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SOME PROPERTIES OF FREE SHIFTS ON INFINITE FREE PRODUCT FACTORS

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1. Introduction.

In this work we study a special class of shift automorphisms on an infinite free product of von Neumann algebras. This is a generalisation of some results in [12]. The automorphisms are shown to be extremely ergodic, i.e. all non-trivial globally invariant von Neumann sub-algebras are *full* factors. As a special case we derive a result of Popa [12]. In this case the automorphisms are known to have Connes-Størmer entropy 0 by [14].

The reduced free product of II_1-factors was first introduced by Ching [2]. Ching shows that for G, H discrete groups $L(G) * L(H) = L(G * H)$, i.e. for group von Neumann algebras, the free product corresponds to the free product of groups.

The concept was generalised to general C^*-algebras independently by Avitzour [1] and Voiculescu [16].

All our von Neumann algebras are assumed to act in separable Hilbert spaces.

2. Preliminaries & Notation.

DEFINITION 2.1. Given a set S, any bijection $\beta: S \to S$ with the property that the orbit of every element under β is infinite, is said to be *free*.

We have the following result on free bijections.

LEMMA 2.2. *Let F be a set and let $\beta: F \to F$ be a free bijection. Let β act on subsets $S \subset F$ in the usual way: $\beta(S) = \{\beta(s): s \in S\}$.*
Assume $S, T \subset F$ are finite.

(1) *For each $n \in \mathbf{Z}$, $n \neq 0$; β^n is a free bijection.*

(2) *There exists $N \in \mathbf{N}$ such that for $|n| \geqq N$ we have*

$$\beta^n(S) \cap T = \emptyset.$$

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(3) *There exists an* $N \in \mathbf{N}$ *such that for* $|p| \geqq N$ *and* $n, m \in \mathbf{Z}$ *we have*

$$\beta^{pn}(S) \cap \beta^{pm}(T) = \emptyset.$$

A factor M is called *full* if $\operatorname{Int} M$ is closed in $\operatorname{Aut} M$.

We will use the algebra of central sequences as introduced by McDuff in [9] and later generalised to infinite von Neumann algebras by Connes in [5, 2.0] and [4, 1.1.2] by the use of *centralising* sequences. In the notation of [5] we let ω denote a free ultrafilter on N and for a von Neumann algebra M we let M_ω denote the corresponding algebra of centralising sequences.

We note that all centralising sequences are central by [5, 2.8].

The property Γ was introduced by Murray and von Neumann in [10]. A von Neumann algebra M is said to possess property Γ if given $\varepsilon > 0$ and $x_1, \ldots, x_n \in M$ there exists a unitary $w \in M$ with $\tau(w) = 0$ such that $\|[w, x_k]\| < \varepsilon$ for $k = 1, \ldots, n$.

We have the following result by Connes (and McDuff), connecting the asymptotic centraliser with the fulness property and property Γ.

LEMMA 2.3. *Let M be a factor with separable predual.*

(1) *M is full iff $M_\omega = \mathbf{C}$.*

(2) *M is full if all central sequences are trivial.*

(3) *If M is a II_1-factor then M is full iff M does not have the property Γ of Murray & von Neumann*

PROOF. This is essentially [5, 3.6, 3.7 and 3.8].

The following lemma, a consequence of the Rohlin lemma of Connes', turns out to be useful. We recall from [3] that an automorphism α is said to be aperiodic if every power α^n, $n \in \mathbf{Z}\setminus\{0\}$ is properly outer.

LEMMA 2.4. *Let M be a factor with separable predual, ω a free ultrafilter on* $\mathbf{N}$, *α an automorphism of M and α_ω the automorphism on M_ω induced by α.*

If α_ω^n is ergodic for all $n \neq 0$, then $M_\omega = \mathbf{C}$.

PROOF. If $M_\omega = \mathbf{C}$ there is nothing to prove. The proof is by contradiction. Assume $M_\omega \neq \mathbf{C}$ and a_ω^n is ergodic for all $n \neq 0$. Then a_ω is aperiodic. By [4, 2.1.2] α^n is not centrally trivial for any $n \in \mathbf{Z}\setminus\{0\}$. By [4, 2.1.4], for any $n > 1$ we can find a partition of unity $\{F_1, \ldots, F_n\} \subset M_\omega$ such that $\alpha_\omega(F_k) = F_{k+1}$ for $k = 1, \ldots, n$ (with $F_{n+1} = F_1$). That is, $\alpha_\omega^n(F_k) = F_k$. But then α_ω^n is not ergodic. This is a contradiction.

We will use the concept of a neighbourhood of infinity. For $\mathbf{Z}$ this is an interval $[n, \infty) \bigcap \mathbf{Z}$; for an ultrafilter ω it is a set $F \in \omega$.

3. Abstract definition of the free product.

We will use an abstract definition of the reduced free product taken from [17, 1.5].

DEFINITION 3.1. Let M be a von Neumann algebra with a faithful normal state τ. Let $M_k, k \in Z$ be von Neumann subalgebras of M each containing the unit $I \in M$. $\{M_k\}_k$ is called a *free family* of von Neumann algebras (relative to τ) if whenever $x_i \in M_{k_i}$, $k_i \neq k_{i+1}$ for $i = 1, \ldots, n$ with $\tau(x_i) = 0$ we have $\tau(x_1 x_2 \cdots x_n) = 0$.

If in addition $M = (\cup_{k \in Z} M_k)''$, we say that M is the *reduced free product* of the M_k (with respect to τ). We denote by τ_k the restriction of τ to M_k. We shall occasionally write $*(M_k, \tau_k)$ or $M = *M_k$ and $\tau = *\tau_k$

For von Neumann algebras M_i with faithful normal states, the construction in [16,] of the free product guarantees the existence of a von Neumann algebra M and faithful normal representations of M_i such that we have the situation in the definition.

We will henceforth assume that M is a von Neuman algebra as described in the definition above. We will see that the free shift which we define on M will asymptotically move any element onto its orthogonal complement and from this deduce properties of the central sequences in M.

4. Canonical form.

We want to write elements in the free product as a sum of elements from the orthogonal sub-algebras which generate the free product. Indeed, finding a general canonical form may probably be done, but we restrict the canonical form to a dense *-algebra.

DEFINITION 4.1. Let M^0 be the *-algebra generated by the M_k's. An element $x \in M^0$ is called a *monomial* if $x = x_1 x_2 \ldots x_n$ where $x_i \in M_{n_i}$, $x_i \neq 0$ and $n_i \neq n_{i+1}$ for $1 \leqq i \leqq n$. If in addition $\tau(x_i) = 0$ for each i, x is called an *irreducible monomial.*

We do not exclude the case where $x_i \in M_{n_i} \cap M_{n_{i+1}}$. However we shall see shortly that this can only happen for x_i scalars.

It is clear that any element in M^0 is a finite sum of monomials and that any monomial may be written as a finite sum of *irreducible* monomials and a scalar.

We note the following result.

LEMMA 4.2. *If $x = x_1 \cdots x_n$ is an irreducible monomial, then*

(1) $\|x\|_\tau = \|x_1\|_\tau \cdots \|x_n\|_\tau$.

(2) *If* $[x_1, x_2] = 0$ *then either* x_1 *or* x_2 *is* 0.

PROOF. For $y \in M$ let y' denote the non-scalar part of y. That is, $y' = y - \tau(y)$.

Proof of (1). We have

$$\|x\|_\tau^2 = \tau(x_n^* \cdots x_2^* x_1^* x_1 x_2 \cdots x_n)$$
$$= \tau(x_1^* x_1)\tau(x_n^* \cdots x_2^* x_2 \cdots x_n) + \tau(x_n^* \cdots x_n^*(x_1^* x_1)' x_2 \cdots x_n).$$

The last summand is zero by freeness. Thus the lemma follows by induction on the length n of the monomial.

Proof of (2). Assume $[x_1, x_2] = 0$, we have

$$\|x_1 x_2\|_\tau^2 = \tau((x_1 x_2)^*(x_1 x_2)) = \tau(x_2^* x_1^* x_1 x_2) = \tau(x_2^* x_1^* x_2 x_1) = 0$$

by freeness. Thus, $0 = \|x_1 x_2\|_\tau = \|x_1\|_\tau \|x_2\|_\tau$ by (1). That is, either x_1 or x_2 is zero.

Note that $M_n \cap M_m = \mathbf{C}$ for $m \neq n$. To see this, let $x \in M_m \cap M_n$, we may write $x = x' + \tau(x)I$, then $x' \in M_m \cap M_n$ is an irreducible monomial, even $x'x'$ is an irreducible monomial, thus by (2), $x' = 0$.

LEMMA 4.3. *If τ_k, $k \in \mathbf{Z}$ are tracial states on M_k, so is $\tau = *\tau$ on $*M_k$.*

PROOF. By continuity and linearity it is sufficient to show that for any irreducible monomials x, y we have $\tau(xy) = \tau(yx)$. For any $z \in M$, denote by $z' = z - \tau(z)$. By linearity, we may assume $\tau(x) = \tau(y) = 0$.

Assume $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ irreducible monomials. We have $\tau(xy) = \tau(x_1 \cdots x_n y_1 \cdots y_m)$ and $\tau(yx) = \tau(y_1 \cdots y_m x_1 \cdots x_n)$.

Assume $m \geqq n$. For these expressions to be non-zero, we must have x_{i+1} in the same algebra M_k as y_{m-1} for each i and that $n = m$.

To see this, note that if y_m is not in the same algebra as x_1, $\tau(yx)$ vanishes by freeness, otherwise we may rewrite

$$\tau(yx) = \tau(y_m x_1)\tau(y_1 \cdots y_{m-1} x_2 \cdots x_n) + \tau(y_1 \cdots y_{m-1}(y_m x_1)' x_2 \cdots x_n)$$

The last expression vanishes by freeness. We repeat the rewriting with y_{m-1} and x_2 and will either end up with y_{m-i} in a different algebra than x_{i+1} for some i (in which case $\tau(yx)$ is zero) or

$$\tau(yx) = \tau(y_m x_1)\tau(y_{m-1} x_2) \cdots \tau(y_{m-n+1} x_n)\tau(y_{m-n} y_{m-n-1} \cdots y_1).$$

The last factor, if its exists, i.e. $m > n$, is zero by freeness, thus we must have $n = m$ to ensure $\tau(yx) \neq 0$.

The computations for $\tau(xy)$ are similar, thus if $n \neq m$ or y_{m-i} is not in the same algebra as x_{i+1} for some i, we get $\tau(xy) = \tau(yx) = 0$.

Assuming $n = m$ and $y_{n-i}, x_{i+1} \in M_{k_{i+1}}$ for each i, we get by the above computations

$$\tau(xy) = \tau(x_n y_1)\tau(x_{n-1} y_2) \cdots \tau(x_1 y_n)$$

and

$$\tau(yx) = \tau(y_1 x_n)\tau(y_2 x_{n-1}) \cdots \tau(y_n x_1).$$

We have $y_{n-i} x_{i+1}, x_{i+1} y_{n-i} \in M_{k_{i+1}}$ for each i. Since τ restricts to τ_i on each M_i, we get

$$\tau(xy) = \tau_{k_n}(x_n y_1) \cdots \tau_{k_1}(x_1 y_n)$$

and

$$\tau(yx) = \tau_{k_n}(y_1 x_n) \cdots \tau_{k_1}(y_n x_1).$$

But the τ_{k_i}'s are traces, thus $\tau(xy) = \tau(yx)$.

The *support* of an element in M^0 will be a subset of H, the "free semigroup" in idempotent generators indexed by $\mathbf{Z}$.

Let H be the (free) semigroup with presentation $\{\sigma_i; \sigma_i^2 = \sigma_i\}_{i \in \mathbf{Z}}$ and unit e. That is, H consists of words over the alphabet $\{\sigma_i\}_{i \in \mathbf{Z}}$ where no letter is doubled, and a null word e. The multiplication is juxtaposition combined with the operation $\sigma_i \sigma_i \mapsto \sigma_i$. We will always consider elements of H in their canonical form (i.e. with no subwords of the form $\sigma_i \sigma_i$).

To any irreducible monomial $x = x_1 \cdots x_n \in M^0$ with $x_i \in M_{n_i}$ we may assign an element $h = \sigma(x_1 \cdots x_n) = \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_n}$. We will call $\{h\} \subset H$ the *support* of $x_1 \cdots x_n$.

The element h simply records the sequence of algebras in which the individual "factors" x_i are elements.

To prove that this "support" does not depend on the particular representation of x as an irreducible monomial, assume $x = x_1 \cdots x_n = y_1 \cdots y_m$ where $x_i \in M_{n_i}$ and $y_j \in M_{m_j}$.

We have $\|x\|_\tau^2 = \tau(x^* x) = \tau(y_m^* y_{m-1}^* \cdots y_1^* x_1 \cdots x_n)$. If y_1^* is not in the same sub-algebra as x_1 we have that $y_m^* y_{m-1}^* \cdots y_1^* x_1 \cdots x_n$ is an irreducible monomial, thus by freeness $\|x\|_\tau = 0$.

If y_1^* is in the same sub-algebra as x_1 (i.e. $n_1 = m_1$), we can write $\|x\|_r^2 = \tau(y_m^* \cdots y_2^* z_1 x_2 \cdots x_n)$ where $z_1 = y_1^* x_1 \in M_{n_1}$, we may rewrite this:

$\|x\|_r^2 = \tau(y_m^* \cdots y_2^* z_1' x_2 \cdots x_n) + \tau(z_1)\tau(y_m^* \cdots y_2^* x_2 \cdots x_n)$ where $z_1' = z_1 - \tau(z_1)$. The first expression in this sum is 0 by freeness (as z_1 can not be in the same sub-algebra as either y_2 or x_2.)

Continuing in this way we see that either $x = 0$ or y_i is in the same subalgebra as x_i for each i (and $n = m$.) Thus, $x_1 \cdots x_n$ has the same support as $y_1 \cdots y_m$. That is, we may write $\sigma(x)$ for $\sigma(x_1 \cdots x_n)$.

Note that for two irreducible monomials x, y with $\sigma(x) \neq \sigma(y)$, the above argument shows that $\tau(x^* y) = 0$, that is, x and y are τ-orthogonal.

For x a finite sum of irreducible monomials we may write $x = \sum_{h \in H} x_h$ where each x_h is either 0 or a sum of irreducible monomials x_h^i with $\sigma(x_h^i) = h$. Note that x_h is τ-orthogonal to x_g if $h \neq g$.

This representation is unique in the sense that if $\sum_{h \in H} x_h = \sum_{h \in H} y_h$ we have $0 = \sum_{h \in H} (x_h - y_h)$ so by orthogonality $x_h = y_h$. We call this the canonical representation.

As any element $x \in M^0$ can be written as $x = \tau(x) + x'$ where $\tau(x') = 0$ and x' can be written as a sum of irreducible monomials and a scalar (which necessarily must be 0), by putting $x_e = \tau(x)$ we see that the canonical representation is unique for all $x \in M^0$.

We define the support of such a sum to be $\sigma(x) = \cup\{h : x_h \neq 0\}$.

We will occasionally refer to the *set of generators* occurring in a subset $S \subset H$. By this we mean the minimal set of generators σ_i necessary to build the words of S. We introduce the notation $\gamma(S)$ for this set. For a set $F \subset M^0$ we adopt the notation $\gamma(F) = \gamma(\sigma(F))$.

For an element $x \in M^0$ and a subset S of the semigroup H, we define the element x_S as follows. If $x = \sum_{h \in H} x_h$ is the canonical form of x, we define $x_S = \sum_{h \in S} x_h$. For a subset $B \subset M^0$ we denote by B_S the set $\{x_S : x \in B\}$.

We may now record some facts about σ.

LEMMA 4.4. *Let* $x, y \in M^0$.

(1) *If* $\tau(x) = \tau(y) = 0$ *and* $\sigma(x) \cap \sigma(y) = \emptyset$ *then* x *is* τ*-orthogonal to* y.

(2) *If* $\tau(x) = \tau(y) = 0$ *and* $\gamma(x) \cap \gamma(y) = \emptyset$ *we have* $\sigma(xy) \subset \sigma(x)\sigma(y)$.

(3) $x_{\{e\}} = \tau(x)$ *and for* $S, T \subset H$ *with* $S \cap T = \emptyset$, *we have* $x_{S \cup T} = x_S + x_T$ *and* $(x + y)_S = x_S + y_S$.

(4) *If* $S \subset T \subset H$ *we have* $\|x_S\|_\tau \leqq \|x_T\|_\tau$.

Proof of (1). Let $x = \sum x_h$ and $y = \sum y_g$ be the canonical forms. By assumption we never have $x_h \neq 0$ and $y_h \neq 0$ simultaneously, which means that every x_h is orthogonal to every y_g, thus $\sum x_h$ must be orthogonal to $\sum y_g$.

Proof of (2). Let $x = \sum x_h$ and $y = \sum y_g$ be the canonical forms. We have $xy = \sum_{hg} x_h y_g$. Each x_h is a sum of irreducible monomials of the form $x_1 \cdots x_{n_h}$, similarly with y_g. Thus, $x_h y_g$ is a sum of monomials of the form $x_1 \cdots x_n y_1 \cdots y_m$.

By assumption x_n is not in the same sub-algebra M_k as y_1. We therefore have that each monomial in the product $x_h y_g$ is an irreducible monomial with support $\{hg\}$.

Thus, we may write $xy = \sum_{hg} z_{hg}$ where h runs over $\sigma(x)$ and g runs over $\sigma(y)$. Clearly, for $h_1, h_2 \in \sigma(x)$ and $g_1, g_2 \in \sigma(g)$ we have $h_1 g_1 = h_2 g_2$ iff $h_1 = h_2$ and $g_1 = g_2$. That is, an element $f \in H$ occurs only once as such a product hg. Hence $xy = \sum_{hg} z_{hg}$, where h, g runs over $\sigma(x), \sigma(y)$ respectively, is a canonical form of xy.

By definition of σ, we have $\sigma(xy) = \cup\{hg : z_{hg} \neq 0\}$. Hence, $hg \in \sigma(xy)$ implies $x_h y_g \neq 0$, thus we must have $h \in \sigma(x)$ and $g \in \sigma(y)$, i.e. $hg \in \sigma((x)\sigma(y)$.

The proofs of (3) and (4) are also easy applications of definitions.

For S a sub-semigroup of H generated by a set of generators of H, the application $x \mapsto x_S$ is a conditional expectation $E_{M_S^0}: M^0 \to M_S^0$, i.e. we have $(xyz)_S = xy_S z$ whenever $x, y \in M_S^0$.

Thus it is not surprising that the operation $x \to x_S$ may be extended to the whole of M. If $x_n \in (M^0)_{\|x\|}$ (the $\|x\|$-ball in M^0) is a sequence converging strongly to x, define $x_S = \lim (x_n)_S$. The existence and uniqueness of x_S follows from the fact that the strong topology on the unit ball is metrisable and complete by the metric $d(x, y) = \|x - y\|_\tau$. We say x has finite support if there exists a finite $S \subset H$ such that $x_S = x$. The support of x is the intersection of all such S.

We shall have occasion to use the last statement of Lemma 4.4 for general x. Too see that it holds, we let $x_n \to x$ be a bounded sequence, so $x_S = \lim (x_n)_S$. We have $\|x_S\|_\tau = \lim \|(x_n)_S\|_\tau \leqq \lim \|z_n\|_\tau = \|x\|_\tau$.

5. The free shift(s).

Assume M is defined as above.

Let π: $\mathsf{Z} \to \mathsf{Z}$ be a free bijection. For the rest of this work we assume that there are canonical *-isomorphismms β_k: $\mathrm{M}_k \to \mathrm{M}_{\pi(k)}$ such that $\tau \circ \beta_k = \tau$ for all $k \in \mathsf{N}$ and $x \in M_k$.

We also assume that there is an automorphism α on M such that $\alpha|_{M_k} = \beta_k$ with $\tau \circ \alpha = \tau$. This is possible at least if the β_k's are unitarily implemented.

REMARK 5.1. Note that since α is defined by means of the general free bijection π, α^n is defined by the free bijection π^n (by Lemma 2.2); hence "everything" we prove about α is true for α^n as well.

We will study the asymptotic properties of α using a number of results from [5] summarised in Lemma 2.3. Although the study of centralising sequences involves the use of the *-strong topology, we will use the $\|\cdot\|_\tau$-norm. Most of our results are symmetric with respect to the *-operation.

We will work in the dense *-algebra M^0. We show that bounded sequences in M may be approximated from M^0.

LEMMA 5.2. *For a bounded sequence* $(x_n)_n \in l^\infty(\mathsf{N}, M)$ *we may find a bounded sequence* $x_n' \in M^0$ *such that* $x_n - x_n' \to 0$ **-strongly. If* $\tau(x_n) \to 0$ *we may choose* $\tau(x_n') = 0$.

PROOF. Given a sequence $(x_n)_n \in M$ bounded by K. If we restrict to M_K, the K-ball of M, the *-strong topology is defined by the norm $(\|x\|_\tau^*)^2 = \|x\|_\tau^2 + \|x^*\|_\tau^2$ by [15, III, 5.3].

Given $n \in N$, since $(M^0)_K$ is *-strongly dense in $(M)_K$ we may find an $x'_n \in (M^0)_K$ such that $\|x_n - x'_n\|_\tau^* < 1/n$.

Given $\varepsilon > 0$, let $N > 1/\varepsilon$, we have for $n > N$ that $\|x_n - x'_n\|_\tau^* < \varepsilon$; hence $x_n - x'_n \xrightarrow[n\to\infty]{} 0$ *-strongly, and the sequence $(x'_n)_n$ is bounded by K.

If $\tau(x_n) \to 0$, let $x''_n = x'_n - \tau(x'_n)$ and use x''_n as an approximating sequence.

The free bijection π can be viewed as a bijection of the generators σ_i of H and so extends to a (semigroup) automorphism of H by letting $\pi(e) = e$. By abuse of notation we call this automorphism α.

α can be applied to a subset $F \subset H$ in the usual way $\alpha(F) = \{\alpha(f): f \in F\}$.

LEMMA 5.3. *If $x \in M^0$ with $\tau(x) = 0$ and $S \subset H$, we have*

(1) *$\sigma(\alpha(x)) = \alpha(\sigma(x))$ and $\alpha(x_S) = \alpha(x)_{\alpha(S)}$.*

(2) *For $y \in M^0$, $\tau(y) = 0$ there exists $N \in \mathbb{N}$ such that for $|n| > N$ we have $\alpha^n(x)$ orthogonal to y.*

(3) *For $y_1, y_2 \in M^0$, $\tau(y_i) = 0$, there exists $N \in \mathbb{N}$ such that for $|n| > N$ we have $a^n(x)y_1$ orthogonal to y_2.*

PROOF. By linearity, it suffices to consider irreducible monomials to prove (1). Assume $x = x_1 \cdots x_n$ with support $\{\sigma_{i_1} \cdots \sigma_{i_n}\}$.

We have $\alpha(x) = \alpha(x_1) \cdots \alpha(x_n) = \beta_{i_1}(x_1) \cdots \beta_{i_n}(x_n)$ which clearly has support $\{\sigma_{\pi(i_1)} \cdots \sigma_{\pi(i_n)}\} = \alpha(\sigma(x))$.

By the above we have $\alpha(y_S) = \alpha(y)_{\alpha(S)}$ for y an irreducible monomial and for y a scalar. For x afinite sum we have $\alpha((\sum x_i)_S) = \alpha(\sum (x_i)_S) = \sum \alpha((x_i)_S) = \sum \alpha(x_i)_{\alpha(S)} = \alpha(\sum x_i)_{\alpha(S)} = \alpha(\sum x_i)_{\alpha(S)} = \alpha(x)_{\alpha(S)}$.

Proof of (2). Assume $x = \sum_{h\in\sigma(x)} x_h$ and $y = \sum_{k\in\sigma(y)} y_k$ are the canonical forms of x, y respectively.

Let S be the set of generators in the support of x, that is $S = \gamma(x)$. As π acts freely on these generators, we may find $N \in \mathbb{N}$ such that for $|n| > N$ we have $\pi^n(S) \cap \gamma(y) = \emptyset$. By (1) $\sigma(\alpha^n(x)) = \alpha^n(\sigma(x))$, so we get that $\sigma(y)$ and $\sigma(\alpha^n(x))$ have no generators in common and neither contains e, thus because the generators are free we have $\sigma(\alpha^n(x)) \cap \sigma(y) = \emptyset$.

Proof of (3). From the proof of (2) we can find an $N \in \mathbb{N}$ such that for $|n| > N$ we have $\gamma(\alpha^n(x)) \cap \gamma(y) = \emptyset$. From Lemma 4.4 we then have: $\sigma(\alpha^n(x)y) \subset \sigma(\alpha^n(x))\sigma(y)$. But any word in $\sigma(\alpha^n(x))\sigma(y)$ must begin with a generator which does not occur in $\sigma(y)$ thus it has no common elements with $\sigma(y)$.

COROLLARY 5.4. *For every non-zero $n \in \mathbb{Z}$, the automorphism α^n of M is ergodic.*

PROOF. Let $z \in M$ be a fixed point for α such that $\|z\|_\tau = 1$ and $\tau(z) = 0$. We will prove that there is an $N \in \mathbb{N}$ such that $\alpha^N(z) \neq z$.

Given $\varepsilon > 0$ we find $z_0 \in M^0$ such that $\tau(z^0) = 0$, $\|z - \dot{z}^0\|_\tau < \varepsilon$ and $\|z^0\|_\tau \geqq 1 - \varepsilon$.

By Lemma 5.3 (2) we can find $N \in \mathbf{N}$ such that for $|m| > N$ we have $\alpha^m(z^0)$ orthogonal to z^0. For such m we have

$$\begin{aligned}\|\alpha^m(z) - z\|_\tau &= \|\alpha^m(z - z^0) - (z - z^0) + (\alpha^m(z^0) - z^0)\|_\tau \\ &\geqq \big| \|\alpha^m(z - z^0) - (z - z^0)\|_\tau - \|\alpha^m(z^0) - z^0\|_\tau \big| \\ &\geqq \|\alpha^m(z^0) - z^0\|_\tau - 2\varepsilon \\ &= \sqrt{2}\,\|z^0\|_\tau - 2\varepsilon \geqq \sqrt{2} - 4\varepsilon.\end{aligned}$$

Choosing ε small we see that $\alpha^m(z) \neq z$ for any $|m| > N$.

We see from the proof of the corollary that α takes any operator x with $\tau(x) = 0$ approximatively into its orthogonal complement.

6. Asymptotic orthogonality.

We have the following proposition concerning asymptotic orthogonality.

PROPOSITION 6.1. *Assume $(x_n) \in l^\infty(\mathbf{N}, M^0)$ is such that $\tau(x_n) = 0$. Assume further that for each finitely generated $S \subset H$ we have $\|(x_n)_S\|_\tau \xrightarrow[n\to\infty]{} 0$.*
Then for each $y \in M$ with $\tau(y) = 0$ we have

$$\lim_{n\to\infty} \big| \|x_n y - y x_n\|_\tau^2 - (\|x_n y\|_\tau^2 + \|y x_n\|_\tau^2)\big| = 0.$$

This result also holds if all limits are taken in a free ultrafilter ω, in which case we may write

$$\lim_{n\to\omega} \|x_n y - y x_n\|_\tau^2 = \lim_{n\to\infty} \|x_n y\|_\tau^2 + \lim_{n\to\omega} \|y x_n\|_\tau^2$$

as all the limits exists.

PROOF. Assume $y \in M^0$ with $\tau(y) = 0$.
Let S be the sub-semigroup of H generated by the generators of $\sigma(y)$. Denote by S^c the complement of S in H. We have by assumption $\|(x_n)_S y\|_\tau \to 0$ and $\|y(x_n)_S\|_\tau \to 0$. (Limit in either senses.)

Given $\varepsilon > 0$ we may thus find a neighbourhood of infinity F such that for $n \in F$ we have each of

$$\big| \|(x_n)_{S^c} y - y(x_n)_{S^c}\|_\tau^2 - \|x_n y - y x_n\|_\tau^2 \big|,$$

$$\big| \|(x_n)_{S^c} y\|_\tau^2 - \|x_n y\|_\tau^2 \big|,$$

$$\big| \|y(x_n)_{S^c}\|_\tau^2 - \|y x_n\|_\tau^2 \big|$$

less than $\varepsilon/3$.

By Lemma 4.4 we have $y(x_n)_{S^c}$ orthogonal to $(x_n)_{S^c}y$ for all n, thus $\|(x_n)_{S^c}y - y(x_n)_{S^c}\|_\tau^2 = \|(x_n)_{S^c}y\|_\tau^2 + \|y(x_n)_{S^c}\|_\tau^2$.

Thus $|\,\|x_n y - yx_n\|_\tau^2 - (\|x_n y\|_\tau^2 + \|yx_n\|_\tau^2)| < \varepsilon$ for every $n \in F$.

This proves the statement in case $y \in M^0$. For general $y \in M$ with $\tau(y) = 0$ and $\varepsilon > 0$ we may approximate y strongly with elements of zero trace and bounded norm from M^0.

We will use the above result to show that the shift automorphism α on M prevents the existence of central sequences in α-invariant sub-factors of M.

We have the following result, which in conjunction with the previous result tells us that α_ω has no non-trivial fixed points in M_ω.

PROPOSITION 6.2. *Assume* $(x_n) \in l^\infty(\mathrm{N}, M^0)$ *with* $\tau(x_n) \to 0$ *is an asymptotic fixed point for* α, *that is* $\alpha(x_n) - x_n \to 0$ *strongly. Then for any finitely generated* $S \subset H$ *we have* $\|(x_n)_S\|_\tau \xrightarrow[n \to \omega]{} 0$.

PROOF. Assume $\|(x_n)_S\|_\tau$ does not converge to 0 in ω. Given $\varepsilon > 0$, by scaling the x_n we may assume that for each $F \in \omega$, we may find $p \in F$ such that $\|(x_p)_S\|_\tau > 1 + \varepsilon$. We may also assume that $\tau(x_n) = 0$ for all n; hence that $e \notin S$.

By applying Lemma 2.2 to the generators of S we may find $q \in \mathrm{N}$ such that $\alpha^{qi}(S) \cap \alpha^{qj}(S) = \emptyset$ for $i \neq j$.

Given $k \in \mathrm{N}$. As (x_n) is an asymptotic fixed point for each power of α, we may, for each i with $1 \leqq i \leqq k$, find $F_i \in \omega$ such that for $n \in F_i$

$$\begin{aligned} |\,\|\alpha^{qi}((x_n)_S)\|_\tau - \|(x_n)_{\alpha^{qi}(S)}\|_\tau| &\leqq \|\alpha^{qi}((x_n)_S) - (x_n)_{\alpha^{qi}(S)}\|_\tau \\ &= \|(\alpha^{qi}(x_n) - x_n)_{\alpha^{qi}(S)}\|_\tau < \varepsilon. \end{aligned}$$

We may find $p \in \cap F_i$ such that for $1 \leqq i \leqq k$ we have $\|(x_p)_{\alpha^{qi}(S)}\|_\tau > 1$

By orthogonality we have

$$\begin{aligned} \|x_p\|_\tau^2 &\geqq \|(x_p)_{\cup \alpha^{qi}(S)}\|_\tau^2 \\ &= \sum_{i=1,\dots,k} \|(x_n)_{\alpha^{qi}(S)}\|_\tau^2 > k. \end{aligned}$$

That is $\|x_p\|_\tau^2 > k$. As k is an arbitrary integer and the sequence x_n is bounded in the uniform norm (and thus in the $\|\cdot\|_\tau$-norm) this is a contradiction. Thus we must have $\|(x_n)_S\|_\tau \xrightarrow[n \to \omega]{} 0$.

In the above lemma we may replace α by any non-zero power of α. This is merely a consequence of Remark 5.1.

PROPOSITION 6.3. *For each* $x \in M$ *we have* $\alpha^n(x) \xrightarrow[n \to \infty]{} \tau(x)$ *weakly.*

PROOF. By linearity we may consider x with $\tau(x) = 0$.

Let φ: $M \to B(K)$ be the representation engendered by τ, φ is a faithful, normal representation. Let $\xi \in K$ be the canonical cyclic, separating vector.

We have $\tau(x) = (\varphi(x)\xi, \xi)$; hence if x, $y \in M$ are τ-orthogonal vectors we have $(\varphi(x)\xi, \varphi(y)\xi) = 0$.

Define β an automorphism on $\varphi(M)$ as $\beta(\varphi(x)) = \varphi(\alpha(x))$.

For $x \in M$ with $\tau(x) = 0$ we will show $(\beta^n(\varphi(x))\eta, \eta) \xrightarrow[n\to\infty]{} 0$ for any $\eta \in K$.

It is sufficient to consider η in a dense subspace, i.e. we may restrict attention to η of the form $\varphi(y)\xi$ with $y \in M^0$. (Noting that ξ is cyclic for $\varphi(M^0)$.)

Assume $x, y \in M^0$; by Lemma 5.3, we may find $N \in \mathbf{N}$ such that for $n > N$ we have $\alpha^n(x)y$ τ-orthogonal to y. That is $(\beta^n(\varphi(x))\varphi(y)\xi, \varphi(y)\xi) = (\varphi(\alpha^n(x)y)\xi, \varphi(y)\xi) = 0$.

Thus, for $x \in M^0$ we have $\alpha^n(x) \xrightarrow[n\to\infty]{} \tau(x)$ weakly.

Then assume $x \in M$ with $\tau(x) = 0$ and $\|x\| = 1/4$.

Given $U \subset M_1$, a convex weak neighbourhood of 0. Since the $\|\cdot\|_2$ balls in M_1 is a base for the strong topology at 0, and the strong topology is finer than the weak topology, we may find $\varepsilon > 0$ such that the set $V = \{x \in M_1: \|x\|_2 < \varepsilon\} \subset U$.

We may find x' with finite support and $\|x'\| \leqq \|x\|$ such that $\|x - x'\|_\tau < \varepsilon/2$. Then $\alpha^n(2(x - x')) \in V \subset U$ for all n.

Since $\alpha^n(x') \to 0$ weakly, we may find $N \in \mathbf{N}$ such that for $n > N$ we have $\alpha^n(2x') \in U$; hence $\alpha^n(x) = (1/2)\alpha^n(2(x - x')) + (1/2)\alpha^n(2x') \in U$ by convexity of U.

This immediately gives us some asymptotic abelian properties as described in [6].

Corollary 6.4.

(1) *The state τ on M is strongly clustering (with respect to α).*

(2) *The automorphism α is weak asymptotic abelian.*

(3) *The group $\{\alpha^n\}_{n\in\mathbf{Z}}$ is a large group.*

By the above proposition we get for every x, $y \in M$ with $\tau(x) = \tau(y) = 0$

$$\lim_{n\to\infty} \|\alpha^n(x)y\|_\tau = \lim_{n\to\infty} \tau(y^*\alpha^n(x^*x)y) = \|y\|_\tau^2 \|x\|_\tau^2$$

because $\alpha^n(x^*x)$ tends weakly to $\|x\|_\tau^2 I$. Thus choosing y in the centre of M with $\tau(y) = 0$ and using Proposition 6.1, we get $\|x\|_\tau \|y\|_\tau = 0$; hence M is a factor.

If the M_k's are II_1-factors with τ_k traces, we have that τ is a trace by Lemma 4.3; hence using Proposition 6.1 and Proposition 6.3 we get the explicit formula $\lim_{n\to\infty} \|[\alpha^n(x), y]\|_\tau = \sqrt{2}\, \|x\|_\tau \|y\|_\tau$.

Lemma 6.5. *Any α-invariant von Neumann sub-algebra N of M is a factor.*

Proof. Let $z \in Z(N)$ be an element in the centre of N with $\tau(z) = 0$. We will prove that $z = 0$.

Let $x \in N$, $\tau(x) = 0$ and $\|x\| = 1$. By Lemma 5.2 find $x_n \in M^0$ with $\tau(x_n) = 0$ and $\|x_n\| \leqq 1$ such that $x_n - \alpha^n(x) \to 0$ *-strongly.

Let $S \subset H$ be finitely generated.

For $\varepsilon > 0$ find $x^0 \in M^0$ with $\tau(x^0) = 0$ such that $\|x^0 - x\|_\tau < \varepsilon/2$. We clearly have $\alpha^n(x^0)_S \xrightarrow[n \to \infty]{} 0$ strongly.

For large $n \in \mathbf{N}$ we have

$$\begin{aligned}\|(x_n)_S\|_\tau &= \|(x_n - \alpha^n(x^0))_S + \alpha^n(x^0)_S\|_\tau \\ &\leqq \|(x_n - \alpha^n(x^0))_S\|_\tau + \|\alpha^n(x^0)_S\|_\tau \\ &\leqq \|x_n - \alpha^n(x^0)\|_\tau + \|\alpha^n(x^0)_S\|_\tau < \varepsilon.\end{aligned}$$

thus $\|(x_n)_S\|_\tau \xrightarrow[n \to \infty]{} 0$.

Since z is in the centre of N we have

$$\begin{aligned}\|x_n z - z x_n\|_\tau &= \|(\alpha^n(x) - x_n)z - z(\alpha^n(x) - x_n)\|_\tau \\ &\leqq \|(\alpha^n(x) - x_n)z\|_\tau + \|z(\alpha^n(x) - x_n)\|_\tau \xrightarrow[n \to \infty]{} 0.\end{aligned}$$

Using Proposition 6.1 on the bounded sequence x_n and z we then have $\|z x_n\|_\tau \xrightarrow[n \to \infty]{} 0$.

Furthermore

$$|\,\|z x_n\|_\tau - \|z\alpha^n(x)\|_\tau| \leqq \|z(x_n - \alpha^n(x))\|_\tau \xrightarrow[n \to \infty]{} 0;$$

hence $\|z\alpha^n(x)\|_\tau \xrightarrow[n \to \infty]{} 0$.

Again, using the property that z commutes with everything in N, we have

$$\begin{aligned}\|z\alpha^n(x)\|_\tau^2 &= \tau(\alpha^n(x^*)z^*z\alpha^n(x)) \\ &= \tau(\alpha^n(x^*x)z^*z) \xrightarrow[n \to \infty]{} \|x\|_\tau^2\,\|z\|_\tau^2\end{aligned}$$

using the strongly clustering property of τ.

We thus have $\|x\|_\tau\,\|z\|_\tau = 0$, that is $z = 0$.

THEOREM 6.6. *Let π be a free bijection of $\mathbf{Z}$, M a von Neumann algebra acting in a separable Hilbert space. Assume τ is a faithful normal state on M and α an automorphism of M such that $\tau \circ \alpha = \tau$. Assume M is the free product of von Neumann algebras $M_i \subset M$, $i \in \mathbf{Z}$ and that $\alpha|_{M_i}$ is a *-isomorphism $\alpha_i\colon M_i \to M_{\pi(i)}$.*

If N is a globally α-invariant von Neumann subalgebra of M, then N is a full factor not of type I *or $N = \mathbf{C}$.*

PROOF. If $N = \mathbf{C}I$ we are finished, we assume $N \neq \mathbf{C}I$. We know from the previous lemma that N is a factor.

By Corollary 5.4, α is ergodic on M, thus on N; hence $\alpha|_N$ can not be inner on N. That is, N can not be of type I as every automorphism of a type I factor is inner.

Given ω a free ultrafilter on $\mathbf{N}$. By Lemma 2.3, it is sufficient to prove that every ω-centralising sequence of N is trivial.

We have that $\alpha|_N$ defines a *-automorphism α_ω on N_ω. A fixed point $x \in N_\omega$ of α_ω is represented by a sequence $x_n \in N$ such that $\alpha(x_n) - x_n \to 0$ *-strongly.

Also, (x_n) is centralising in N; hence central. That is $\|x_n y - y x_n\|_\tau^* \xrightarrow[n\to\omega]{} 0$ for each $y \in N$. Since M^0 is dense in M, we may, by Lemma 5.2, for each x_n find $x'_n \in M^0$ such that $\|x_n - x'_n\|_\tau < 1/n$ and $\|x'_n\| \leqq \|x_n\|$. We have $\|x'_n y - y x'_n\|_\tau \xrightarrow[n\to\omega]{} 0$ for each $y \in N$ and $\alpha(x'_n) - x'_n \xrightarrow[n\to\omega]{} 0$ *-strongly.

For each $y \in N$ with $\tau(y) = 0$, we get from Proposition 6.1 and Proposition 6.2, $\lim_{n\to\omega} \|y x'_n\|_\tau^2 + \lim_{n\to\omega} \|x'_n y\|_\tau^2 = 0$. Letting y be a unitary we see that $\lim_{n\to\omega} \|x'_n\|_\tau = 0$; hence $\lim_{n\to\omega} \|x_n\|_\tau = 0$. The same argument applies to the sequence (x_n^*) (with $((x'_n)^*)$, thus $x_n \xrightarrow[n\to\omega]{} 0$ *-strongly.

That is, α_ω is ergodic on N_ω.

By Remark 5.1, it is clear that this holds for any power of α. Any power of α_ω is therefore ergodic. By Lemma 2.4, $N_\omega = \mathbf{C}$.

This theorem immediately specialises to

COROLLARY 6.7. *Let $L(\mathsf{F}_Z)$ be the left regular representation of the free group with generators γ_i: $i \in \mathbf{Z}$. Let α be the shift automorphism on $L(\mathsf{F}_Z)$ coming from the bijection $\gamma_i \mapsto \gamma_{i+1}$.*

Then every von Neumann subalgebra of $L(\mathsf{F}_Z)$ globally invariant under α is either $\mathbf{C}$ or a full II_1-factor.

PROOF. We take M_i to be the von Neumann algebra generated by $\lambda(\gamma_i)$ and the free bijection π to be $i \mapsto i+1$. Define α by π: $\alpha(\lambda(\gamma_i)) = \lambda(\gamma_{i+1})$. Let τ be the trace on $L(\mathsf{F}_Z)$. Then $L(\mathsf{F}_N)$ is the free product of the M_i's and α is the free shift; hence we may apply Theorem 6.6.

The above corollary was noted in [12].

REFERENCES

1. D. Avitzour, *Free products of C*-algebras*, Trans. Amer. Math. Soc. 271 (1982), 423–465.
2. W. M. Ching, *Free products of von Neumann algebras*, Trans. Amer. Math. Soc. 178 (1973), 147–163.
3. A. Connes, *Classification of injective factors*, Ann. of Math. 104 (1976), 73–115.
4. A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. (1975), 383–420.
5. A. Connes, *Almost periodic states and factors of type* III_1, J. Funct. Anal. 16 (1974), 415–445.
6. S. Doplicher, D. Kastler and E. Størmer, *Invariant States and Asymptotic Abelianness*, J. Funct. Anal. 3 (1969), 419–434.
7. I. Kovács and J. Szücs, *Ergodic type theorems in von Neumann algebras*, Acta Sci. Math 27 (1966).

8. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. 1–2, Academic Press, 1986.
9. D. McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. 21 (1970), 443–461.
10. F. Murray & J. von Neumann, *On Rings of Operators* IV 44 (1943), 716–808.
11. J. Phillips, *Automorphisms of full* II_1*-factors, with applications to factors of type* III, Duke Math. J. 43 (1976), 375–385.
12. S. Popa, *Maximal injective subalgebras in factors associated with free groups*, Adv. in Math. 50 (1983), 27–58.
13. E. Størmer, *Large groups of automorphisms of* C^**-algebras*, Comm. Math. Phys. 5 (1967).
14. E. Størmer, *Entropy of some automorphisms of the* II_1*-factor of the free group in infinite number of generators*, Invent. Math. 110 (1992), 63–73.
15. M. Takesaki, *Theory of Operator Algebras*, vol. I, Springer-Verlag, Berlin, 1972.
16. D. Voiculescu, *Symmetries of some reduced free product* C^**-algebras*, Lecture Notes in Math. 32 (1985), 556–588.
17. D. Voiculescu, *Circular and Semicircular Systems and Free Product Factors*, Progress in Math. 92 (1990), Birkhäuser, 45–60.

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