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SELECTION FROM UPPER SEMI-CONTINUOUS COMPACT-VALUED MAPPINGS

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Abstract.

The aim of this paper is to show that if axiom M (or the continuum hypothesis) is assumed, then every upper semi-continuous compact-valued map from the space of irrationals to a compact (not necessarily metric) space has a selection, which is measurable in the sense that pre-images of Baire measurable sets are universally measurable. The methods used will yield generalizations and easier proofs of well-known theorems, namely of a selection theorem by Sion [1], and a representation theorem by Ioffe [3].

0. Introduction.

It was conjectured by Jørgen Hoffmann-Jørgensen that all upper semi-continuous compact-valued maps of the irrationals into a compact Hausdorff space, K, have a selection, which is measurable in the sense that pre-images of Baire sets are universally measurable. A result of this kind would have implications in asymptotic likelihood theory and in the theory for continuity of stochastic processes. This note shows that such selections indeed do exist, if a special axiom called axiom M is assumed. Axiom M says that, on the unit interval with the Lebesgue measure, the union of strictly less than continuum many Lebesgue null-sets is a Lebesgue null-set. Axiom M is clearly implied by the continuum hypothesis and also by Martin's axiom, see [5]. First, a general characterisation of minimal usco-maps is given, showing that images of hereditarily separable spaces by such maps are separable. Next, a number of selection results are proved, using a method which is a modfication of that used by Sion in [1], leading to the answer to the original question. Among these results is a generalisation of Sion's selection result for set-valued maps with a simpler proof. Finally, a new proof of

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a representation theorem for set-valued maps by Ioffe in [3] is given. Again, the new proof is simpler than Ioffe's and allows a more general conclusion.

1. Definitions.

All spaces used here will be assumed to be Hausdorff. A usco-map ϕ of X into Y is a set-valued correspondance which is upper semi-continuous and compact-valued. For a set-valued correspondance we define the kernel: $\ker\phi = \{x\colon \phi(x) \neq \emptyset\}$; if ϕ is a usco-map then $\ker\phi$ is closed. The space of all usco-maps of X into Y is given a partial order as follows: $\psi \leqq \phi$ if $\psi(x)$ is a subset of $\phi(x)$ for all $x \in X$ and $\ker\psi = \ker\phi$. A usco-map is said to be minimal if it is minimal in this partial ordering. A selection from a set-valued correspondance, $\phi\colon X \to Y$, is a function, $f\colon \ker\phi \to Y$ such that $f(x) \in \phi(x)$ for all $x \in \ker\phi$. For set-valued correspondances we employ the notation $\phi^{-s}(A) = \{x\colon \phi(x) \subseteq A\}$ and $\phi^{-w} = \{x\colon \phi(x) \cap A \neq \emptyset\}$.

A function $f\colon X \to Y$ is $\mathscr{A} \to \mathscr{B}$ measurable if $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$. If f is only partially defined $f\colon D \to Y$, where $D \subseteq X$, then f is $\mathscr{A} \to \mathscr{B}$ measurable if $D \cap f^{-1}(B) \in \mathscr{A}_D$, where $\mathscr{A}_D$ is the trace of $\mathscr{A}$ on D. On any space, X, the families of sets $\mathscr{F}(X)$, $\mathscr{G}(X)$, $\mathscr{B}o(X)$, $\mathscr{B}a(X)$ and $\mathscr{M}u(X)$ are the families of closed, open, Borel, Baire and universally measurable subsets of X, respectively. A subset of X is universally measurable if it is measurable with respect to any σ-finite Radon measure on X.

The space of irrationals will be identified with $\mathbf{N}^{\mathbf{N}}$ equipped with the product topology. A space is said to be K-analytic if it is the image of $\mathbf{N}^{\mathbf{N}}$ by a usco-map. A Souslin scheme is a map, A, of $\mathbf{N}^{(\mathbf{N})}$, the set of all finite sequences of integers, into 2^X, the set of subsets of X. Performing the Souslin-operation on A yields the set

$$S(A) = \bigcup_{\sigma \in \mathbf{N}^{\mathbf{N}}} \bigcap_{n \in \mathbf{N}} A(\sigma \mid n).$$

The paving $S(\mathscr{F}(X))$ consists of all subsets of X on the form $S(A)$ where A is a closed-valued Souslin scheme. We denote by $\mathscr{B}i(X)$ the σ-algebra $S(\mathscr{F}(X)) \cap CS(\mathscr{F}(X))$ of biSouslin-sets. The paving $\mathscr{S}_1(X)$ is the least σ-algebra containing $S(\mathscr{F}(X))$.

A pair consisting of a space, X, and a σ-algebra, $\mathscr{A}$, *on* X is said to be Blackwell if

$$\ker A = \left\{\sigma\colon \bigcap_n A(\sigma \mid n) \neq \emptyset\right\} \in S(\mathscr{F}(\mathbf{N}^{\mathbf{N}}))$$

for all $\mathscr{A}$-valued Souslin-schemes, A. The pair $(X, \mathscr{A})$ is Blackwell with the selection property if it is Blackwell and, for all $\mathscr{A}$-valued Souslin schemes, there exists a $\mathcal{M}u(\mathbf{N}^{\mathbf{N}}) \to \mathcal{A}$ measurable selection from the correspondance $\sigma \to \bigcap_n A(\sigma \mid n)$.

The weight of a space is the least cardinal, τ, such that the space has a base of cardinality τ. A space, X, is said to be injective if there exists a universally measurable injection of X into the real line. If A is a subset of a space X, then A^c is the complement of A in X.

2. Minimal usco-maps.

The main result of this section is Proposition 2, which gives a necessary and sufficient condition for a usco-map to be minimal. We start with a little lemma.

LEMMA 1. *For a usco-map $\phi: X \to Y$ and an open set $G \subseteq Y$ define for each $x \in X$*

$$\phi(G)(x) = \begin{cases} \phi(x) \backslash G, & \text{if } x \in \operatorname{int}(\phi^{-w}(G^c) \cup \ker \phi^c), \\ \phi(x), & \text{if } x \in \operatorname{cl}(\phi^{-s}(G) \cap \ker \phi). \end{cases}$$

Then $\phi(G)$ is usco and $\phi(G) \leqq \phi$.

PROOF. Let F be a closed subset of Y, then

$$\phi(G)^{-w}(F) = [\phi^{-w}(F \cap G^c) \cap \operatorname{int}(\phi^{-w}(G^c) \cup \ker \phi^c)]$$

$$\cup [\phi^{-w}(\mathrm{F}) \cap \operatorname{cl}(\phi^{-s}(G) \cap \ker \phi)] = \phi^{-w}(F \cap G^c) \cup [\phi^{-w}(F) \cap \operatorname{cl}(\phi^{-s}(G) \cap \ker \phi)],$$

since

$$\phi^{-w}(F \cap G^c) \subseteq [\phi^{-w}(F \cap G^c) \cap \operatorname{int}(\phi^{-w}(G^c) \cup \ker \phi^c)] \cup [\phi^{-w}(F) \cap \operatorname{cl}(\phi^{-s}(G) \cap \ker \phi)]$$

We conclude that $\phi(G)$ is usco and the rest of the lemma is immediate.

PROPOSITION 2. *A usco-map, $\phi: X \to Y$, is minimal, if and only if,*

$$(*) \qquad \phi^{-w}(G) \subseteq \operatorname{cl}(\phi^{-s}(G) \cap \ker \phi)$$

for each open set $G \subseteq Y$.

PROOF. Assume $(*)$ holds for all open subsets of Y and let $\psi: X \to Y$ be a usco-map such that $\psi \leqq \phi$ and $\psi(y) \neq \phi(y)$. Since $\psi(y)$ is compact we can find an open set U such that $\psi(y) \subseteq U$ and $\phi(y) \cap [\operatorname{cl}(U)]^c \neq \emptyset$. Then

$$\begin{aligned} y &\in \phi^{-w}([\operatorname{cl}(U)]^c) \cap \psi^{-s}(U) \\ &\subseteq \operatorname{cl}(\phi^{-s}([\operatorname{cl}(U)]^c) \cap \ker \phi) \cap \psi^{-s}(U), \quad \text{by } (*), \\ &\subseteq \operatorname{cl}(\psi^{-s}([\operatorname{cl}(U)]^c) \cap \ker \psi) \cap \psi^{-s}(U), \quad \text{since } \psi \leqq \phi. \end{aligned}$$

Hence $\psi^{-s}([\mathrm{cl}(U)]^c) \cap \ker\psi \cap \psi^{-s}(U)$ is nonempty which is a contradiction, and we conclude that ϕ is minimal.

Now assume that (*) does not hold for the open subset G of Y. That is, there exists $y \in \mathrm{int}(\phi^{-w}(G^c) \cup \ker\phi^c)$ such that $\phi(y)\backslash G$ is a proper non-empty subset of $\phi(y)$. Using Lemma 1 we conclude that ϕ is not minimal.

We shall now use this characterization to give some properties of minimal usco-maps. Recall that a function is said to have the Baire property if the pre-image of every open set is an open set modulo a set of the first category. Also recall that a family of sets is said to be T_0-separating if there, for any pair of distinct points, exists a set from the family that contains one of the points but not the other. We do not require that the separating set can be chosen such that it contains, say, the first of the points of the pair.

COROLLARY 3. *Let* $\phi: X \to Y$ *be a minimal usco-map.*

(i) *If* $\ker\phi$ *is separable, then* $\phi(X)$ *is separable.*

(ii) *Any selection from* ϕ *has the Baire-property.*

(iii) *If there exists a countable* T_0*-separating family of open sets in* Y*, then the set* $\{x: \#\phi(x) > 1\}$ *is of the first category in* X.

PROOF. (i) Let $\mathrm{cl}\{x_n\} = \ker\phi$ and choose points $y_n \in \phi(x_n)$. If $G \cap \phi(X) \neq \emptyset$ for an open set $G \subseteq Y$, then $\emptyset \neq \phi^{-w}(G) \subseteq \mathrm{cl}(\phi^{-a}(G) \cap \ker\phi)$ by Proposition 2. Hence the open set $\phi^{-s}(G)$ has nonempty intersection with $\ker\phi$ and we find x_n such that $y_n \in \phi(x_n) \subseteq G$.

ii) Let f be a selection from ϕ. Then

$$\phi^{-s}(G) \cap \ker\phi \subseteq f^{-1}(G) \subseteq \phi^{-w}(G) \subseteq \mathrm{cl}(\phi^{-s}(G)) \cap \ker\phi$$

by Proposition 2, and hence $f^{-1}(G)$ has the Baire-property for any open subset G of Y.

(iii) Let $\{G_n\}$ be a countable T_0-separating family of open subsets of Y. Then

$$\{x: \#\phi(x) > 1\} = \bigcup_n (\phi^{-w}(G_n)\backslash\phi^{-s}(G_n)) \subseteq \bigcup_n (\mathrm{cl}(\phi^{-s}(G_n))\backslash\phi^{-s}(G_n)),$$

The latter set, as a countable union of sets of the first category, is of the first category.

Let a usco-map, $\phi: X \to Y$, be given. Given a well-ordering of a base for the open subsets of Y we explicitly construct a minimal usco-map $\psi: X \to Y$ such that $\psi \leqq \phi$.

Let $\{G_\gamma: \gamma < \Omega\}$ be a wellordering of a base for the open subsets of Y and use Lemma 1 to define $\phi_1 = \phi(G_1)$ and, still using the lemma, define inductively for $\gamma < \Omega$: $\phi_\gamma = \left(\bigcap_{\beta<\gamma} \phi_\beta\right)(G_\gamma)$. Let $\psi = \bigcap_{\gamma<\Omega} \phi_\gamma$.

By Lemma 1 each ϕ_γ is usco, contained in ϕ and has the same kernel so this is also true for ψ. By Proposition 2 the map ψ is minimal, for

$$\psi^{-w}(G) = \bigcup_{\{\gamma: G_\gamma \subseteq G\}} \psi^{-w}(G_\gamma) \subseteq \bigcup_{\{\gamma: G_\gamma \subseteq G\}} \varphi_\gamma^{-w}(G_\gamma)$$

$$\subseteq \bigcup_{\{\gamma: G_\gamma \subseteq G\}} \mathrm{cl}\left(\left(\bigcap_{\beta<\gamma} \phi_\beta\right)^{-s} (G_\gamma) \cap \ker \psi\right)$$

$$\subseteq \mathrm{cl}\left(\bigcup_{\{\gamma: G_\gamma \subseteq G\}} \psi^{-s}(G_\gamma) \cap \ker \psi\right) \subseteq \mathrm{cl}(\psi^{-s}(G) \cap \ker \psi)$$

by the construction and the fact that $\psi \subseteq \phi_\gamma \subseteq \bigcap_{\beta<\gamma} \phi_\beta$.

3. Selection.

We shall now consider another way of cutting compact-valued (not necessarily usco) correspondances down. The approach used here will be very much like that of Sion in [1], but the results we shall obtain will be more general. The proofs in the rest of this note will depend on the properties of the following construction.

For a compact-valued correspondance $\phi: X \to Y$ and an open set $G \subseteq Y$ we define for each x in X

$$\phi_G(x) = \begin{cases} \phi(x)\backslash G, & \text{if } \phi(x)\backslash G \neq \emptyset, \\ \phi(x), & \text{otherwise.} \end{cases}$$

Let $\{G_\gamma: \gamma < \Omega\}$ be a T_0-separating family of open subsets of Y, and define $\phi_1 = \phi_{G_1}$ and, for each $\gamma < \Omega$, define inductively: $\phi_\gamma = \left(\bigcap_{\beta<\gamma} \phi_\beta\right)_{G_\gamma}$. Let $\psi = \bigcap_{\gamma<\Omega} \phi_\gamma$, then we have the following consequences.

(i) $\ker \phi = \ker \psi$.

(ii) $\# \psi(x) \leqq 1 \; \forall x \in X$.

(iii) $\psi^{-s}(G_\gamma) = \left(\bigcap_{\beta<\gamma} \phi_\beta\right)^{-s} (G_\gamma) = \bigcup_{\beta<\gamma} \phi_\beta^{-s}(G_\gamma)$ for all $\gamma < \Omega$.

(iv) For any open set $G \subseteq Y$ we have

$$\phi_\gamma^{-s}(G)$$

$$= \left(\bigcap_{\beta<\gamma} \phi_\beta\right)^{-s} (G) \cup \left(\left(\bigcap_{\beta<\gamma} \phi_\beta\right)^{-s} (G \cup G_\gamma) \cap \left[\left(\bigcap_{\beta<\gamma} \phi_\beta\right)^{-s} (G_\gamma)\right]^c\right)$$

$$= \bigcup_{\beta<\gamma} \phi_\beta^{-s}(G) \cup \left(\bigcap_{\beta<\gamma} \phi_\beta^{-s}(G \cup G_\gamma) \cap \bigcap_{\beta<\gamma} [\phi_\beta^{-s}(G_\gamma)]^c\right).$$

Define $f: \ker \phi \to Y$ by $\{f(x)\} = \psi(x)$ for all x in $\ker \phi$. From the construction we immediately get the following generalisation of Sion's result. By ω_1 we denote the first uncountable ordinal.

PROPOSITION 4. *Let $\phi: X \to Y$ be a compact-valued correspondance, $\{G_\gamma: \gamma < \omega_1\}$ be a T_0-separating family of open subset of Y such that every open subset of Y is a countable union of sets from this family, and such that the family is stable under finite unions. Let $\mathscr{H}$ be a σ-algebra on X such that $\phi^{-s}(G_\gamma) \in \mathscr{H}$ for all $\gamma < \omega_1$. Then ϕ has a $\mathscr{H} \to \mathscr{B}o$ measurable selection.*

PROOF. Let ψ be a selection from ϕ given by the construction above and let $\{f(x)\} = \psi(x)$ for all $x \in \ker \phi$. It is sufficient to prove that $f^{-1}(G_\gamma) \in \mathscr{H}$ for every $\gamma < \omega_1$. By (iii) we have

$$f^{-1}(G_\gamma) = \bigcup_{\beta < \gamma} \phi_\beta^{-s}(G_\gamma) \cap \ker(\phi).$$

The result follows by (iv) and transfinite induction since $\{G_\gamma: \gamma < \omega_1\}$ is stable under finite unions.

In [1] Sion requires that Y be regular. Following [6], Proposition 1-6-2 we find that on a measurable space (X, Σ), where Σ is countably generated, the universally Σ-measurable sets are stable under the union of strictly less than continuum sets when axiom M is assumed. The proof of the next proposition is similar to the proof of Proposition 4.

PROPOSITION 5. *Let X be of countable weight and assume that the weight of Y is strictly less than continuum. Let $\phi: X \to Y$ be a compact-valued correspondance such that $\phi^{-s}(G)$ is universally measurable for all open subsets G of Y. Assume axiom* M. *Then ϕ has a $\mathscr{M}u(X) \to \mathscr{B}o(Y)$ measurable selection.*

Using Propositions 4 and 5 we obtain the next two propositions.

PROPOSITION 6. *Let Y be K-analytic, hereditarily Lindelöf and of weigth less than or equal to $\aleph_1$. Then $(Y, \mathscr{G}i(Y))$ is Blackwell with the selection property.*

REMARK. If the K-analytic space Y is regular and hereditarily Lindelöf, then all open subsets of Y are Souslin-$\mathscr{F}$ sets. This again implies that $\mathscr{B}o(Y) = \mathscr{B}i(Y)$ and Y is hereditarily Lindelöf.

PROOF. Write $Y = \phi(\mathbf{N}^\mathbf{N})$ where Φ is usco and let F be an $\mathscr{F}(Y)$-Souslin scheme. It suffices to consider $\mathscr{F}(Y)$-Souslin schemes since $\mathscr{B}i(Y)$ is contained in $S(\mathscr{F}(Y))$. The map $\psi: \mathbf{N}^\mathbf{N} \times \mathbf{N}^\mathbf{N} \to Y$, defined by $\psi(\sigma, \tau) = \phi(\sigma) \cap \bigcap_n F(\tau \mid n)$, is usco and $\ker F = \pi_2(\ker \psi)$, where π_2 is the projection of $\mathbf{N}^\mathbf{N} \times \mathbf{N}^\mathbf{N}$ onto the second coordinate. Let f be a Borel measurable selection from ψ and let $g: \mathbf{N}^\mathbf{N} \to \mathbf{N}^\mathbf{N}$ be $\mathscr{S}_1 \to \mathscr{S}_1$ measurable such that $(g(\tau), \tau) \in \ker \psi$ for all τ in $\ker F$ (see Theorem 2.2.11 p. 348 in [2]). (Recall that $\mathscr{S}_1$ is the smallest σ-algebra containing $S(\mathscr{F}(\mathbf{N}^\mathbf{N}))$.) Then $f(g(\tau), \tau)$ is an $\mathscr{S}_1 \to \mathscr{B}o(Y)$ measurable selection from F.

The proof of the next proposition is similar to the proof of Proposition 6.

PROPOSITION 7. *Let Y be K-analytic weight strictly less than continuum and assume axiom* M. *Then* $(Y, \mathscr{B}i(Y))$ *is Blackwell with the selection property.*

If we are willing to accept weaker measurability properties of selections, this allows us to weaken the conditions of Propositions 4 and 5.

PROPOSITION 8. *Let X be Lindelöf and let Y be of weight less than or equal to the first uncountable ordinal. Then every usco-map of X into Y has a* $\mathscr{B}o(X) \to \sigma(\mathscr{F}(Y) \cap \mathscr{G}_\delta(Y))$ *measurable selection.*

REMARK. Note that $\mathscr{B}a(Y)$ is generated by a family of closed $\mathscr{G}_\delta$-sets.

PROOF. Let $\phi\colon X \to Y$ be usco and let $\{G_\gamma\colon \gamma < \omega_1\}$ be a base for Y. By our construction we have a selection f from ϕ such that $f^{-1}(G_\gamma) \in \mathscr{B}o(X)$ for all γ. Let $F = \bigcap_n G_n$ be a closed G_δ in Y. Then $F \cap \phi(X)$ is Lindelöf, and we can, for each n, find basic open sets such that $F \cap \phi(X) \subseteq \bigcup_m G_{mn} \cap \phi(X) \subseteq G_n \cap \phi(X)$, implying $F \cap \phi(X) = \bigcap_n \bigcup_m G_{mn} \cap \phi(X)$. Now $f^{-1}(F) = \bigcap_n \bigcup_m f^{-1}(G_{mn}) \in \mathscr{B}o(X)$.

PROPOSITION 9. *Let X be of countable weight and let Y be of weight less than or equal to continuum. Assume axiom* M. *Then every usco-map of X into Y has a* $\mathscr{M}u(X) \to \sigma(\mathscr{F}(Y) \cap \mathscr{G}_\delta(Y))$ *measurable selection.*

PROOF. Substitute $\mathscr{M}u(X)$ for $\mathscr{B}o(X)$ and 2^{w_0} for ω_1 in the proof for Proposition 8.

Theorem 10. *Let X be separable and Lindelöf and let Y be regular. Then every usco-map of X into Y with nonempty values has a selection, f, with the following measurability properties.*

(i) (CH) *f is* $\mathscr{B}o(X) \to \mathscr{B}a(Y)$ *measurable.*

(ii) (CH) *If Y also is hereditarily Lindelöf, then f is* $\mathscr{B}o(X) \to \mathscr{B}o(Y)$ *measurable. Let, in addition, X be of countable weight.*

(iii) (M) *f is* $\mathscr{M}u(X) \to \mathscr{B}a(Y)$ *measurable.*

(iv) (M) *If Y also is hereditarily Lindelöf, then f is* $\mathscr{M}u(X) \to \mathscr{B}o(Y)$ *measurable.*

PROOF. Let $\phi\colon X \to Y$ be a minimal usco-map with nonempty values. Then $\mathrm{cl}(\phi(X))$ is separable and regular, hence, by [4], Theorem 1.5.6., the weight of $\mathrm{cl}(\phi(X))$ is less than or equal to continuum, and so (i) and (iii) follow from Propositions 8 and 9. If Y is hereditarily Lindelöf, then (ii) and (iv) follow from Propositions 4 and 5.

Theorem 10 (iii) implies that, under axiom M, all compact Hausdorff spaces with the Baire σ-algebra are Blackwell with the selection property. But in the case where the range space is compact we can obtain conclusions (i) and (iii) of Theorem 10 with weaker conditions on the domain space.

THEOREM 11. *Let Y be compact of weight τ and let $\phi: X \to Y$ be usco. Then ϕ has a selection, f with the following measurability properties.*
(i) *If $\tau \leqq \aleph_1$, then f is $\mathscr{B}o(X) \to \mathscr{B}a(Y)$ measurable.*
(ii) *If $\tau \leqq 2^{\aleph_0}$ and axiom M is assumed, then f is $\mathscr{M}u(X) \to \mathscr{B}a(Y)$ measurable.*

REMARK. If X is separable then the weight of $\mathrm{cl}(\phi(X))$ is less than or equal to continuum.

PROOF. Let $C(Y)$ be the space of continuous functions from Y to R equipped with the topology of uniform convergence. Then there is a dense subset, $\mathscr{C}$, of $C(Y)$ of cardinality τ. (Use the Stone-Weierstraß Theorem, to be found e.g. in [7].) The family of sets $\mathscr{U}_2 = \{\{f < a\}: a \in \mathsf{Q}, f \in \mathscr{C}\}$ generates the Baire σ-algebra on Y and is of cardinality τ, and $\mathscr{B}a(Y)$ is contained in the σ-algebra generated by $\mathscr{U}$.

If $\tau \leqq \aleph_1$, then by property (iii) of the construction of the selection we have $f^{-1}(\mathscr{U}) \subseteq \mathscr{B}o(X)$ and (i) follows. If $\tau \leqq 2^{\aleph_0}$, then by axiom M we have $f^{-1}(\mathscr{U}) \subseteq \mathscr{M}u(X)$ and (ii) follows.

4. Representation.

Finally, we shall prove a representation theorem analoguous to that of [3]. In comparison to Theorem 2 and Corollary 2.1 in [3], Theorem 13 below gives only a bit more information about the measurability properties we can require selections to have. The main reason for including Theorem 13 in the present paper is that it shows how the method of selection that we have employed here can be applied to obtain representations. Furthermore, we can avoid using $\cup$-homomorphisms. We shall first prove a set-theoretical lemma.

LEMMA 12. *Let Y be regular of weight τ, let $\mathscr{U}_1$ be a base for Y of cardinality τ and let*

$$\mathscr{U} = \mathscr{U}_1 \cup \{[\mathrm{cl}(U)]^c: U \in \mathscr{U}_1\} = \{U(\gamma): \gamma < \tau\}\}.$$

Let $\Sigma \subseteq \tau^\tau$ be the set of bijections from τ to τ.
For any $y \in Y$ there exists $\sigma \in \Sigma$ such that

$$y \in U(\sigma(\gamma)) \Rightarrow \exists \beta < \gamma: y \in U(\sigma(\beta))^c \subseteq U(\sigma(\gamma)).$$

PROOF. Let $\gamma < \tau$ and assume $\{\sigma^\beta: \beta < \gamma\} \subseteq \Sigma$ have been defined so that:
(i) $y \in U(\sigma^\beta(\eta)), \eta \leqq \beta \Rightarrow \exists \xi < \eta: y \in U(\sigma^\beta(\xi))^c \subseteq U(\sigma^\beta(\eta))$; and
(ii) $\forall \eta < \tau \exists \xi < \gamma$: $\quad \sigma^\alpha(\eta) = \sigma^\beta(\eta) \quad \forall \alpha, \beta < \xi$
and $\quad \sigma^\beta(\eta) = \sigma^\xi(\eta) \quad \forall \xi \leqq \beta < \gamma$.

According to (ii) we can define $\psi \in \Sigma$ by $\psi(\eta) = \lim_{\beta < \gamma} \sigma^\beta(\eta)$.

If $y \in U(\gamma)$ and for no $\beta < \gamma$ we have $y \in U(\psi(\beta))^c \subseteq U(\gamma)$, find ξ such that $y \in \mathrm{cl}(U) \subseteq U(\gamma)$, where $U \in \mathscr{U}_1$ and $U(\xi) = [\mathrm{cl}(U)]^c$. In this case we let

$$\sigma^{\gamma}(\eta) = \begin{cases} \xi, \text{ if } \eta = \gamma, \\ \gamma, \text{ if } \eta = \xi \\ \psi(\eta), \text{ otherwise.} \end{cases}$$

Otherwise let $\sigma^{\gamma} = \psi$.

Finally put $\sigma = \lim_{\gamma < \tau} \sigma^{\tau}$.

THEOREM 13. *Let $\phi\colon X \to Y$ be usco with nonempty values, let Y be regular and let τ be the smallest ordinal corresponding to the weight of* $\mathrm{cl}(\phi(X))$. *Let $\mathscr{U}_1$ be a base for* $\mathrm{cl}(\phi(X))$ *and let $\mathscr{U} = \mathscr{U}_1 \cup \{[\mathrm{cl}(U)]^c\colon U \in \mathscr{U}_1\} = \{U(\gamma)\colon \gamma < \tau\}$. Let the space τ^{τ} have the topology induced by the base consisting of sets of the form $\{\sigma' : \sigma' \mid \gamma = \sigma \mid \gamma\}$, $\sigma \in \tau^{\tau}$, $\gamma < \tau$. Let $\Sigma \subseteq \tau^{\tau}$ be the set of bijections from τ to τ.*

There exists a function $h\colon X \times \Sigma \to Y$ such that

(i) *$h(x, \Sigma) = \phi(x)$ for all $x \in X$.*

(ii) *$\sigma \to h(x, \sigma)$ is continuous from Σ to Y for all $x \in X$.*

(iii) *If $\tau \leqq \omega_1$ and $\mathscr{H}$ is a σ-algebra on X such that $\phi^{-s}(U) \in \mathscr{H}$ for all $U \in \mathscr{U}$ and such that every open set in Y is a countable union of sets from $\mathscr{U}$, then $x \to h(x, \sigma)$ is a $\mathscr{H} \to \mathscr{B}o$ measurable selection of ϕ for all $\sigma > \tau$.*

(iv) *If X is of countable weight, τ is strictly less than continuum, $\Phi^{-s}(G)$ is universally measurable for all open subsets G of Y and if axiom* M *is assumed, then $x \to h(x, \sigma)$ is a $\mathscr{M}u \to \mathscr{B}o$ selection of ϕ for all $\sigma < \tau$.*

(v) *If X is Lindelöf and if τ is less than or equal to ω_1, then $x \to h(x, \sigma)$ is a $\mathscr{B}o \to \sigma(\mathscr{F} \cap \mathscr{G}_{\delta})$ measurable selection of ϕ for all $\sigma < \tau$.*

(vi) *If X of countable weight, if τ is less than or equal to continuum and if axiom* M *is assumed, then $x \to h(x, \sigma)$ is a $\mathscr{M}u \to \sigma(\mathscr{F} \cap \mathscr{G}_{\delta})$ measurable selection of ϕ for all $\sigma < \tau$.*

(vii) *If X is separable and Lindelöf, then, for all $\sigma < \tau$, $x \to h(x, \sigma)$ is a selection of ϕ with the following measurability properties.*

(CH) *The selection is $\mathscr{B}o \to \mathscr{B}a$ measurable.*

(CH) *If Y is also hereditarily Lindelöf, then the selection is $\mathscr{B}o \to \mathscr{B}o$ measurable.*

(M) *If X is of countable weight, then the selection is $\mathscr{M}u \to \mathscr{B}a$ measurable.*

(M) *If X is of countable weight and Y us hereditarily Lindelöf, then the selection is $\mathscr{M}u \to \mathscr{B}o$ measurable.*

(viii) *If Y is compact and $\tau \leqq \omega_1$, then the selection is $\mathscr{B}o \to \mathscr{B}a$ measurable. If Y is compact, axiom* M *is assumed and $\tau \leqq 2^{\omega_0}$, then the selection is $\mathscr{M}u \to \mathscr{B}a$ measurable.*

PROOF. Define $\phi_1^{\sigma} = \phi_{U(1)}$, $\phi_{\gamma}^{\sigma} = \left(\bigcap_{\beta < \gamma} \phi_{\beta}^{\sigma}\right)_{U(\sigma(\gamma))}$ and $\{h(x, \sigma)\} = \bigcap_{\gamma < \tau} \phi_{\gamma}^{\sigma}(x)$. This is the construction used in Propositions 4, 5, 8 and 9 and Theorems 10 and 11 and hence (iii) to (viii) follows. To prove (i) note that $h(x,\Sigma)\subseteq\phi(x)$ for all $x\in X$ by construction. Let $y\in\phi(x)$, then by Lemma 12 there exists $\sigma\in\Sigma$ such that $y\in U(\sigma(\gamma))\Rightarrow\exists\beta<\gamma\colon y\in U(\sigma(\beta))^c\subseteq U(\sigma(\gamma))$ and hence $h(x,\sigma)=y$. If $h(x,\sigma)\in U(\sigma(\gamma))$, then $h(x,\sigma')\in U(\sigma(\gamma))$ for all $\sigma'\in\{\sigma'\colon\sigma'\mid\gamma=\sigma\mid\gamma\}$ and this proves (ii).

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