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ON KLEIN SURFACES AND DIHEDRAL GROUPS

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1. Introduction.

In this paper we study the following problem. Given an NEC group Γ and the dihedral group D_p, with p a prime, how many non conjugate normal subgroups of Γ has D_p as quotient group? This is equivalent to asking how many non-biconformally equivalent Klein surfaces that are coverings of the orbifold whose fundamental group is Γ admit D_p as a group of automorphisms. A related question is the classification of actions of D_p on a Riemann surface. Natanzon [9] gives a classification for D_2-actions on Riemann surfaces.

This paper is a generalization to NEC groups of the paper of Lloyd [6].

2. Klein surfaces and their groups.

Let H be the hyperbolic plane. A *non-Euclidean crysstallographic* NEC *group* is a discrete subgroup Γ of automorphisms of H (Iso H) with compact quotient space (2-orbifold). Equivalently, Γ is a group that acts properly discontinuously on H. If Γ is an NEC group containing orientation-reversing elements, then Γ is called a *proper* NEC group; otherwise Γ is called a *Fuchsian* group and is a subgroup of $Iso^+ H$, where $\mathrm{Iso}^+ H$ denotes the subgroup of Iso H formed by the orientation-preserving automorphisms of H. If Γ is a proper NEC group then $\Gamma \cap \mathrm{Iso}^+ H = \Gamma^+$ is a Fuchsian group called the *canonical Fuchsian group* of Γ.

DEFINITION ([8]). A *2-orbifold M* is a connected Hausdorf space which admits a *folding atlas* $\mathscr{A}$ formed by *folding charts* (U_i, ϕ_i, G_i, A_i) where U_i is an open subset of C, G_i is a finite group, and the mapping $\phi_i\colon A_i \to U_i$ is such that $A_i/G_i \approx U_i$, with the following compatibility condition that for all $x \in A_i$ and $y \in A_j$ such that $\phi_i(x) = \phi_j(y)$, there exist open subsets V_i of A_i and V_j of A_j, and a diffeomorphism $\phi\colon V_i \to V_j$ such that $\phi_j = \phi_j \phi$.

Moreover, if $p \in M$, $p = \phi_i(x)$, $x \in A_i$, then the group $\mathrm{Stb}(p) = \{g \in G_i; xg = x\}$

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depends only on p and is independent of the choice of x or U_i, so we can distinguish the following points on the 2-orbifold M.
p is a *regular point* if $\mathrm{Stb}(p) = I_d$,
p is a *cone point* if $\mathrm{Stb}(p)$ is a cyclic rotation group of order n,
p belongs to a *mirror line* if $\mathrm{Stb}(p)$ is a cyclic rotation group of order 2 generated by one reflection,
p is a *corner point* if if $\mathrm{Stb}(p)$ is a dihedral group of order $2n$ generated by 2 reflections.
The above number n is called the *order of p*.

DEFINITION ([8]). Let M, N be 2-orbifolds and let $h: N \to M$ be a continuous onto mapping. h is called an *(orbifold-)covering* if there exists a folding atlas $\mathscr{A} = \{(U_i, \phi_i, G_i, A_i)\}$ for M such that for every connected component V of $h^{-1}U_i$ there exists a folding chart $f_i: A_i \to V$ in the maximal atlas of N such that $hf_i = \phi_i$.

EXAMPLE. Let S be a Riemann surface, S has a 2-orbifold structure whose points are all regular. If now G is a group acting properly discontinuously on S, then the quotient space S/G has orbifold structure and the projection map $\pi: S \to S/G$ is an (orbifold-)covering.

A surface S with boundary is a 2-orbifold without cone or corner points. The connected components of the boundary correspond to the mirror lines of the 2-orbifold.

A 2-orbifold M is called a *good 2-orbifold* if M has a covering which is a surface. We denote by $\mathscr{U}$ the universal covering of the orbifold M ($\mathscr{U}$ is either the 2-sphere S^2, the Euclidean plane E^2 or the hyperbolic plane H). A good 2-orbifold M is the quotient of $\mathscr{U}$ by a group Γ acting properly discontinuously on $\mathscr{U}$. The group Γ is called the *fundamental group of M* and we write $\Gamma = \pi(M)$ since, if M is a closed surface, then Γ is the fundamental group of M. Notice that the fundamental group, as a 2-orbifold, of a surface M with boundary is not the fundamental group of M as a 2-manifold.

EXAMPLE. Let $M = H/\Gamma$ be a good hyperbolic 2-orbifold. The 2-sheeted covering M^+ of M given by H/Γ^+, where Γ^+ is the canonical Fuchsian subgroup of Γ, is called *the complex double* of M.

The algebraic structure of Γ or, equivalently, the geometrical structure of the quotient 2-orbifold $M = H/\Gamma$ is determined by the signature:

$$(2.1) \quad s(M) = s(\Gamma) = (g; \pm; [m_1, \dots m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{p1}, \dots, n_{ps_p})\}$$

where H/Γ is a 2-orbifold lying on a compact surface of genus g and having p mirror lines. If the orbifold is orientable we use the sign $+$, and the sign $-$ if it is non-orientable. The integers $m_1, \dots, m_r$ are called the *periods* of $s(\Gamma)$: these are the orders the cone points of H/Γ. The bracket $(n_{k1}, \dots, n_{ks_k})$ is called the kth *period* *cycle*, it is associated to the kth mirror line of H/Γ. The integers $n_{k1}, \ldots, n_{ks_k}$, called *link periods*, are the orders of the corner points on the kth mirror line.

Associated with the signature of a 2-orbifold $M = H/\Gamma$ is a presentation for the NEC group Γ with:

GENERATORS: $x_1, \ldots, x_r, c_{k,0}, \ldots, c_{k,s_k}, \quad 1 \leqq k \leqq p,$
(If H/Γ is non-orientable) ((1) a_h, $1 \leqq h \leqq g$,
(If H/Γ is orientable) (2) a_h, b_h, $1 \leqq h \leqq g$.

RELATORS: $x_i^{m_i}$, $1 \leqq i \leqq r$, $c_{k,j}^2$, $1 \leqq k \leqq p$, $0 \leqq j \leqq s_k$,
$(c_{k,j-1}c_{k,j})^{n_{kj}}$, $1 \leqq k \leqq p$, $1 \leqq j \leqq s_k$, $c_{k,0} = e_k^{-1}c_{k,s_k}e_k$, $1 \leqq k \leqq p$,

$$\text{(2.2)} \qquad (1) \prod_{i=1}^{r} x_i \prod_{k=1}^{p} e_k \prod_{h=1}^{g} a_h^2 \quad \text{or} \quad (2) \prod_{i=1}^{r} x_i \prod_{k=1}^{p} e_k \prod_{h=1}^{g} [a_h, b_h].$$

(Where $[a_h, b_h]$ is the commutator of a_h and b_h)

Let Γ be an NEC group. Γ is called a *surface group* if $s(\Gamma) = (g, \pm, [-], \{\{-\})$, and called a *bordered surface group* if $s(\Gamma) = (g, \pm, [-], \{(-), \ldots, (-)\})$. Shortening we say that Γ is a surface group in both cases.

Every good 2-orbifold $M = \mathscr{U}/\Gamma$ admits a finite-sheeted (orbifold-)covering S that is a surface. This allows us to generalize the Euler characteristic to a 2-orbifold M:

$$\chi(M) = 2 - \alpha g - p - \sum_i \left(1 - \frac{1}{m_i}\right) - \tfrac{1}{2}\sum_{k,j}\left(1 - \frac{1}{n_{kj}}\right),$$

where $\alpha = 1$ if $\mathscr{U}/\Gamma$ is non-orientable and $\alpha = 2$ if $\mathscr{U}/\Gamma$ is orientable.

Let $\mathscr{U}$ be the universal covering of a 2-orbifold M. Then $\mathscr{U} = S^2$ if $\chi(M) < 0$, $\mathscr{U} = E^2$ if $\chi(M) = 0$ and $\mathscr{U} = H$ if $\chi(M) > 0$.

The hyperbolic area $\mu(M)$ of a 2-orbifold $M = H/\Gamma$ depends only on $s(\Gamma)$. It is calculated by the Gauss-Bonnet formula:

$$\text{(2.3)} \qquad \mu(\Gamma) = \mu(M) = -2\pi\chi(M),$$

Let Γ be an NEC group with quotient orbifold M. If Γ' is a subgroup of Γ of finite index $[\Gamma : \Gamma'] = n$ in Γ, then Γ' is an NEC group whose quotient orbifold M' is an n-sheeted covering of M, whose monodromy map is the representation of the action of Γ on the Γ-cosets. We have the Riemann-Hurwitz formula:

$$\text{(2.4)} \qquad \mu(\Gamma') = [\Gamma : \Gamma']\mu(\Gamma).$$

Let Γ be an NEC group with quotient 2-orbifold $M = H/\Gamma$. Then $M^+ = H/\Gamma^+$ is a 2-orbifold without mirror lines called the *complex double* of M. The *algebraic genus* g_a of M is defined to be the topological genus of its complex double, i.e.

$$g_a = \alpha g + k - 1, \tag{2.5}$$

where $\alpha = 1$ if M is non-orientable and $\alpha = 2$ if M is orientable, g is the genus of M and k is the number of mirror lines in M.

Let G be a finite group. If G acts faithfully on a Klein surface S as a group of automorphisms, where $S \cong H/\Lambda$, then $G \cong \Gamma/\Lambda$, where Λ is a normal subgroup of an NEC group Γ. Now, the following are equivalent:

a) The number $\phi(\Gamma, G)$ of normal surface subgroups Λ of an NEC group Γ such that $\Gamma/\Lambda \cong G$.

We say that an epimorphism $\phi: \Gamma \to G$ is a *surface kernel* epimorphism if Ker ϕ is a group without other elliptic elements than reflections, i.e. $H/\text{Ker}\,\phi$ is a surface. Then the number $\phi(\Gamma, G)$ given in a) yields us

b) the number of non-equivalent surface kernel epimorphisms from Γ onto G.

If we consider that two orbifold-coverings of the orbifold $M = H/\Gamma$ are isomorphic if and only if their fundamental groups are conjugate in Γ (the fundamental group of M) [8], then we can express a) or b) with geometrical words as follows:

c) the number $\phi(\Gamma, G)$ of non equivalent surfaces that are regular (orbifold-)coverings of H/Γ, where G is the group of transformations of the covering.

After the following lemma, our aim is to calculating the above number $\phi(\Gamma, G)$ when G is a dihedral group D_p, with p a prime number. In this case all link-periods in Γ must be equal to the prime p. This is the natural generalization to NEC groups of [6].

LEMMA 2.1. *If Γ is a proper* NEC *group with some link-period of odd order p, then G does not admit a cyclic group C_p as a quotient group by a surface group.*

PROOF. Let $c_{k,j-1}$ and $c_{k,j}$ be the reflections associated to the odd link-period n_{kj}. Let $\phi: \Gamma \to C_p$ be an epimorphism from Γ to C_p, where $C_p = \langle u/u^p = 1_d \rangle$, As $c_{k,j-1}$ and $c_{k,j}$ are elements of order 2 in Γ, both $c_{k,j-1}$ and $c_{k,j}$ must be in Ker ϕ. Then $c_{k,j-1}c_{k,j}$ is an elliptic element in Ker ϕ and Ker ϕ is not a surface group.

Lemma 2.1 does not apply for $p = 2$.

LEMMA 2.2. *Let Γ be an* NEC *group with $s(\Gamma) = (0, +, [-], \{(2,\dots,2),\dots, (2,\dots,2)\})$, where the r period cycles are of even length. Then there are 2^{2r-1} non equivalent surface coverings of $\mathcal{U}/\Gamma$ which admit C_2 as a group of automorphisms.*

NOTE. $\mathcal{U} = S^2$ if $s(\Gamma) = (0, +, [-], \{(2,2)\})$ and $\mathcal{U} = E^2$ if $s(\Gamma) = (0, +, [-], \{(2,2,2,2)\})$.

PROOF. Let us write $C_2 = \{1_d, u\}$. As Aut C_2 is trivial, to prove the lemma we must count $\phi(\Gamma, C_2)$, i.e. in how many ways can we define $\phi: \Gamma \to C_2$ on the generators $c_{k,j}$ and e_k of Γ such that ϕ is a surface kernel epimorphism?

Now, we can define ϕ for the generators of each period cycle in two ways:

i) $\phi(c_{k,2j}) = 1_d = \phi(c_{k,s_k})$, $\phi(c_{k,2j+1}) = u$, $\phi(e_k) = 1_d$ or $\phi(e_k) = u$, $1 \leqq j \leqq (s_k - 2)/2$, or

ii) $\phi(c_{k,2j}) = u = \phi(c_{k,s_k})$, $\phi(c_{k,2j+1}) = 1$, $\phi(e_k) = 1_d$ or $\phi(e_k) = u$, $1 \leqq j \leqq (s_k - 2)/2$, but $\phi(e_k)$ is given by the condition $\prod \phi(e_k) = 1_d$. So $\phi(\Gamma, C_2) = 2, 4^{r-1} = 2^{2r-1}$.

3. Generating epimorphisms $\phi: \Gamma \to D_p$, with p an odd prime.

We are only interested in those NEC groups Γ whose quotient orbifold lies on the sphere and without conic points. So $s(\Gamma) = (0, +, [-], \{C_k\}_{k=1,r})$, where each period cycle C_k have s_k link-periods equal to p and a presentation for Γ is the following:

$$\Gamma = \langle e_k, c_{k,j}, k = 1, r, r, j = 0, s_k / c_{k,j}^2, (c_{k,j-1}c_{k,j})^p, c_{k,0}e_k^{-1}c_{k,s}e_k, \prod e_k \rangle.$$

$D_p = \langle c, u / c^2 . u^p, cucu \rangle$. If p is a prime distinct from 2, there is one conjugacy class of elements of order 2 and one conjugacy class of elements of order p. If $p = 2$, there are 3 conjugacy classes of elements of order 2, namely $\{c\}$, $\{u\}$ and $\{cu\}$.

We consider in this paragraph the cases when p is an odd prime.

If $\phi: \Gamma \to D_p$ is a surface kernel epimorphism, then $o(\phi(c_{k,j}))$ divides 2 ($o(z)$ denotes the order of an element z in D_p), $\phi(c_{k,j-1}c_{k,j}) = u^y$, for some $y \in \{1, .. p-1\}$ and $\prod \phi(e_k) = 1_d$. So the conditions to be satisfied by the images of the generators of G are:

(3.1) i) $g_{kj} = \phi(c_{k,j-1}) = cu^y$, $y \in \{0, .. p-1\}$, $\phi(c_{k,j}) = cu^x$, where $x \neq y$,

(3.2) ii) $g_k = \phi(e_k)$ is such that $\phi(c_{k,0})(\phi(e_k))^{-1}\phi(c_{k,s})\phi(e_k) = 1_d$ and $\prod \phi(e_k) = 1_d$.

Since a set $G_\phi = \{g_{kj}, g_k / \text{satisfying (3.1) and (3.2)}\}$ is a generating set for D_p, the number $\phi(\Gamma, D_p)$ is the number of orbits of the sets G_ϕ under the action of Aut D_p. But Aut $D_p = C_p \rtimes C_{p-1}$, so $|\text{Aut } D_p| = p(p-1)$.

THEOREM 3.1. *Let Γ be an NEC group with $s(\Gamma) = (0, +, [-], \{(p, \ldots, p)\})$, where there are $s \geqq 3$ link-periods. Then there are* $\phi(\Gamma, D_p) = (p-1)^{s-2} - \sum_{j=0}^{s-3} \binom{s-2}{j} p^{s-3-j}(-1)^j$ *non equivalent surface coverings of $\mathscr{U}/\Gamma$ and the surfaces admit D_p as a group of automorphisms.*

Again, $\mathscr{U} = E^2$ if $s(\Gamma) = (0, +, [-], \{(3, 3, 3)\})$, otherwise $\mathscr{U} = H$.

PROOF. We begin by calculating the number Q_s of different sets $G_\phi = \{\phi(c_i), 0 \leqq i \leqq s-1\}$, where $c_i, 0 \leqq i \leqq s$ are the generating reflections of Γ. Notice that $c_0 = c_s$.

For $s = 3$, we have that $\phi(c_0) = cu^x$, $\phi(c_1) = cu^y$, with $y \neq x$ and $\phi(c_2) = cu^z$, with $z \neq x$ and $z \neq y$. So there are $Q_3 = p(p-1)(p-2)$ different sets.

For $s = 4$, we have $\phi(c_0)$, $\phi(c_1)$ and $\phi(c_2) = cu^z$ as for $s = 3$. For $\phi(c_3)$, we have, respectively, $p-1$ choices if $z = x$ or $p-2$ if $z \neq x$. So $Q_4 = p(p-1)[(p-1) + (p-2)^2]$.

For $s \geqq 5$, we have $p-2$ choices for $\phi(c_{s-1})$ if $\phi(c_{s-2})$ is distinct from $\phi(c_0)$ and $p-1$ choices if $\phi(c_{s-2})$ is equal to $\phi(c_0)$. But $\phi(c_{s-2})$ is equal to $\phi(c_0)$ if $\phi(c_{s-3})$ is distinct from $\phi(c_0)$. So Q_s satisfies the equation:

$$Q_s = (p-2)Q_{s-1} + (p-1)Q_{s-2}. \tag{3.3}$$

Equation (3.3) is a homogeneous difference equation with characteristic polynomial

$$r^2 - (p-2)r - (p-1). \tag{3.4}$$

The zeros of (3.4) are $p-1$ and -1. So, the general solution is $Q_s = (p-1)^s A + (-1)^s B$. Using Q_3 and Q_4 to determine the constants A and B, we get $A = 1$, $B = p-1$. So

$$Q_s = (p-1)^s + (-1)^s(p-1) \tag{3.5}$$

i.e. $Q_s = [(p-1)^{s-1} + (-1)^s](p-1)$, where $(p-1)^{s-1} + (-1)^s = p \sum_{k=0}^{s-2} \binom{s-1}{k} p^{s-2-k}(-1)^k$, so

$$Q_s = (p-1)p \sum_{k=0}^{s-2} \binom{s-1}{k} p^{s-2-k}(-1)^k =$$

$$= \left[(p-1)^{s-2} - \sum_{j=0}^{s-3} \binom{s-2}{j} p^{s-3-j}(-1)^j\right](p-1)p.$$

$$\phi(\Gamma, D_p) = \frac{Q_s}{p(p-1)} = (p-1)^{s-2} - \sum_{j=0}^{s-3} \binom{s-2}{j} p^{s-3-j}(-1)^j.$$

NOTE 1. By the Riemann-Hurwitz formula and using the representation of D_p as permutation group, there are different biconformal structures on a non-orientable surface or orientable surface of genus $(s-2)(p-1)$ or $(s-2)(p-1)/2$ respectively without boundary components. The subgroups Λ of Γ associated to them have signature $s(\Lambda) = ((s-2)(p-1), -, [-], \{-\})$ or $s(\Lambda) = ((s-2)(p-1)/2, +, [-], \{-\})$.

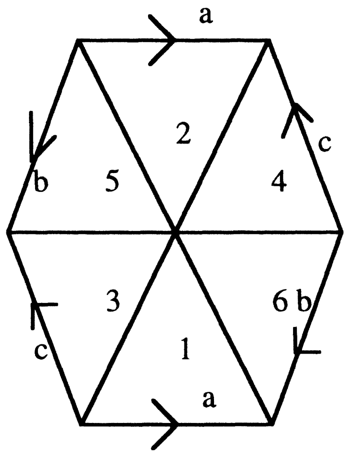

Figure 1.

The minimal genus surfaces with D_p as a group of automorphism occur when Γ is an NEC group with signature $s(\Gamma) = (0, +, [-], \{(p, p, p)\})$. The number of such non equivalent surface coverings is $p - 2$ according to theorem 3.1. In particular, there is a unique biconformal structure for a torus admitting D_3 as a group of automorphisms. A fundamental region for this torus is shown in figure 1. Its fundamental group Λ is a normal subgroup of an NEC group with signature $s(\Gamma) = (0, +, [-], \{(3, 3, 3)\})$ and with the following permutation representation: $\phi: \Gamma \to \Sigma_6$ defined by $f(c_0) = (1,2)(3,4)(5,6)$, $f(c_1) = (1,3)(2,5)(4,6)$, $f(c_2) = (1,6)(3,5)(2,4)$. Notice that $(1,2)(3,4)(5,6)$, and $(1,5,4)(2,3,6) = f(c_0)f(c_1)$ generate D_3. The map ϕ is also the monodromy map of the (orbifold-)covering $F: E^2/\Lambda \to E^2/\Gamma$.

NOTE 2. We can extrapolate to $Q_2 = p(p-1)$, $Q_1 = 0$, $Q_0 = p$. They have geometrical interpretation. For instance, the signature $s(\Gamma) = (0, +, [-], \{(p)\})$ is not admissible since the orbifold lying on a disc with one corner point is not a good orbifold.

With the same calculations as in theorem 3.1:

COROLLARY 3.1. *Let Γ be an* NEC *group with* $s(\Gamma) = (0, +, [-], \{(p, \ldots, p), \ldots, (p, \ldots, p)\})$, *where at least one period cycle has* 2 *or more link-periods and all cycles are non empty. The number $\phi(\Gamma, D_p)$ of non-equivalent epimorphisms from Γ onto D_p is $p^{r-2}(p-1)^{s-1}$, where s is the number of link-periods and r is the number of period cycles in* $s(\Gamma)$.

PROOF. We assume that the last period cycle has more than 1 link-period. Let s_i be the number of link-periods in the i-th cycle, with $\sum s_i = s$. Now $\phi(c_{i,s_i})$ is conjugate, but not necessarily equal to $\phi(c_{i,0})$. So for all cycles except the last one the number of choices of $\{\phi(c_{i,j}), 0 \leqq j \leqq s, 1 \leqq i \leqq r-1\}$ are $Q'_i = p(p-1)^{s_i}$.

To calculate the number of choices for the last period cycle we must distinguish two cases:

a) If $\phi(e_r) = 1$ from the relator $\prod \phi(e_i) = 1$, then $\phi(c_{i,0}) = \phi(c_{i,s_i})$ and $Q'_r = Q_{s_r}$, where $Q_s = (p-1)p \sum_{k=0}^{s-2} \binom{s-1}{k} p^{s-2-k}(-1)^k$ is given in theorem 3.1.

b) If $\phi(e_r) \neq 1$, then $\phi(e_r) = u^z$, with $z \neq 0$. If $\phi(c_{r,0}) = cu^y$, then $\phi(c_{r,s_r}) = cu^x$, where x satisfies the equation $x - y = -2z \bmod (p)$.

If $s_r = 2$, then $Q''_r = p(p-2)$, $(\phi(c_{r,0}) = cu^y, \phi(c_{r,2}) = cu^x, \phi(c_{r,1}) = cu^{x'}$, with x' distinct from x and $y)$

if $s_r = 3$, then $Q''_r = p(p-1) + p(p-2)^2$,

For $s_r \geqq 4$, we have $p-2$ choices for $\phi(c_{r,s_r-2})$ is distinct from $\phi(c_{r,s_r})$ and $p-1$ choices if $\phi(c_{r,s_r-2})$ is equal to $\phi(c_{r,s_r})$. But $\phi(c_{r,s_r-2})$ is equal to $\phi(c_{r,s_r})$ if $\phi(c_{r,s_r-3})$ is equal to $\phi(c_{r,s_r})$. So Q''_r satisfies the equation (3.3) with characteristic polynomial (3.4). The general solution is $Q''_r = (p-1)^s A' + (-1)^s B'$. Using Q''_r for $s_r = 2$, $s_r = 3$ to determine the constants A' and B', we get $A' = 1$, $B' = -1$. So

$$Q''_r = (p-1)^{s_r} + (-1)^{s_r+1} = p \sum_{j=0}^{s_r-1} \binom{s_r}{j} p^{s_r-1-j}(-1)^j \tag{3.6}$$

So the number of choices for the last cycle is:

$$Q'_r = \frac{Q_{s_r}}{p} + \frac{(p-1)Q''_r}{p} = (p-1)\left[\sum_{j=0}^{s_r-2} \binom{s_r-1}{j} p^{s_r-2-j}(-1)^j + \right.$$

$$\left. + \sum_{j=0}^{s_r-1} \binom{s_r}{j} p^{s_r-1-j}(-1)^j\right] = (p-1)\left[\binom{s_r}{0} p^{s_r-1}(-1)^0 + \right.$$

$$\left. + \sum_{j=1}^{s_r-1} \binom{s_r-1}{j} p^{s_r-1-j}(-1)^j\right] = (p-1)(p-1)^{s_r-1} = (p-1)^{s_r},$$

and $\phi(\Gamma, D_p) = \dfrac{\Pi Q'_i}{p(p-1)} = \dfrac{p^{r-1}(p-1)^s}{p(p-1)} = p^{r-2}(p-1)^{s-1}$.

Corollary 3.2. *Let Γ be an* NEC *group with $s(\Gamma) = (0, +, [-], \{(p, \ldots, p), \ldots, (p, \ldots, p)\})$, with s link-periods in r period cycles, where at least one period cycle is empty. Then there are $\phi(\Gamma, D_p) = (p-1)^{s-1}p^{r-2}$ non equivalent surface coverings of $\mathcal{U}/\Gamma$ which admit D_p as a group of automorphisms.*

Proof. We can assume that the empty period cycles are the $r - r'$ last ones and each of the r' first period cycles has s_i link-periods, with $\sum s_i = s$. So for all non-empty period cycles the number of choices of $\{\phi c_{i,j}, 0 \leqq j \leqq s, 1 \leqq i \leqq r'\}$ are

$Q_i' = p(p-1)^{s_1}$. $\phi(e_r)$ is given as the commutator of $\phi(c_{i,0})$ and $\phi(c_{i,s_i})$. The number of choices for all empty period cycles except the last one is p.

Finally, $\phi(c_r)$ and $\phi(e_r)$ are fixed by the relators $\prod \phi(e_i) = 1_d$, $\phi(c_r)(\phi(e_r))^{-1}$ $\phi(c_r)\phi(e_r) = 1_d$. Therefore the number of sets G_ϕ is $p^{r-1}(p-1)^s$, and $\phi(\Gamma, D_p) = p^{r-2}(p-1)^{s-1}$.

REMARK. To calculate the number $\phi(\Gamma, D_p)$ of non equivalent surface coverings of H/Γ with D_p as the group of covering-transformations is slightly different from theorem 3.1 for the groups Γ when all period cycles have exactly one link-period. If Γ is such a group, then, as in note 2, $s(\Gamma) = (0, +, [-], \{(p), \ldots, (p)\})$, with at least 2 period cycles. The generators of Γ are $c_{i,0}$, $c_{i,1}$, e_i, $1 \leqq i \leqq r$, where r is the number of period cycles. To calculate the different sets $G_\phi = \{g_{kj}, g_k/\text{satisfying (3.1) and (3.2)}\}$, we must consider that $\phi(c_{i,0}) \neq \phi(c_{i,1})$, $1 \leqq i \leqq r$, so $\phi(e_i) \neq 1_d$.

The case when $r = 2$, $\phi(c_{1,0}) = cu^x$, $\phi(c_{1,1}) = cu^y$, with $y \neq x$, $\phi(e_1) = u^z$ is the commutator of cu^x and cu^y, so $z \neq 0$. We have choices for $\phi(c_{2,0})$, but $\phi(c_{2,1})$ and $\phi(e_2)$ are fixed. The number of sets G_ϕ is $I_2 = p^2(p-1)$, and $\phi(\Gamma, D_p) = p$.

The case when $r = 3$. $\phi(c_{1,0}) = cu^x$, $\phi(c_{1,1}) = cu^y$, with $y \neq x$, $\phi(e_1) = u^z$, where u^z is the commutator of cu^x and cu^y, so $z \neq 0$. To choose $\phi(e_2)$ and $\phi(e_3)$, we must satisfy the condition

(3.7) $$\phi(e_2)\phi(e_3) = u^{-z}, \text{ with } \phi(e_2) \neq 1_d \text{ and } \phi(e_3) \neq 1_d.$$

$\phi(c_{2,1})$ and $\phi(c_{3,1})$ are given by the condition $c_{i,0} = e_i^{-1} c_{i,1} e_i$.

Condition (3.7) is equivalent to the following: counting ordered pairs (z', z'') of numbers between 1 and $p-1$ such that $z' + z'' = -z \bmod (p)$.

There are $(p-2)$ such pairs, hence $I_3 = p^3(p-1)(p-2)$, and $\phi(\Gamma, D_p) = p^2(p-2)$.

If $r \geqq 4$, then $\phi(c_{i,0}) = cu^x$, $\phi(c_{i,1}) = cu^y$, with $y \neq x$, $\phi(e_i) = u^z$, $1 \leqq i \leqq r-2$, where u^z is the commutator of cu^x and cu^y, so $z \neq 0$. If $\prod_{i=1}^{r-2} e_i = 1_d$, then we do as in the case $r = 2$ for the two last cycles. If $\prod_{i=1}^{r-2} e_i \neq 1_d$, then we do as in the case $r = 3$ for the last cycles.

Hence $$I_r = p^{r-2}(p-1)^{r-2}p^2\left[\frac{(p-1)(p-2)}{p} + \frac{(p-1)}{p}\right] = p^{r-1}(p-1)^{r-1}$$ $(p-2+1) = p^{r-1}(p-1)^r$. Therefore $\phi(\Gamma, D_p) = p^{r-2}(p-1)^{r-1}$.

NOTE 3. For the NEC groups Γ in theorem 3.1 and corollaries 3.1, 3.2 and the previous remark, Ker ϕ is a normal surface subgroup of index $2p$ in Γ. This is just twice the minimal index for surfaces subgroups calculated in [3].

4. Surfaces with D_2 as a group of automorphisms.

If $p = 2$, there are 3 conjugacy classes of elements of order 2, namely $\{c\}$, $\{u\}$ and $\{cu\}$. If $\phi: \Gamma \to D_2$ is a surface kernel epimorphism, then $o(\phi(c_{k,j}))$ divides 2 ($o(z)$ denotes the order of an element z in D_2), $\phi(c_{k,j-1}c_{k,j})$ must be c, u or cu, and $\prod \phi(e_k) = 1_d$. So the conditions to be satisfied by the images of the generators of G are:

i) $g_{kj} = \phi(c_{k,j-1}) = z$, $z \in \{c, u, cu, 1_d\}$, $\phi(c_{k,j}) = z'$, $z' \in \{c, u, cu, 1_d\}$, where $z \neq z'$,

ii) $g_k = \phi(e_k)$ is such that $\phi(c_{k,0})(\phi(e_k))^{-1}\phi(c_{k,s})\phi(e_k) = 1_d$ and $\prod \phi(e_k) = 1_d$.

Since a set $G_\phi = \{g_{kj}, g_k/\text{satisfying i) and ii)}\}$ may generate D_2 or any of the 3 cyclic subgroups of order 2, we consider the sets $G'_\phi = \{g_{kj}, g_k/\text{satisfying i) and ii), generating } D_2\}$. The number $\phi(\Gamma, D_2)$ is the number of orbits of the sets G'_ϕ under the action of $\operatorname{Aut} D_2$. But $\operatorname{Aut} D_2 = S_3$, so $|\operatorname{Aut} D_2| = 6$.

LEMMA 4.1. *Let Γ be an NEC group with all link-periods equal to 2. If Γ has some period cycle with exactly one link-period, then D_2 is not a quotient group of Γ by a surface group.*

PROOF. Let $\phi: \Gamma \to D$ an epimorphism from Γ to D_2. Let C_k be the period cycle with exactly one link-period. Γ has, among others, the generators c_{k0}, c_{k1} and e_k with the relation $c_{k,0} = e_k^{-1}c_{k,1}e_k$. But the elements of order 2, $\phi(c_{k,0})$ and $\phi(c_{k,1})$, are conjugate in D_2 if and only if $\phi(c_{k,0}) = \phi(c_{k,1})$. So $\phi(c_{k,0}c_{k,1}) = 1$, and $\operatorname{Ker}\phi$ is not a surface group.

In the following, we consider signatures of NEC groups where there are at least two link-periods in each cycle.

THEOREM 4.1. *Let Γ be an* NEC *group with $s(\Gamma) = (0, +, [-], \{(2, \ldots, 2)\})$, where there are s link-periods all equal to 2. The number of non equivalent surface coverings of $\mathscr{U}/\Gamma$ which admit D_2 as a group of automorphisms is:*

a) $$\phi(\Gamma, D_2) = 2\sum_{j=0}^{s-2}\binom{s-1}{j}4^{s-2-j}(-1)^j \text{ if } s \text{ is odd, or}$$

b) $$\phi(\Gamma, D_2) = 2\sum_{j=0}^{s-2}\binom{s-1}{j}4^{s-2-j}(-1)^j - 1 \text{ if } s \text{ is even.}$$

NOTE. $\mathscr{U} = S^2$ if $s(\Gamma) = (0, +, [-], \{(2, 2)\})$ or $s(\Gamma) = (0, +, [-], \{(2, 2, 2)\})$, $\mathscr{U} = E^2$ if $s(\Gamma) = (0, +, [-], \{(2, 2, 2, 2)\})$.

PROOF. We must distinguish the cases when the number of link-periods s is odd or even.

a) s is odd. Then $G_\phi = G'_\phi$. The number Q_s of different sets $G_\phi = \{\phi(c_i),$

$0 \leqq i \leqq s-1\}$, where $c_i, 0 \leqq i \leqq s$ $(c_0 = c_s)$, are the generating reflections of Γ, is calculated as in theorem 3.1, but now $p = 4$. So

$$Q_s = (4-1)^s + (-1)^s(4-1) = 12\sum_{j=1}^{s-2}\binom{s-1}{j}p^{s-2-j}(-1)^j, \text{ and}$$

$$\phi(\Gamma, D_2) = \frac{Q_s}{6} = 2\sum_{j=1}^{s-2}\binom{s-1}{j}p^{s-2-j}(-1)^j.$$

b) s is even. Then the number of sets G'_ϕ is the number of sets G_ϕ minus 3 times the number C_s of different sets that generate any of the cyclic subgroups of D_2. We have calculated this number C_s in lemma 2.2. We have $C_s = 2$.

So $$Q_s = (4-1)^s + (-1)^s(4-1) - 6 = 6\left[2\sum_{j=1}^{s-2}\binom{s-1}{j}p^{s-2-j}(-1)^j - 1\right].$$

Therefore, $$\phi(\Gamma, D_2) = \frac{Q_s}{6} = 2\sum_{j=1}^{s-2}\binom{s-1}{j}p^{s-2-j}(-1)^j - 1.$$

COROLLARY 4.1. *Let Γ be an* NEC *group with* $s(\Gamma) = (0, +, [-], \{(2,\ldots,2),\ldots, (2,\ldots,2)\})$, *where there r period cycles, each of them with s_i link-periods equal to* 2. *The number of non equivalent surface coverings of $\mathscr{U}/\Gamma$ that admit D_2 as a group of automorphisms is:*

a) $$\phi(\Gamma, D_2) = \left(3^{r-1}\prod_{i=1}^{r}[3^{s_i-1} + (-1)^{s_i}]\right)\Big/2 \text{ if } s_i \text{ is odd for some } i, \text{ or}$$

b) $$\phi(\Gamma, D_2) = \left(3^{r-1}\prod_{i=1}^{r}[3^{s_i-1} + (-1)^{s_i}]\right)\Big/2 - 4^{r-1} \text{ if all } s_i \text{ are even.}$$

PROOF. First of all, as two elements $\phi(c_{i,0})$ and $\phi(c_{i,s_i})$ of order 2 in D_2 are conjugate if and only if $\phi(c_{i,0}) = \phi(c_{i,s_i})$, the number of sets G_ϕ for the ith cycle is Q_{s_i}, where Q_{s_i} is given in theorem 4.1.

a) Some s_i is odd. Then $G_\phi = G'_\phi$, $Q_{s_i} = 3[3^{s_i-1} + (-1)^{s_i}]$, we notice that 4, and hence 2, divides $3^{s_i-1} + (-1)^{s_i}$.

So $$\phi(\Gamma, D_2) = \frac{\Pi Q_{s_i}}{6} = \left(3^{r-1}\prod_{i=1}^{r}[3^{s_i-1} + (-1)^{s_i}]\right)\Big/2.$$

b) All s_i are even. Then the number of sets G'_ϕ is the number of sets G_ϕ minus 3 times the number C_r of different sets that generate any of the cyclic subgroups of D_2. We have calculated this number C_r in lemma 2.2. We have $C_r = 2^{2r-1}$.

So
$$\phi(\Gamma, D_2) = \frac{3^r \prod_{i=1}^{r} [3^{s_i - 1} + (-1)^{s_i}] - 3(2^{2r-1})}{6} =$$
$$= \left(3^{r-1} \prod_{i=1}^{r} [3^{s_i - 1} + (-1)^{s_i}]\right) \Big/ 2 - 4^{r-1}.$$

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