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WEIGHTED CONVOLUTION ALGEBRAS WITHOUT BOUNDED APPROXIMATE IDENTITIES

M. DESPIĆ, F. GHAHRAMANI[1] and S. GRABINER

Abstract.

We characterize the multipliers and derivations on any weighted convolution algebra without bounded approximate identities and the multipliers of the standard ideals in any weighted convolution algebra. We also characterize compact and weakly compact multipliers for any weighted convolution algebra. It is shown that the algebra of compact multipliers can be identified with a radical Banach algebra of absolutely continuous measures, and for some weights the algebra of compact multipliers can have elements outside of the corresponding weighted convolution algebra.

1. Introduction.

Suppose that w is a *weight* function on the positive half-line $(0, \infty)$ satisfying $w(s + t) \leqq w(s)w(t)$ and that w is continuous. Let $L^1(w)$ be the space of all equivalence classes of functions on $(0, \infty)$ integrable with respect to $w\,dt$, where dt denotes the Lebesgue measure on $(0, \infty)$. With convolution product

$$(1.1) \qquad (f * g)(x) = \int_0^x f(x - y)g(y)\,dy \quad (f, g \in L^1(w), \text{ a.e. } x \in (0, \infty))$$

$L^1(w)$ is a Banach algebra. The algebras $L^1(w)$ have been the subject of much study in recent years (see for example [1], [2],[4], [5], [7], [8], [9], [10], [11], [12], [13], [14] and [18]). When w is bounded near 0, $L^1(w)$ has a bounded approximate identity, for example the sequence $e_n = n\chi_{(0, 1/n)}, n = 1, 2, \ldots$, is one of them. However, that condition is not necessary for the existence of a bounded approximate identity. In fact, it has been shown by S. Ouzomgi in [16] that $L^1(w)$ has a bounded approximated identity if and only if $\liminf_{t \to 0} w(t) < \infty$.

Bounded approximate identities have shown to be powerful tools in the study of multipliers, derivations and automorphisms of weighted convolution algebras with bounded weights. In this paper we show that the known results concerning the multipliers, and derivations of those $L^1(w)$ with a bounded weight extend to

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any $L^1(w)$. Furthermore, we characterize the multiplier algebra of any standard ideal, compact and weakly compact multipliers, and compact derivations on $L^1(w)$. These results are new even for convolution algebras with bounded approximate identities.

We make use of the so called "Weighted Volterra type" algebras defined below. Suppose that $a > 0$ and w is a continuous positive weight function on $(0, a)$. We let $L^1_w(0, a)$ be the space of equvalence classes of functions integrable with respect to $w\,dt$. Again, with convolution as defined in (1.1) $L^1_w(0, a)$ is a Banach algebra.

For a Radon measure μ on R^+ let $\alpha(\mu)$ be the infimum of the support of μ, with $\alpha(0) = \infty$. Then the Titchmarsh convolution theorem states that $\alpha(\mu * \nu) = \alpha(\mu) + \alpha(\nu)$: for a proof see [4].

Due to continuity of w, if $\varepsilon > 0$, then w is bounded on $[\varepsilon, a)$, so that $L^1_w(0, a)$ contains the prime ideal $\{f \in L^1(0, a) : \alpha(f) > 0\}$. When w is bounded on $(0, a)$ the Banach spaces $L^1_w(0, a)$ and $L^1(0, a)$ are the same spaces with equivalent norms, so in this case $L^1_w(0, a)$ is just the familiar Volterra algebra $L^1(0, a)$. So we assume that w is unbounded (near 0). Then we have $L^1_w(0, a)$ properly contained in $L^1(0, a)$. We also note that if w is a weight function on $(0, \infty)$, then $L^1_w(0, a)$ can be identified with quotient of the algebra $L^1(w)$ by the standard ideal $I_a = \{f \in L^1(w) : \alpha(f) \geqq a\}$.

We start with identifying the multipliers of $L^1_w(0, a)$.

LEMMA 1.2. *Suppose that T is a multiplier on $L^1_w(0, a)$ and let $I_{w,\varepsilon} = \{f : f \in L^1_w(0, a)$ and $\alpha(f) \geqq \varepsilon\}$. Then*

a) *T is continuous.*

b) *$I_{w,\varepsilon}$ is invariant under T.*

PROOF. a) Continuity of T follows from an application of the closed graph theorem. In fact, if $f_n \to 0$ and $T(f_n) \to g$, then for $h \in L^1_w(0, a)$ with $\alpha(h) = 0$ we have $g * h = \lim T(f_n) * h = \lim f_n * T(h) = 0$. Hence $g = 0$, by the Titchmarsh convolution theorem.

b) Suppose first that $\alpha(f) > \varepsilon$. Then $f = \delta_\varepsilon * g$, for some $g \in L^1(0, a)$ with $\alpha(g) > 0$. We have $g = g_1 * g_2$, for some g_1 and g_2 in $L^1(0, a)$ by the Cohen factorization theorem, where $\alpha(g_1) > 0$ or $\alpha(g_2) > 0$. Suppose $\alpha(g_1) > 0$. Then g_1 and $g_2 * \delta_\varepsilon$ are in $L^1_w(0, a)$. Hence

$$T(f) = T(g_1 * g_2 * \delta_\varepsilon) = T(g_1) * g_2 * \delta_\varepsilon,$$

from which it follows that

$$\alpha(T(f)) = \alpha(T(g_1)) + \alpha(g_2 * \delta_\varepsilon) \geqq \varepsilon.$$

This together with the continuity of T shows that T maps $I_{w,\varepsilon}$ into $I_{w,\varepsilon}$.

THEOREM 1.3. *A linear map T on $L^1_w(0, a)$ is a multiplier if and only if there exists* *a Radon measure* μ *on* $[0,a)$ *with* $T(f) = f * \mu$, *for every* f *in* $L^1_w(0,a)$ *and where* μ *satisfies*

$$\|T\| = \sup\left\{\frac{1}{w(x)}\int_{[0,a-x]} w(x+y)\,d|\mu|(y) : x \in (0,a)\right\} < \infty.$$

PROOF. Suppose that T is a multiplier on $L^1_w(0,a)$. For every ε with $0 < \varepsilon < a$, let S_ε be an operator on $L^1(0,a)$ defined by $S_\varepsilon(f) = T(\delta_\varepsilon * f)$. Then S_ε is a continuous operator on $L^1(0,a)$, since it is the composition of multiplication by δ_ε from $L^1(0,a)$ to $L^1(\varepsilon,a)$ followed by T on $L^1_w(\varepsilon,a) = L^1(\varepsilon,a)$. Now we use this to prove that S_ε is a multiplier. Let f and g be in $L^1(0,a)$ with $\alpha(g) > 0$. Then, since $\delta_\varepsilon * f$ and g are in $L^1_w(0,a)$ we have

$$S_\varepsilon(f * g) = T(\delta_\varepsilon * f * g) = T(\delta_\varepsilon * f) * g = S_\varepsilon(f) * g. \tag{1.4}$$

Since the set of all g with $\alpha(g) > 0$ is dense in $L^1(0,a)$, by continuity of S_ε we have $S_\varepsilon(f * g) = S_\varepsilon(f) * g$, for all f and g in $L^1(0,a)$. This shows that S_ε is a multiplier on $L^1(0,a)$. Thus by a characterization of multipliers of $L^1(0,a)$ (see [15, Remark 7]) there exists a measure ν_ε on $[0,a)$ such that $S_\varepsilon(f) = f * \nu_\varepsilon$, for every $f \in L^1(0,a)$. Hence

$$T(\delta_\varepsilon * f) = f * \nu_\varepsilon \quad (f \in L^1(0,a)). \tag{1.5}$$

Now we show that $\alpha(\nu_\varepsilon) \geqq \varepsilon$. Suppose that f is a non-zero element of $L^1_w(0,a)$ with $\alpha(f) = 0$. Then from (1.5), by the Titchmarsh convolution theorem and by Lemma 1.2b), we have

$$\alpha(f) + \alpha(\nu_\varepsilon) \geqq \varepsilon, \tag{1.6}$$

showing that $\alpha(\nu_\varepsilon) \geqq \varepsilon$. Thus, there exists a Radon measure μ_ε on $[0,a)$ with $\delta_\varepsilon * \mu_\varepsilon = \nu_\varepsilon$. Then from (1.5) we have

$$T(\delta_\varepsilon * f) = f * \delta_\varepsilon * \mu_\varepsilon \quad (f \in L^1(0,a)). \tag{1.7}$$

Now from (1.7)

$$T(g) = g * \mu_\varepsilon \quad (g \in I_{w,\varepsilon}), \tag{1.8}$$

since $L^1(0,a) * \delta_\varepsilon = I_{w,\varepsilon}$.

Suppose ε' is such that $0 < \varepsilon' < \varepsilon$. Then from (1.8), with ε replaced by ε',

$$T(g) = g * \mu_{\varepsilon'} \quad (g \in I_{w,\varepsilon'}). \tag{1.9}$$

From (1.8) and (1.9) it follows that if $0 < \varepsilon' < \varepsilon$, then $\mu_\varepsilon = \mu_{\varepsilon'}$ on the interval $[0, a - \varepsilon]$. Hence there exists a unique Radon measure μ on $[0,a)$ which coincides with μ_ε on $[0, a - \varepsilon)$, for every $\varepsilon > 0$. Thus from (1.8) we have

$$T(g) = g * \mu \quad (\alpha(g) > 0,\ g \in L^1_w(0,a)). \tag{1.10}$$

Now suppose that f and g are any elements of $L^1_w(0, a)$, but $\alpha(g) > 0$. Then from (1.10) we have

$$T(f)*g = T(f*g) = f*g*\mu = (f*\mu)*g. \tag{1.11}$$

Since in (1.11) g is arbitrary, an application of the Titchmarsh convolution theorem shows that

$$T(f) = f*\mu \quad (f \in L^1_w(0, a)). \tag{1.12}$$

Now to prove the assertion concerning the norm $\|T\|$, let $M_w[\varepsilon, a) = \{v \in M_w(0, a) : \varepsilon \leqq \alpha(v)\}$. We note that $M_w[\varepsilon, a) = M[\varepsilon, a)$, since w is bounded away from 0 on $[\varepsilon, a)$. First we show that $\bar{T}_\mu$ given by $\bar{T}_\mu(v) = v*\mu$, maps $M_w[\varepsilon, a)$ into $M_w[\varepsilon, a)$. To this end, let $v \in M_w[\varepsilon, a)$. Then for every $f \in L^1(0, a)$ we have $f*v*\mu \in L^1_w(\varepsilon, a)$, since $f*v \in L^1_w(\varepsilon, a)$. Thus, $f*v*\mu \in L^1(0, a)$ and $f \mapsto f*v*\mu$ is a multiplier on $L^1(0, a)$. Hence $v*\mu$ must have a finite total variation.

Now, if $x \geqq \varepsilon$, then $\frac{1}{w(x)}\delta_x*\mu = \text{wk*-lim}\left(\frac{1}{w(x)}\delta_x*\mu*e_n\right)$ where $e_n = n\chi_{[0,1/n]}$, $n = 1, 2, \ldots$ and $\text{wk*} = \sigma(M_w[\varepsilon, a), C_0([\varepsilon, a)))$.

Thus,

$$\begin{aligned} \left\|\frac{1}{w(x)}\delta_x*\mu\right\|_w &\leqq \underline{\lim}\left\|\frac{1}{w(x)}\delta_x*e_n*\mu\right\|_w = \underline{\lim}\left\|T\left(\frac{1}{w(x)}\delta_x*e_n\right)\right\|_w \\ &\leqq \|T\|\,\underline{\lim}\left\|\frac{1}{w(x)}\delta_x*e_n\right\|_w = \|T\|\frac{1}{w(x)}\underline{\lim}\int_0^{\frac{1}{n}} nw(x+y)\,dy \\ &= \|T\|, \text{ by continuity of } w \text{ at } x. \end{aligned} \tag{1.13}$$

Since ε is arbitrary, from (1.13) we have

$$\sup\left\{\frac{1}{w(x)}\int_{[0,a-x)} w(x+y)\,d|\mu|(y) : x \in (0, a)\right\} \leqq \|T\|. \tag{1.14}$$

Conversely, suppose that for some Radon measure μ satisfying (1.14) we have $T(f) = f*\mu$ $(f \in L^1_w(0, a))$. It suffices to show that $f*\mu \in L^1_w(0, a)$. We have

$$\begin{aligned} \int_0^a w(x)\,d|\mu*f|(x) &\leqq \int_0^a w(x)\,d(|\mu|*|f|)(x) \\ &= \int_0^a \int_{[0,a-x)} w(x+y)\,d|\mu|(y)\,|f(x)|\,dx \\ &\leqq \left(\int_{(0,a)} w(x)\,|f(x)|\,dx\right) \\ &\quad \times \sup\left\{\frac{1}{w(x)}\int_{[0,a-x)} w(x+y)\,d|\mu|(y) : x \in (0, a)\right\}. \end{aligned} \tag{1.15}$$

Hence,

$$(1.16)\qquad \|T(f)\|_w = \|\mu * f\|_w \leqq \|f\|_w \sup\left\{\frac{1}{w(x)}\int_{[0,a-x)} w(x+y)\,d|\mu|(y) : x\in(0,a)\right\}.$$

Together with (1.13) this shows that $\|T\| = \sup\left\{\frac{1}{w(x)}\int_{[0,a-x)} w(x+y)\,d|\mu|(y):\right.$ $\left. x\in(0,a)\right\}$, and the proof is complete.

COROLLARY 1.17. (a) *A linear operator T is a multiplier of $L^1(w)$ if and only if there exists a Radon measure μ on $[0,\infty)$ such that $T(f) = f * \mu$ $(f\in L^1(w))$ and such that μ satisfies the growth condition*

$$(1.18)\qquad \|T\| = \sup\left\{\frac{1}{w(x)}\int_{[0,\infty)} w(x+y)\,d|\mu|(y) : x\in(0,\infty)\right\} < \infty.$$

(b) *A linear mapping T is a multiplier of $M(w)$ if and only if there exists a Radon measure μ satisfying* (1.18) *and $T(v) = v * \mu$, for every v in $M(w)$.*

PROOF. (a) Suppose that T is a multiplier of $L^1(w)$ and for $a > 0$, $I_a = \{f\in L^1(w) : \alpha(f) \geqq a\}$. Then from the equation $f * Tg = T(f * g) = T(f) * g$ and by the Titchmarsh convolution theorem we have $T(I_a) \subset I_a$. Now by identifying $L^1_w(0,a)$ with $L^1(w)/I_a$, we obtain a mapping T_a: $L^1_w(0,a) \to L^1_w(0,a)$, defined by $T_a(f + I_a) = T(f) + I_a$. It is straightforward to verify that T_a is a multiplier of $L^1_w(0,a)$, and $\|T_a\| \leqq \|T\|$, for all $a > 0$. Hence by Theorem 1.2 for $a > 0$, there exists a Radon measure μ_a on $[0,a)$ such that $T_a(f) = f * \mu_a$ and

$$(1.19)\qquad \sup\left\{\frac{1}{w(x)}\int_{[0,a-x)} w(x+y)\,d|\mu_a|(y) : x\in(0,a)\right\} = \|T_a\|.$$

Also it can be easily verified that if $a < b$, then $\mu_b\,|\,[0,a) = \mu_a$. Thus, there exists a unique Radon measure μ on $[0,\infty)$ such that $\mu_a = \mu\,|\,[0,a)$, so that, $T(f) = f * \mu(f\in L^1(w))$. Since $\|T_a\| \leqq \|T\|$ we have the growth condition

$$(1.20)\qquad \sup\left\{\frac{1}{w(x)}\int_{[0,\infty)} w(x+y)\,d|\mu|(y) : 0 < x < \infty\right\} \leqq \|T\|.$$

The reversed inequality follows from calculations similar to (1.15) and (1.16).

(b) Suppose that T is a multiplier of $M(w)$. Again an application of the closed graph theorem shows that T is continuous. Next we show that $L^1(w)$ is invariant under T. For $a > 0$ and $f\in L^1(w)$ we have $T(\delta_a * f) = T(\delta_a) * f\in L^1(w)$. Since every function in $L^1(w)$ vanishing outside a compact subset of $(0,\infty)$ is of the form $\delta_a * f$, where $a > 0$ and $f\in L^1(w)$, the set of all $\delta_a * f$ is dense in $L^1(w)$. Since T is continuous, we have $T(L^1(w)) \subset L^1(w)$. Thus, by part (a) there exists a Radon measure μ such that $T(f) = f * \mu$, and μ satisfies (1.18). Now if ν is any measure in $M(w)$ and f is a non-zero element of $L^1(w)$, then $T(\nu) * f = T(\nu * f) = \nu * f * \mu$, and we have $T(\nu) = \nu * \mu$, by the Titchmarsh convolution theorem. Conversely, suppose that μ is a measure satisfying (1.18). Then it can be easily verified that for every $\nu \in M(w)$, $\int_{[0,\infty)} w(x)\, d|\nu * \mu|(x) < \infty$. Hence ρ_μ is a multiplier of $M(w)$.

It was shown in [7, Theorem 1.4] that when w is bounded near 0, the multiplier algebra of $L^1(w)$ can be identified with $M_w[0,\infty)$. The following generalizes that result.

COROLLARY 1.21. *Suppose that $L^1(w)$ has a bounded approximate identity. Then a linear map T is a multiplier of $L^1(w)$ if and only if for some measure $\nu \in M_w(0,\infty)$ and a complex number ζ, $T(f) = f * (\nu + \zeta\delta_0)$ $(f \in L^1(w))$.*

PROOF. The if part being obvious we assume that T is a multiplier of $L^1(w)$. Then there exists a Radon measure μ on $[0,\infty)$ such that $T(f) = f * \mu$ and μ satisfies (1.18). Let $\zeta = \mu(\{0\})$. Then $\mu = \nu + \zeta\delta_0$, with $\nu(\{0\}) = 0$. We identify ν with its restriction to $(0,\infty)$ and we show that $\nu \in M_w(0,\infty)$. Since $L^1(w)$ has a bounded approximate identity we have $\liminf_{x\to 0^+} w(x) = a < \infty$ [16]. Let $(x_n) \subset (0,\infty)$ be a sequence such that $x_n \to 0$ and $w(x_n) \to a$. From (1.18)

$$|\zeta| + \frac{1}{w(x_n)} \int_{(0,\infty)} w(x_n + y)\, d|\nu|(y) = \frac{1}{w(x_n)} \int_{[0,\infty)} w(x_n + y)\, d|\zeta\delta_0 + \nu|(y) \leqq \|T\|.$$

Thus

$$\int_{(0,\infty)} w(x_n + y)\, d|\nu|(y) \leqq (\|T\| - |\zeta|) w(x_n). \tag{1.22}$$

Now by Fatou's Lemma, from (1.22) we have

$$\int_{(0,\infty)} w(y)\, d|\nu|(y) \leqq (\|T\| - |\zeta|) a,$$

proving the claim about ν.

REMARK 1.23. The following example shows that when $L^1(w)$ does not have a bounded approximate identity then the multiplier algebra can be larger than the unitization of $M_w(0,\infty)$.

Suppose that w is non-increasing on $(0,\infty)$ and $\lim_{x\to 0} w(x) = \infty$, for example $w(x) = e^{-\eta(x)}$, where $\eta(x) = (x^3 - 1)/x$ (see [1, page 80]). Then $L_w(0,1) \subsetneqq L^1(0,1)$. Let $f \in L^1(0,1) \backslash L^1_w(0,1)$. Extend f to $(0,\infty)$ by defining it zero on $[1,\infty)$. Then the extended f is not in the unitization $M_w(0,\infty) \oplus \mathbf{C}$, but, ρ_f does define a multiplier on $L^1(w)$. In fact, since w is non-increasing we have

$$\frac{1}{w(x)}\int_0^\infty w(x+y)\,|f(y)|\,dy = \int_0^1 \frac{w(x+y)}{w(x)}|f(y)|\,dy$$
$$\leqq \int_0^1 |f(y)|\,dy$$

and (1.18) is fulfilled.

In [3] W. G. Bade, H. G. Dales and K. B. Laursen have given a description of the multiplier algebra of the unique primary ideal of $l^1(Z_0, w)$. They show that some of these algebras have "unusual" properties and therefore belong to a class of Banach algebras which are suitable as (counter) examples for various purposes. In the same sense, it might be interesting to look at the multiplier algebra of any standard ideal $I_a = \{f \in L^1(w) : \alpha(f) \geqq a\}$. Below, we give their description.

Corollary 1.24. *A linear mapping T is a multiplier of I_a if and only if there exists a Radon measure μ on $[0,\infty)$ with $T(f) = f * \mu$ $(f \in I_a)$ and*

$$\|T\| = \sup\left\{\frac{1}{w(x)}\int_{[0,\infty)} w(x+y)\,d|\mu|(y) : x \in [a,\infty)\right\} < \infty.$$

Proof. Suppose that T is a multiplier on I_a. First we show that for any f in I_a,

$$T(f * \delta_a) = T(f) * \delta_a. \tag{1.25}$$

To prove this suppose that f and g are non-zero elements of I_a. Then

$$T(f * \delta_a) * g = T(f * \delta_a * g) = T(f) * \delta_a * g, \tag{1.26}$$

and (1.25) follows from (1.26) and by the Titchmarsh convolution theorem.

Now define S on $L^1(w)$ by $S(f) = T(f * \delta_a)$. We claim that S is a multiplier on $L^1(w)$. To prove this, suppose that $f, g \in L^1(w)$, then from (1.22)

$$\begin{aligned} S(f * g) * \delta_a &= T(f * g * \delta_a) * \delta_a = T(f * \delta_a * g * \delta_a) \\ &= T(f * \delta_a) * g * \delta_a = S(f) * g * \delta_a. \end{aligned} \tag{1.27}$$

Hence $S(f * g) = S(f) * g$, by the Titchmarsh convolution theorem, showing that S is a multiplier of $L^1(w)$. By Corollary 1.17, there exists a Radon measure ν on $[0,\infty)$ such that

$$T(f * \delta_a) = S(f) = f * \nu \quad (f \in L^1(w)) \tag{1.28}$$

and

$$\sup\left\{\frac{1}{w(x)}\int_{[0,\infty)} w(x+y)\,d|\nu|(y) : x \in (0,\infty)\right\} < \infty. \tag{1.29}$$

Now in equation (1.28) we let f be a non-zero element of $L^1(w)$ with $\alpha(f) = 0$.

Then, since $T(f * \delta_a) \in I_a$, by the Titchmarsh convolution theorem we have $a \leqq \alpha(v)$. Hence $v = \delta_a * \mu$, for a unique Radon measure μ on $[0, \infty)$. Thus from (1.28)

$$(1.30) \qquad T(g) = g * \mu \quad (g \in \delta_a * L^1(w)).$$

From (1.29)

$$(1.31) \qquad \sup\left\{\frac{1}{w(x)} \int_{[0,\infty)} w(x + y + a)\, d|\mu|(y) : x \in (0, \infty)\right\} < \infty.$$

Now if f is any element of I_a, then from (1.25) and (1.30) we have

$$(1.32) \qquad T(f) * \delta_a = T(f * \delta_a) = f * \delta_a * \mu.$$

Hence $T(f) = f * \mu$ $(f \in I_a)$, as required.

To calculate the norm of the multiplier T; first we note that if $e_n = n\chi_{[0, 1/n]}$, $n = 1, 2, \ldots,$ and $\varepsilon > 0$, then wk*-lim $e_n * \delta_\varepsilon = \delta_\varepsilon$. In fact, if f is any continuous function on $(0, \infty)$, then

$$(1.33) \qquad \langle e_n * \delta_\varepsilon, f\rangle = \int_0^{1/n} f(x + \varepsilon)\, dx \to f(\varepsilon) = \langle \delta_\varepsilon, f\rangle,$$

by continuity of f. Next we show that if $x > a$, then $\delta_x * \mu \in M(w)$ and $\delta_x * \mu =$ wk*-lim $e_n * \delta_x * \mu$. That $\delta_x * \mu \in M(w)$ can be seen from (1.31). Now suppose $x - a = 2\varepsilon$. Then

$$(1.34) \qquad e_n * \delta_x * \mu = e_n * \delta_\varepsilon * \delta_{a+\varepsilon} * \mu \xrightarrow{\text{wk*}} \delta_\varepsilon * \delta_{a+\varepsilon} * \mu = \delta_x * \mu.$$

Now a calculation similar to (1.13) shows that

$$(1.35) \qquad \sup\left\{\frac{1}{w(x)} \int_{[0,\infty)} w(x + y)\, d|\mu|(y) : x \in [a, \infty)\right\} \leqq \|T\|.$$

Conversely, if the supremum in (1.35) is finite, then a simple calculation shows that

$$(1.36) \quad \|f * \mu\| \leqq \|f\| \sup\left\{\frac{1}{w(x)} \int_{[0,\infty)} w(x + y)\, d|\mu|(y) : x \in [a, \infty)\right\} \quad (f \in I_a).$$

Hence $f \mapsto f * \mu$ is a multiplier on I_a. Together (1.35) and (1.36) show that

$$\|T\| = \sup\left\{\frac{1}{w(x)} \int_{[0,\infty)} w(x + y)\, d|\mu|(y) : x \in [a, \infty)\right\}.$$

2. Compact Multipliers.

In [1] W. G. Bade and H. G. Dales have described elements f of $L^1(w)$ for which $g \mapsto g * f$ is a compact operator. Here we characterize all the compact multipliers of $L^1(w)$.

Recall that when w is continuous, a necessary and sufficient condition for the absence of a bounded approximate identity is $\lim_{x\to 0} w(x) = \infty$.

Also we recall that the wk*-topology of $M_w(0, \infty)$ is the topology $\sigma(M_w(0, \infty), C_{0,1/w}(0, \infty))$ where $C_{0,1/w}(0, \infty)$ is the space of all continuous functions φ on $(0, \infty)$ satisfying

$$\lim_{x\to 0} \frac{\varphi(x)}{w(x)} = \lim_{x\to\infty} \frac{\varphi(x)}{w(x)} = 0,$$

with $\|\varphi\| = \sup\left\{\left|\frac{\varphi(x)}{w(x)}\right| : x\in(0, \infty)\right\}$.

LEMMA 2.1. a) *Suppose that $L^1(w)$ does not have a bounded approximate identity. Then if (μ_i) is a net in $M_w(0, \infty)$ with $\mu_i \xrightarrow{\text{wk*}} \mu$, we have $\mu_i * v \xrightarrow{\text{wk*}} \mu * v$, for every v in $M_w(0, \infty)$.*

b) *For any $L^1(w)$ if $\{\mu_i : i\in I\}$ is a net in $M_w(0, \infty)$ with* $\inf\{\alpha(\mu_i) : i \in I\} > 0$, *then $\mu_i \xrightarrow{\text{wk*}} \mu$ implies $\mu_i * v \xrightarrow{\text{wk*}} \mu * v$, for every v in $M_w(0, \infty)$.*

PROOF. a) Suppose $\mu_i \xrightarrow{\text{wk*}} \mu$. Then for $v \in M_w(0, \infty)$ and $\varphi \in C_{0,1/w}(0, \infty)$ we have

$$\langle \mu_i * v, \varphi\rangle = \int_{(0,\infty)}\int_{(0,\infty)} \varphi(x + y)\, dv(y)\, d\mu_i(x). \tag{2.2}$$

First we show that $\psi(x) = \int_{(0,\infty)} \varphi(x + y)\, dv(y)$ is in $C_{0,1/w}(0, \infty)$. Now for any sequence $(x_n) \subset (0, \infty)$ if $x_n \to x \in (0, \infty)$, then

$$\begin{aligned}\frac{1}{w(x_n)}\int_{(0,\infty)} \varphi(x_n + y)\, dv(y) &= \int_{(0,\infty)} \frac{\varphi(x_n + y)}{w(x_n)}\, dv(y)\\ &\to \int_{(0,\infty)} \frac{\varphi(x + y)}{w(x)}\, dv(y),\end{aligned} \tag{2.3}$$

by the dominated convergence theorem, since $\left|\frac{\varphi(x_n + y)}{w(x_n)}\right| = \left|\frac{\varphi(x_n + y)}{w(x_n + y)}\right| \times \frac{w(x_n + y)}{w(x_n)} \leqq \|\varphi\|\, w(y)$, showing that ψ is continuous. Similarly, if $x_n \to \infty$ we have by the dominated convergence theorem $\frac{1}{w(x_n)}\int_{(0,\infty)} \varphi(x_n + y)\, dv(y) \to 0$.

Thus $\lim_{x\to\infty} \frac{1}{w(x)}\psi(x) = 0$.

Suppose now $x_n \to 0$. We have

$$\left|\frac{1}{w(x_n)}\int_{(0,\infty)}\varphi(x_n+y)\,d\nu(y)\right| \leqq \|\varphi\|\int_{(0,\infty)}\frac{w(x_n+y)}{w(x_n)}\,d|\nu|(y) \to 0,$$

since $w(x_n) \to \infty$, and by the dominated convergence theorem. Thus $\frac{1}{w(x)}\psi(x) \to 0$, as $x \to 0$, and $\psi \in C_{0,1/w}(0,\infty)$. From (2.2) we now have

$$\lim_i \langle \mu_i * \nu, \varphi\rangle = \int_{(0,\infty)}\int_{(0,\infty)}\varphi(x+y)\,d\nu(y)\,d\mu(x)$$
$$= \langle \mu * \nu, \varphi\rangle.$$

b) Suppose that w is any weight, $(\mu_i) \subset M_w(0,\infty)$ with $\inf_i \alpha(\mu_i) = a > 0$, and $\nu \in M(w)$. If $\varphi \in C_{0,1/w}(0,\infty)$, then we have (2.2). Similar to what we did in part a) we can show that if $\psi(x) = \int_{(0,\infty)}\varphi(x+y)\,d\nu(x)$, then ψ is continuous and $\lim_{x\to\infty}\frac{\psi(x)}{w(x)} = 0$. Now since $\mu_i([0,a]) = 0$, for all i, by redefining ψ on $[0,a]$, we can assume that $\lim_{x\to\infty}\frac{\psi(x)}{w(x)} = 0$, and the result follows.

THEOREM 2.4. *For a multiplier $\rho_\mu: f \mapsto f * \mu$ on $L^1(w)$ the following are equivalent:*

a) *ρ_μ is compact.*

b) *ρ_μ is weakly compact.*

c) *μ is absolutely continuous and if $L^1(w)$ has a bounded approximate identity then*

$$\lim_{x\to\infty}\frac{1}{w(x)}\int_0^\infty w(x+y)\,d|\mu|(y) = 0 \tag{2.5}$$

while if $L^1(w)$ does not have a bounded approximate identity in addition to (2.5) *one also has*

$$\lim_{x\to 0}\frac{1}{w(x)}\int_0^\infty w(x+y)\,d|\mu|(y) = 0. \tag{2.6}$$

d) *ρ_μ is a compact multiplier on $M(w)$.*

e) *ρ_μ is a weakly compact multiplier on $M(w)$.*

PROOF. (a) $\Rightarrow$ b) is trivial.

(b) $\Rightarrow$ (c). Suppose that ρ_μ is weakly compact. Let $e_n = n\chi_{[0,\frac{1}{n}]}$, $n = 1, 2, \ldots$. Then by the weak compactness of ρ_μ for every $x > 0$, there exists a subsequence (e_{n_i}) of (e_n) and a measure ν in $M(w)$ such that

$$\frac{1}{w(x)}\delta_x * e_{n_i} * \mu \xrightarrow{\text{weakly}} \nu. \tag{2.7}$$

Then if f is a non-zero element of $L^1(w)$ from (2.7) we have

$$\frac{1}{w(x)}\delta_x * e_{n_i} * f * \mu \xrightarrow{\text{weakly}} f * \nu. \tag{2.8}$$

Now from (1.34) and (2.8) we have $\frac{1}{w(x)}\delta_x * f * \mu = f * \nu$. Hence by the Titchmarsh convolution theorem $\frac{1}{w(x)}\delta_x * \mu = \nu$. From this and (2.7) it follows that for $x > 0$,

$$\frac{1}{w(x)}\delta_x * e_{n_i} * \mu \xrightarrow{\text{weakly}} \frac{1}{w(x)}\delta_x * \mu. \tag{2.9}$$

Hence $\delta_x * \mu$ is in $L^1(w)$, since $L^1(w)$ is weakly closed in $M(w)$. Thus μ is absolutely continuous.

To obtain the growth conditions in the statement c) we first assume that $L^1(w)$ does not have a bounded approximate identity. Let $x > 0$, and (e_n) be as before. Then

$$\left\|\frac{1}{w(x)}\delta_x * e_n\right\| = \frac{n}{w(x)}\int_0^{1/n} w(x+y)\,dy \to 1, \text{ as } n \to \infty, \tag{2.10}$$

by continuity of w at x. Let N_x be the least positive integer such that if $n > N_x$, then $\left\|\frac{1}{w(x)}\delta_x * e_n\right\| < 2$. Since ρ_μ is weakly compact, the set

$$E = \bigcup_{x>0}\left\{\frac{1}{w(x)}\delta_x * e_n * \mu : n > N_x\right\}$$

is weakly conditionally compact. From (2.9) we have the set $F = \left\{\frac{1}{w(x)}\delta_x * \mu : x > 0\right\}$ is contained in the weak closure of E. Hence F is weakly conditionally compact. The mapping $d\nu(t) \to w(t)\,s\nu(t)$ is a linear isometric isomorphism from $M_w(0,\infty)$ onto $M(0,\infty)$. Hence by the Dieudonné-Grothendeick characterization of weakly conditionally compact subsets of $M(X)$ for a locally compact subspace X [6, Theorem 4.22.1(4)] the set $\left\{\frac{1}{w(x)}\delta_x * |\mu| : x > 0\right\}$ is weakly conditionally compact. Since in separable Banach spaces bounded sets are metrizable there exists a sequence $(x_i) \subset R^+$ and $f \in L^1(w)$ such that $x_i \to \infty$ and

$$\frac{1}{w(x_i)}\delta_{x_i} * |\mu| \xrightarrow{\text{weakly}} f. \tag{2.11}$$

Then for $a > 0$,

$$\frac{1}{w(x_i)}\delta_{x_i} * \delta_a * |\mu| \xrightarrow{\text{weakly}} f * \delta_a. \tag{2.12}$$

Now $\frac{1}{w(x)}\delta_x \xrightarrow{\text{wk*}} 0$, as $x \to \infty$. Hence from (2.12) and by Lemma 2.1. a) we have $f * \delta_a = 0$, since $\delta_a * |\mu| \in L^1(w)$, whence, $f = 0$. Thus weak-$\lim_{x\to\infty} \frac{1}{w(x)}\delta_x * |\mu| = 0$. Therefore,

$$\lim_{x\to\infty} \frac{1}{w(x)} \int_0^\infty w(y)\, d((\delta_x) * |\mu|)(y) = \lim_{x\to\infty} \frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu|(y) = 0,$$

since $f \mapsto \int_0^\infty w(y) f(y)\, dy$ is a continuous linear functional on $L^1(w)$. A similar argument shows that the other limit in part c) has to be 0. Thus,

$$\lim_{x\to 0} \frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu|(y) = \lim_{x\to\infty} \frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu|(y) = 0. \tag{2.13}$$

Now suppose that $L^1(w)$ has a bounded approximate identity. Then if ρ_μ is weakly compact a similar argument to the first case shows that μ is absolutely continuous hence by Remark 1:21, $\mu \in L^1(w)$. Then equivalence of (b) and (c) is in [1, Theorem 2.2].

(c) $\Rightarrow$ (a). Suppose that μ is absolutely continuous and satisfies (2.13). For $n = 1, 2, \ldots$, let $f_n = \chi_{[\frac{1}{n}, n]}\mu$. Then f_n is in $L^1(w)$ and from (2.13) by [1, Theorem 2.2] ρ_{f_n} is a compact multiplier. So it suffices to show that (ρ_{f_n}) converges to ρ_μ in the operator norm topology. Let $\varepsilon > 0$, and from (2.13) choose $[a, b] \subset (0, \infty)$ such that $\frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu|(y) < \varepsilon$ when x is not in $[a, b]$. Then from Corollary 1.17 we have

$$\begin{aligned} \|\rho_\mu - \rho_{f_n}\| &= \sup\left\{\frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu - f_n|(y) : x \in (0, \infty)\right\} \\ &\leqq \varepsilon + \sup\left\{\frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu - f_n|(y) : x \in [a, b]\right\} \end{aligned} \tag{2.14}$$

For $n = 1, 2, \ldots$, define $G_n(x)$ by

$$G_n(x) = \frac{1}{w(x)} \int_0^\infty w(x+y)\, d|\mu - f_n|(y)$$
$$= \frac{1}{w(x)} \left(\int_0^{1/n} w(x+y)\, d|\mu|(y) + \int_n^\infty w(x+y)\, d|\mu|(y) \right).$$

Then each G_n is continuous on $[a, b]$. In fact for x in $[a, b]$ we have

$$G_n(x) = \frac{1}{w(x)} \int_0^\infty w\left(x - \frac{a}{2} + y\right) d(|\mu - f_n| * \delta_{a/2})(y)$$

and $|\mu| * \delta_{a/2} \in L^1(w)$. Thus continuity of G_n follows by the dominated convergence theorem with $w(y)$ as the dominating function. Now $G_{n+1}(x) \leqq G_n(x)$ and G_n converges to 0 pointwise on $[a, b]$. Thus, by Dini's theorem, G_n converges uniformly to 0 on $[a, b]$. Hence from (2.14) it follows that $\rho_{f_n} \to \rho_\mu$, and ρ_μ is compact, as required.

The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (b) are obvious. Thus to complete the proof of the theorem it suffices to show that (a) $\Rightarrow$ (d). If ρ_μ is compact then by an argument analogous to that in the proof of (b) $\Rightarrow$ (c) we see that the set $\left\{\frac{1}{w(x)} \delta_x * \mu : 0 < x < \infty\right\}$ is conditionally compact. Hence the set $K = \left\{\frac{\alpha}{w(x)} \delta_x * \mu : 0 < x < \infty, |\alpha| = 1\right\}$ is conditionally compact. Thus, by Mazur's theorem, the closed convex hull $\overline{co}\, K$ is also compact. Now if v is in the unit ball of $M(w)$ with $\alpha(v) > 0$, then there exists a sequence (v_i) in the unit ball of $M(w)$ with $\alpha(v_i) \geqq \alpha(v)$ and, such that each v_i is a convex combination of measures of the form $\frac{\alpha}{w(x)} \delta_x$, with $0 < x < \infty$ and $|\alpha| = 1$, and $v_i \to v$ in the weak*-topology of $M(w)$. Then it follows that $v * \mu \in \overline{co}\, K$. Hence ρ_μ is compact on $M(w)$ as required.

We recall that a weight function w is called *regulated* at a if for every $y > a$, $\lim_{x \to \infty} \frac{w(x+y)}{w(x)} = 0$. Let $\alpha_w = \inf\{a \in (0, \infty) : w \text{ is regulated on } [a, \infty)\}$. It was shown in [1] that $L^1(w)$ contains non-zero elements f with ρ_f compact if and only if $\alpha_w < \infty$, and in this case $\alpha_w \leqq \alpha(f)$.

It follows from the argument for (c) $\Rightarrow$ (a) in Theorem 2.4:

COROLLARY 2.15. *The ideal of all compact multipliers of $L^1(w)$, is the uniform closure of the regular representation of I_{α_w}, and thus it is a radical Banach algebra.*

REMARK 2.16. The example introduced in Remark 1.23 with $w(x) = e^{-\eta(x)}$ and $\eta(x) = \frac{x^3 - 1}{x}$ can be used to furnish a compact multiplier ρ_f with f outside of $L^1(w)$. First we note that w is regulated at 0 so that every element of $L^1(w)$ acts compactly on $L^1(w)$ by multiplication [1, Theorem 2.2]. Now let f be as in Remark 1.23 and let $f_n = f \cdot \chi_{[1/n,1]}$, $n = 1,2,\dots$. Then $f \in L^1(w)$, whence ρ_{f_n} acts compactly on $L^1(w)$. Now by (1.18)

$$\|\rho_{f_n} - \rho_f\| = \sup\left\{\frac{1}{w(x)}\int_0^{1/n} w(x+y)\,|f(y)|\,dy : x \in (0,\infty)\right\}$$

$$\leqq \int_0^{1/n} |f(y)|\,dy \to 0, \text{ as } n \to \infty,$$

since $f \in L^1(0,1)$. Hence ρ_f is a compact multiplier, with $f \notin L^1(w)$.

3. Derivations.

In this section we give a characterization of derivations of $L^1(w)$. First we note that by [17; Remark 3 (a)] derivations on $L^1(\omega)$ are continuous. If $L^1(w)$ is semisimple, then 0 is the only derivation on $L^1(w)$. Therefore, for the remainder of this section, we assume that $L^1(w)$ is radical.

THEOREM 3.1. *Suppose that $L^1(w)$ is any radical weight. A linear map D is a derivation on $L^1(w)$ if and only if there exists a locally finite measure μ on $[0,\infty)$ such that $D(f) = xf * \mu$, where*

$$\|D\| = \sup\left\{\frac{x}{w(x)}\int_{[0,\infty)} w(x+y)\,d|\mu|(y) : x \in (0,\infty)\right\}.$$

PROOF. For w bounded near 0, this was proved in [7]. So we assume that w is not necessarily bounded. Especially, $L^1(w)$ may or may not have a bounded approximate identity. The first step is to extend D to the multiplier algebra (recall that the multiplier algebra can be identified with a space of measures satisfying (1.18)). We extend D as follows: for every multiplier μ define an operator T_μ on $L^1(w)$ by

$$T_\mu(f) = D(f * \mu) - D(f) * \mu. \tag{3.2}$$

Then a routine calculation shows that for f_1, f_2 in $L^1(w)$, $T_\mu(f_1 * f_2) = T_\mu(f_1) * f_2$. Thus T_μ is a multiplier on $L^1(w)$. Thus, by Corollary 1.17, there exists a measure $\bar{D}(\mu)$ such that $T_\mu(f) = f * \bar{D}(\mu)$, for every $f \in L^1(w)$, or equivalently

$$D(\mu * f) - \mu * D(f) = \bar{D}(\mu) * f, \quad (f \in L^1(w)). \tag{3.3}$$

From (3.3) and by another application of the Titchmarsh convolution theorem it easily follows that $\bar{D}$ is a derivation on the multiplier algebra $\mathrm{Mul}(L^1(w))$ whose restriction to $L^1(\omega)$ is D. Then the remainder of the proof follows the same lines as the proof for the characterization of derivations of $L^1(w)$, when $L^1(w)$ has a bounded approximate identity.

4. Compact derivations.

THEOREM 4.1. *Suppose that D is a derivation on $L^1(w)$ given by $D(f) = xf * \mu$. Then the following are equivalent.*

(a) *D is a compact derivation on $L^1(w)$.*

(b) *D is a weakly compact derivation on $L^1(w)$.*

(c) *μ is absolutely continuous and*

$$\lim_{x\to 0} \frac{x}{w(x)} \int_0^\infty w(x+y)\, d|\mu|(y) = \lim_{x\to\infty} \frac{x}{w(x)} \int_0^\infty w(x+y)\, d|\mu(y) = 0. \tag{4.2}$$

(d) *D has an extension to a compact derivation on $M(w)$.*

(e) *D has an extension to a weakly compact derivation on $M(w)$.*

PROOF. (a) $\Rightarrow$ (b) is obvious. The proof of (b) $\Rightarrow$ (c) follows the same lines as (b) $\Rightarrow$ (c) in the proof of Theorem 2.1. Now we prove that (c) $\Rightarrow$ (d). Suppose that μ is absolutely continuous and (4.2) holds. Then we first show that the set $E = \left\{ \frac{\alpha x}{w(x)} \delta_x * \mu : x \in (0,\infty), |\alpha| \leqq 1 \right\}$ has a norm compact closure. Suppose $\left(\frac{\alpha_n x_n}{w(x_n)} \delta_{x_n} * \mu \right)$ is a sequence of elements of E. By passing to a subsequence we can assume that (α_n) converges to a number α. Now if (x_n) has a subsequence (x_{n_k}) with either $x_{n_k} \to \infty$ or $x_{n_k} \to 0$, then from (4.2) it follows that norm-lim $\frac{\alpha_{n_k} x_{n_k}}{w(x_{n_k})} \delta_{x_{n_k}} * \mu = 0$. Otherwise (x_n) will be bounded away from 0 and ∞. Then there will be a subsequence (x_{n_k}) of (x_n) converging to some positive x. For every $a > 0$, the Radon measure $\delta_a * \mu$ belongs to $L^1(w)$, since $\delta_a * \mu = \frac{1}{a} D(\delta_a)$ and μ was assumed to be absolutely continuous. Hence if $0 < a < x$, then

$$\delta_{x_{n_k}} * \mu = \delta_{x_{n_k} - a} * \delta_a * \mu \xrightarrow{\|\cdot\|} \delta_{x-a} * \delta_a * \mu = \delta_x * \mu,$$

whence norm-lim $\frac{\alpha_n x_n}{w(x_n)} \delta_{x_n} * \mu = \frac{\alpha x}{w(x)} \delta_x * \mu$.

Having shown that in any case $\left(\frac{\alpha_n x_n}{w(x_n)} \delta_{x_n} * \mu \right)$ has a convergent subsequence we conclude that E has a compact closure. Thus by Mazur's theorem the closed convex hull $\overline{\text{co}}(E)$ is compact. Now suppose that ν is any measure in the unit ball of $M(w)$. Then there exists a net (ν_i) with each ν_i a finite sum $\sum \alpha_n \frac{\delta_{x_n}}{w(x_n)}$, ($|\alpha_n| \leqq 1$, $\|\nu_i\| \leqq \|\nu\|$) and $\nu_i * f \xrightarrow{\|\cdot\|} \nu * f$, for every $f \in L^1(w)$. Then from compactness of $\overline{\text{co}}\, E$ and by the Titchmarsh convolution theorem we have $D(\nu) \in \overline{\text{co}}\, E$, which shows that D is compact on $M(w)$.

Since we obviously have (d) $\Rightarrow$ (a), (d) $\Rightarrow$ (e) $\Rightarrow$ (b) $\Rightarrow$ (c), the equivalence of (a), (b), (c), (d) and (e) follows.

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UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2

UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2

POMONA COLLEGE
CLAREMONT
CALIFORNIA 91711-6312