to a direct summand of a direct sum of copies of R_{RG}.*

COROLLARY 8. *Let R be a ring and G a group of automorphisms of R such that R_{RG} is quasi-projective. Consider the following conditions:*

1. *R_{R^G} is flat*
2. *$R \otimes_{R^G} R \simeq \varinjlim M_i$ where M_i is a finite direct sum of copies of R_{RG}.*
3. *$R \otimes_{R^G} R \simeq \varinjlim M_i$ is a direct summand of direct sum of copies of R_{RG}.*

In general (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If G is finitely generated, then all the conditions are equivalent.*

If R_{RG} is not quasi-projective, neither Corollary 7 nor Corollary 8 hold as the following example shows.

EXAMPLE 9. Let A be a noetherian ring, $R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ and G the group of automorphism of R generated by the inner automorphism associated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $R^G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in A \right\}$.

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $r_{R^G}(e_2) = \{r \in R \mid e_2 r = 0\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R^G$ is not a direct summand of $R^G_{R^G}$, $e_2 R^G$ is not projective. Therefore, neither is R, and, since R^G is noetherian, R_{R^G} is not flat either. But $(R \otimes_{R^G} R)_{RG} \simeq (e_1 R^G \oplus e_2 R^G) \otimes_{R^G} R \simeq e_1 R \oplus e_2 R = R_{RG}$.

Last corollary leads to the question of when R_{RG} is quasi-projective. That question has been considered in [5] and [16]. A simple checking of that conditions is given in the following Proposition:

PROPOSITION 10. *The following conditions are equivalent:*

1. *R_{RG} is quasiprojective.*

2. *For every G-invariant right ideal I and every* $r \in R$, *such that* $r^g - r \in I$ *for all* $g \in G$ *there exists an* $a \in R^G$ *such that* $r - a \in I$.

3. *For every* $r \in R$, $(r + R^G) \cap \sum_{g,h \in G} (r^g - r^h)R \neq \emptyset$.

PROOF. (1) $\Rightarrow$ (2) Let I be a G-invariant ideal and $r \in R$, such that $r^g - r \in I$ for all $g \in G$. Then I is a submodule of R_{RG} and $r + I \in (R/I)^G$. The map $f: R \to R/I$ given by $f(x) = (r + I)x$ $(x \in R)$ is a homomorphism of right RG-modules. Therefore, there exists $g \in \mathrm{End}_{RG}(R)$, such that $f = p \circ g$, where $p: R \to R/I$ is the canonical epimorphism. Put $a = g(1) \in R$. Then, $a \in R^G$ and $p(a) = f(1) = r + I$, as required.

(2) $\Rightarrow$ (3) Plainly $I = \sum_{g,h \in G} (r^g - r^h)R$ is a G-invariant right ideal and $r^g - r \in I$, for all $g \in G$. By hypothesis there is $a \in R^G$, such that $r - a \in I$. Then $r - a \in (r + R^G) \cap \sum_{g,h \in G} (r^d - r^h)R$.

(3) $\Rightarrow$ (1) Let I be a submodule of R and f: $\mathrm{Hom}_{RG}(R, R/I)$. By Lemma 1 there exists an $r + I \in (R/I)^G$ such that $f(x) = (r + I)x$ for all $x \in R$. Let $a \in R^G$ such that $r - a \in \sum_{g,h \in G} (r^g - r^h)R$ and $g: R \to R$ given by $g(x) = ax$ for all $x \in R$. Then $g \in \mathrm{End}_{RG}(R)$ and $p \circ g = f$.

Working Sufficient Conditions.

Theorem 3 (resp. Theorem 5) gives necessary and sufficient conditions for R_{RG} to be projective (resp. flat) which have been improved in Corollary 7 (resp. Corollary 8) for the case of R_{RG} being quasi-projective. But, maybe it is not easy to check if a given module is a direct summand of a direct sum of copies of R_{RG} (resp. a direct limit of modules which are finite direct sums of copies of R_{RG}). Therefore it would be nice to have easier checking sufficient conditions. The rest of the paper is aimed to find such conditions.

If G is finite there is a map, called the trace map, $\mathrm{tr}: R \to R^G$, given by $\mathrm{tr}(r) = \sum_{g \in G} r^g$.

We start with a necessary and sufficient condition for the case when G is finite and there is an element in R of trace 1. To do that we need the following well known Lemma.

LEMMA 11 ([3]). *The following conditions are equivalent:*

1. R_{RG} *is projective.*
2. G *is finite and* $1 \in \mathrm{tr}(R)$.

PROOF. The same as in [3, Proposition 1.7]. Note that in the given reference the condition G to be finite is originally assumed, but the same proof shows that if R_{RG} is projective, then G is finite.

PROPOSITION 12. *Let R be a ring and G a group of automorphisms of R. Consider the following conditions:*

1. *G is finite, $1 \in \operatorname{tr}(R)$ and R_{RG} is projective.*
2. *$R \otimes_{R^G} R$ is projective as right RG-module.*
3. *There exists a projective right RG-module M, such that $M^G \simeq R_{R^G}$ and $M = M^G R$.*
4. *G is finite and R_{R^G} is projective.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

PROOF. $(1) \Rightarrow (2)$ Assume that G is finite, $1 \in \operatorname{tr}(R)$ and R_{R^G} is projective. By Theorem 3, $R \otimes_{R^G} R$ is isomorphic to a direct summand of a direct sum of copies of R_{RG}. By Lemma 11, R_{RG} is projective and hence $(R \otimes_{R^G} R)_{RG}$ is projective.

$(2) \Rightarrow (3)$ Let $f\colon (R^G)^{(I)} \to R$ be an epimorphism in mod-R^G. Then $f \otimes 1\colon (R^G)^{(I)} \otimes_{R^G} R \to R \otimes_{R^G} R$ is an epimorphism in mod-RG. Therefore, $f \otimes 1$ splits and hence the following is a commutative diagram with exact rows:

$$\begin{array}{ccccc} (R^G)^{(I)} & \xrightarrow{\;f\;} & R & \to & 0 \\ {\scriptstyle u_{(R^G)^{(I)}}}\downarrow & & \downarrow{\scriptstyle u_R} & & \\ ((R^G)^{(I)} \otimes_{R^G} R)^G & \xrightarrow[(f \otimes 1)^G]{} & (R \otimes_{R^G} R)^G & \to & 0 \end{array}$$

By Lemma 1 $u_{(R^G)^{(I)}}$ is an isomorphism. Therefore, u_R is surjective and, by Lemma 2 it is an isomorphism. Thus, $M = R \otimes_{R^G} R$ satisfies the conditions of (3).

$(3) \Rightarrow (4)$ Let $f\colon R^{(M^G)} \to M$ be the map given by $f((r_m)_{m \in M^G}) = \sum_{m \in M^G} m r_m$. By Lemma 1, f is a homomorphism of right RG-modules, which is an epimorphism by the assumption $M = M^G R$. Therefore, f splits and hence M_{RG} is a direct summand of a direct sum of copies of R_{RG}. By Theorem 3, R_{R^G} is projective.

On the other hand, there exists an $N \in$ mod-RG, such that $M \oplus N \simeq RG^{(I)}$ for some set I. Then $M^G \oplus N^G \simeq ((RG)^G)^{(I)}$. If G is infinite, then $(RG)^G = 0$ and hence $M^G = 0$ which yields a contradiction with (3).

The implication $(4) \Rightarrow (2)$ of last Corollary does not hold if R has not an element of trace 1 as the following example shows, even if R_{RG} and ${}_{RG}$R are quasi-projective.

EXAMPLE 13. Let K be a field of characteristic 2 and F a Galois extension of K of order 2. Let $G = \langle \sigma \rangle$ the Galois group of F over K. Let $R = K \times F$ and $g \in \operatorname{Aut}(R)$ given by $(a, b)^g = (a, b^\sigma)$. Since $R^G = K \times K$ is semisimple, R_{R^G} is projective. But $(R \otimes_{R^G} R)_{RG} \simeq R^2$ is not projective because $1 \notin \operatorname{tr}(R)$.

Let R be any ring.r.gl.dim(R) (resp. w.dim(R)) will denote the right global (resp. weak) dimension of R. Recall that R is said to be right hereditary (resp.

semihereditary) if every (resp. finitely generated) right ideal of R is projective. Right hereditary rings are those which satisfies r.gl.dim$(R) \leqq 1$.

A ring R is said to be biregular if for every $r \in R$, there is a central idempotent $e \in R$, such that $RrR = Re$.

COROLLARY 14. *Let R be a ring and G a finite group of automorphisms of R. R is projective as right R^G-module if some of the following conditions hold:*

1. *$R\pi R$ is a projective as a right RG-module, where $\pi = \sum_{g \in G} g \in RG$.*
2. *RG is right hereditary.*
3. *R is right hereditary and R has a central element of trace* 1.
4. *RG is right semihereditary and R_{RG} is finitely generated.*
5. *R is right semihereditary, has a central element of trace* 1 *and R_{RG} is finitely generated.*
6. *R_{RG} is a generator.*
7. *RG is biregular.*

PROOF. (1) Let $M = R\pi R$. Then $M^G = R\pi = (RG)^G \simeq R_{RG}$ and $M = M^G R$. Thus, R_{RG} is projective, by Proposition 12.

(2) is a direct consequence of (1).

(3) is a direct consequence of (2) and [12, Proposition 2.3].

(4) By (1), it is enough to prove that $R\pi R_{RG}$ is finitely generated. Let X be a finite set of generators of R_{RG}. Let $r \in R$ and set $r = \sum_{x \in X} x a_x$ for some $a_x \in R^G$. Then $r\pi = \sum_{x \in X} x\pi a_x$. Therefore, $\{x\pi \mid x \in X\}$ is a finite set of generators of $R\pi R_{RG}$.

(5) The same argument used to prove [12, Proposition 2.3] proves that if R is semihereditary and R has an element of trace 1, then RG is semihereditary. Therefore, (5) is a consequence of (4).

(6) From Lemma 1 it is easy to deduce that if R_{RG} is a generator then $RG = (RG)^G R = R\pi R$. It only remains to apply (1).

(7) is a direct consequence of (1).

COROLLARY 15. *Let R be a ring and G a finitely generated group of automorphisms of R. The following conditions are sufficient for R_{RG} to be flat.*

1. *$R\hat{\pi}R$ is a flat as right RG-module, where $\hat{\pi} = \sum_{g \in G} g \in \widehat{RG}$.*
2. *G is finite and RG has weak dimension at most* 1.
3. *G is finite and R has weak dimension at most* 1 *and a central element of trace* 1.

PROOF. (1) Let $f\colon R^{(R)} \to R\hat{\pi}R$ be the map given by $f((r_s)_{s \in R}) = \sum_{s \in R} s\hat{\pi}r_s$. f is an epimorphism of right RG-modules. Since $R\hat{\pi}R_{RG}$ is flat f is a pure epimorphism. Moreover, by Lemma 4, R_{RG} is finitely presented. By [17, Theorem 7.34.2] $R\hat{\pi}R_{RG}$ is isomorphic to a direct limit of finite direct sums of copies R_{RG}. Then $M = R\hat{\pi}R$ satisfies the conditions of Theorem 5(4).

(2) Is a direct consequence of (1) because if G is finite, then $R\bar{\pi}R$ is a submodule of RG.

(3) Consider R as a G-graded ring by setting $R_g = Rg$, for all $g \in G$. If R has a central element x of trace 1, then $\{x^g \mid g \in G\}$ is a separability system of RG in the sense of [13]. Then, by [13, Proposition 3.5] and [4, Theorem 2.8], $\text{w.dim}(RG) = \text{gr.w.dim}(\text{RG}) = \text{w.dim}(\text{R}) \leqq 1$, where "gr.w.dim" means "graded weak dimension". It only remains to apply (2).

Examples and Comments.

Note that if $R\bar{\pi}R_{RG}$ is projective then R_{RG} is projective (the same proof as the given for Theorem 14(1) proves this fact). But this is included in Theorem 14(1). Indeed, in that case, as a consequence of Proposition 12, G has to be finite. Similarly, the proof of Corollary 14(6), does not use that G is finite, but it is not difficult to see that if R_{RG} is a generator, then G has to be finite.

S. Jøndrup [8] has proved that R_{RG} is projective if R is hereditary, G is finite with invertible order in R and one of the following conditions hold: R is semiprime and right noetherian; R is PI ring and is finitely generated algebra over a subring of its center; R is artinian algebra; R is commutative von Neumann regular; R is reduced von Neumann regular and finitely generated over its center; R is reduced von Neumann regular PI ring and the order of G is a power of 2; R is reduced von Neumann regular and G is solvable. Corollary 14(3) generalizes these results.

If the assumption R_{RG} to be finitely generated on (4) or (5) of Corollary 14 is removed, then R_{RG} could not be projective as Examples (4) and (5) on [8] shows. More examples of this kind can be found in [7] where there are examples of regular right self-injective rings R of arbitrary type with 2 invertible in R, which have a group of automorphisms G of order 2 and R_{RG} is not projective.

It worth to mention that if G is finite, RG is hereditary if and only if R is hereditary and $\text{r.gl.dim}(RG) < \infty$ (cf. [18, Lemma 2.2]). A recent characterization of when RG has finite right global dimension for R right FBN and left coherent appears in [18]. On the other hand, if G is infinite cyclic, then $\text{r.gl.dim}(R) \leqq \text{r.gl.dim}(RG) \leqq \text{r.gl.dim}(RG) + 1$.

The previous comment can be used, in combination with Corollary 14(2) or Corollary 15(2), to compute $\text{r.gl.dim}(RG)$ in some cases. For example, if R and G are as in Example 9 $\text{r.gl.dim}(RG) = \infty$, if K has non-zero characteristic, and $\text{r.gl.dim}(RG) = 2$, if K has characteristic 0. Because, otherwise RG would be hereditary and by Corollary 15, R_{RG} would be flat which is not the case.

Example 9 shows that the conditions R right hereditary and R_{RG} finitely generated does not implies projectivity nor even flatness on R_{RG}.

Assume that G is finite and RG is von Neumann regular. By Corollary 14, if

R_{RG} is finitely generated, then it is projective. If R is right self-injective as well, the converse holds by [7, Theorem 11]. Nevertheless, if R is not right self-injective the converse does not hold as the following example shows.

EXAMPLE 16. Let A be a commutative von Neumann regular ring and I a projective nonfinitely generated ideal of A. Let $R = \begin{pmatrix} A & I \\ I & A \end{pmatrix}$ and G the group of automorphisms of R generated by the inner automorphisms associated to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $R^G \simeq A$ and $R_{RG} \simeq A^2 \oplus I^2$ is projective but not finitely generated because I_A is projective but not finitely generated.

Recall that a ring extension A/B is called Frobenius if A_B is finitely generated and projective and $A \simeq \operatorname{Hom}(A_B, B_B)$ as B-A-bimodules. Moreover, A/B is a Frobenius extension if and only if there is a homomorphism of B-B-modules $h: A \to B$, and elements $r_1, \ldots, r_n, s_1, s_2, \ldots, s_n \in A$, such that $a = \sum_{i=1}^n r_i h(s_i a) = \sum_{i=1}^n h(ar_i)s_i$, for every $a \in A$.

Kitamura [9] has proved that if G is finite, RG is biregular and $1 \in \operatorname{tr}(R)$, then R/R^G is a Frobenius extension. Corollary 14(7) shows that the condition $1 \in \operatorname{tr}(R)$, in Kitamura's result is not needed, to deduce that R_{RG} is projective. Next Proposition shows that $1 \in \operatorname{tr}(R)$ is not needed at all.

PROPOSITION 17. *Let G a finite group of automorphism of a ring such that RG is biregular. Then R/R^G is a Frobenius extension.*

PROOF. Let e be a central element of RG such that $R\pi R = eRG$. Set $e = \sum_{i=1}^n r_i \pi s_i = \sum_{i=1}^n \sum_{g\in G} r_i s_i^{g^{-1}} = \sum_{i=1}^n \sum_{g\in G} g r_i^g s_i$. Let $r \in R$. Then

$$r\pi = er\pi = \sum_{i=1}^n \sum_{g\in G} \sum_{h\in G} r_i s_i^{g^{-1}} grh = \sum_{i=1}^n \sum_{g\in G} \sum_{h\in G} r_i s_i^{g^{-1}} r^{g^{-1}} gh =$$

$$= \sum_{i=1}^n r_i \sum_{\sigma\in G} \sum_{\tau\in G} (s_i r)^\tau \sigma = \sum_{i=1}^n r_i \operatorname{tr}(s_i r)\pi$$

Therefore, $r = \sum_{i=1}^n r_i \operatorname{tr}(s_i r)$. Similarly one prove that $r = \sum_{i=1}^n \operatorname{tr}(rr_i)s_i$.

We finish with an example which shows that we can not avoid to assume that G is finitely generated in Theorem 5 and Corollary 15.

EXAMPLE 18. An infinitely generated group G of automorphisms of a ring R such that, RG is von Neumann regular but R_{RG} is not flat.

Let K be a countable field of characteristic different from 2. Let us denote by N_0 the set of nonnegative integers and by N the set of positive integers. Let A be the ring of square matrices indexed by N_0 with entries in K which has only finitely many nonzero entries in each row. For every $n \in N$, let α_n be the inner automorphism of A defined by $a_n = 1 + e_n \in A$ where e_n is the matrix having 1 at the

$(0, n)$th entry and 0 elsewhere. Let H be the group of automorphisms of A generated by $\{\alpha_n \mid n \in N\}$. It is not hard to see that $A^H = K1 + \sum_{n \in N} Ke_n$.

Let A^N be the product ring. The nth entry of an $a \in A^N$ will be denoted by a_n.

Now we are going to give a set of automorphisms of A^N. To do so we are going to introduce some notation.

Let G be any group and G^0 the monoid obtained by adjoining a zero to G. That is, $G^0 = G \cup \{0\}$, where 0 is a symbol not representing any element of G and the multiplication in G^0 extends the multiplication in G by defining $a0 = 0a = 0$ for all $a \in G$.

Let X be any set. For every family $\{a_x \mid x \in X\}$ of elements of G^0 such that $a_x \neq 0$ for at most one $x \in X$, we set $\sum_{x \in X} a_x = 0$ if $a_x = 0$ for all $x \in X$ and $\sum_{x \in X} a_x = a_x$ if $a_x \neq 0$. Let S be the set of matrices indexed by X with entries in G^0 which have at most one nonzero entry in each column. The (x, y)th entry of an element $\alpha \in S$ will be denoted by $\alpha(x, y)$. Then S become a monoid with the standard product of matrices; that is, the product of two elements $\alpha, \beta \in S$ is defined by $(\alpha, \beta)(x, y) = \sum_{z \in X} \alpha(x, z)\beta(z, y)$. We will denote that monoid by $M_X(G)$.

Let $S = M_N(\mathrm{Aut}(A))$. For every $s \in S$, the map $M_s \colon A^N \to A^N$ given by $M_s(r)_n = \sum_{m \in N} r_m s(m, n)$, is a ring endomorphism of A^N. Furthermore, the mapping $s \to M_s$ is a homomorphism of monoids from S to the monoid of ring endomorphisms of A^N. In particular, M restricts to a group homomorphism from the subgroup $U(S)$ of units of S to $\mathrm{Aut}(A^N)$. It is not hard to see that $U(S)$ is formed by the elements in S having exactly one nonzero entry in each row and each column. This notation has the advantage that we can give automorphisms of A^N by giving matrices so that the action of those automorphisms can be represented by standard matrix multiplications and the composition of two of those automorphisms can be computed by a standard matrix product. In the remainder this fact will be used often without specifically refering to it. Some computations are left to the reader.

For a $\beta \in \mathrm{Aut}(A)$, let βI_n denote the $n \times n$ matrix having β at every entry of the diagonal and zeroes elsewhere

For every $n \in N$, let $s_n \in U(S)$ be the matrix which has a block decomposition as follows

$$s_n = \begin{pmatrix} A_n & 0 & \dots \\ 0 & A_n & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

here $A_n = \begin{pmatrix} 0 & \alpha_n I_{2^{n-1}} \\ \alpha_n^{-1} I_{2^{n-1}} & 0 \end{pmatrix} \in M_{2^n}(\mathrm{Aut}(A))$. By using standard matrix multiplications it is not hard to see that $s_n s_m = s_m s_n$ and $s_n^2 = 1$ for all $n, m \in N$. Therefore the group G of automorphisms of A^N generated by $\{g_n = M_{s_n} \mid n \in N\}$ is commutative and $g^2 = 1$ for all $g \in G$. Thus, G is locally finite and the order of every finite subgroup of G is a power of 2.

Let R be the subring of A^N formed by the elements $r \in A^N$, such that there exist two numbers n, m, such that $r_k = r_{k+2^m}$ for all $k \geqq n$. It is not hard to see that R is G-invariant. Thus, we can consider G as a group of automorphisms of R. By using the fact that A is von Neumann regular we deduce that R is von Neumann regular too. Therefore, RG is von Neumann regular (cf. [1]).

It only remains to prove that R_{RG} is not flat.

Let $r \in R^G$. Let n, m be numbers such that $r_k = r_{k+2^m}$ for all $k \geqq n$. Then, for every $l \geqq m$ and every $k \geqq n$, $r_k = (r^{g_l})_k = (r_{k+2^l})^{\alpha_l} = (r_k)^{\alpha_l}$. In other words, $r_k \in A^{H_m}$ for every $k \geqq n$, being H_m the group of automorphisms of A generated by $\{\alpha_m, \alpha_{m+1}, \dots\}$. But, if $k \in N$, there exists an $l \geqq m$, such that $k + 2^l \geqq n$ and hence $r_k = (r_{k+2^l})^{\alpha_n} = r_{k+2^l}$. In other words, the sequence r is periodic of period 2^m and its entries belong to A^{H_m}. Furthermore, if $k - 1 = a_0 + a_1 2 + a_2 2^2 + \dots$, with $a_i \in \{0, 1\}$, then $r_k = (r_1)^{\alpha_1^{-a_0}\alpha_2^{-a_1}\cdots}$. For every $k \in N$, let $h_k = \alpha_1^{-a_0}\alpha_2^{-a_1}\dots$ where $k - 1 = a_0 + a_1 2 + a_2 2^2 + \dots$, with $a_i \in \{0, 1\}$. We conclude that

$$R^G = \{r \in R \mid \text{there exists an } m \in N_0, \text{such that } r_1 \in A^{H_m} \text{ and } r_k = (r_1)^{h_k} \text{ for all } k \in N\}$$

For any $m \in N$ let $R_m = \{r \in R \mid r_1 \in A^{H_m}$ and $r_k = (r_1)^{h_k}$ for all $k \in N\}$. Thus $R^G = \varinjlim R_m$ where m runs in N. But the map f_m: $A^{H_m} \to R_m$ given by $f_m(a)_k = a^{h_k}$ is a ring isomorphism and $f_m(A^H) = R_1$. By straightforward computations one has that $A^{H_m} = \oplus_{k=0}^{2m-1} x_k A^G$ where x_k is the matrix having 1 at the $(k, 0)$-entry and 0 elsewhere. Furthermore, $x_k A^G \simeq A^G$ as a right R^G-module. Therefore, A^{H_m} is free as a right A^G-module and, hence R_m is free as a right R_1-module. Thus, R^G is flat as a right R_1-module. If R_{R^G} were flat then R_{R_1} will be flat too and we are going to see that this is not true.

The claim is equivalent to see that R_{A^H} is not flat. But $\{r \in R \mid r_n = 0$ for all $n \neq 0\}$ is isomorphic to a direct summand of R_{A^H} which is isomorphic to A as a right A^H-module. Thus, it is enough to see that A_{A^H} is not flat. Furthermore, A is isomorphic to a countable direct product of copies of A^H. Recall that a ring S is said to be left coherent if every finitely generated right ideal of S is finitely presented. It is well known that S is coherent if and only a direct product of $|S|$ copies of S is flat as right S-module [2, page 243]. Thus, since A^H is countable, if A_{A^H} were flat, then A^H would be coherent (note that A^H is commutative) and this is not the case because the annihilator of e_1 in A^H is not finitely generated.

Remark. In last example we have assumed that the field K has characteristic different from 2 to be able to check that the conditions of [1, Proposition 1.1] hold. Actually this assumption is not needed because, with some more work it is possible to check that the conditions of [1, Proposition 2.9] hold without any assumption on the characteristic of K.

REFERENCES

1. R. Alfaro, P. Ara and A. del Río, *Regular skew group rings*, to appear in J. Austral. Math. Soc.
2. F. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer Verlag, Berlin. 1974.
3. M. Auslander, I. Reiten and S. O. Smalø, *Galois actions on rings and finite Galois coverings*, Math. Scand. 65 (1989), 5–32.
4. E. D. Dade, *Group-graded rings and modules*, Math. Z. 174 (1980), 241–262.
5. J. J. García and A. del Río, *Group acting on fully bounded noetherians rings*, Comm. Algebra 22 (1994), 1495–1505.
6. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London 1079.
7. D. Handelman and G. Renault, *Actions of groups on self-injective rings*, Pacific J. Math. 89 (1980), 60–80.
8. S. Jøndrup, *When is the ring a projective module over the fixed point ring?*, Comm. Algebra, 16 (1988), 1971–1992.
9. Y. Kitamura, *A note on fixed rings*, Math. J. Okayama Univ. 31 (1989), 85–91.
10. M. Lorenz, *On the global dimension of fixed rings*, Proc. Amer. Math. Soc. 106 (4) (1989), 923–932.
11. S. Montgomery, *Fixed Rings of Finite Automorphism Groups of Associative Rings*, Lect. Notes in Math. 818, (1980).
12. E. Nauwellaerts and F. Van Oystaeyen, *Finite generalized crossed products over tame and maximal orders*, J. Algebra 101 (1986), 61–68.
13. M. D. Rafael, *Separable functors revisited*, Comm. Algebra, 18 (1990), 1445–1459.
14. M. Sato, *Fuller's Theorem on Equivalences*, J. Algebra 52 (1979), 412–420.
15. B. Stenström, *Rings of Quotients*, Springer Verlag, Berlin, 1975.
16. B. Wilke, *Zur Modelstruktur von Ringen über Schiefgruppenringen*, Ph.D. Thesis.. Düsseldorf, 1994.
17. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Sc. Publ. 1991.
18. Zhong Yi, *Homological dimension of skew group rings and crossed products*, to appear in J. Algebra.

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE MURCIA
30071 MURCIA
SPAIN
email adelrio@fcu.um.es