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ON EXCEPTIONAL SETS FOR SUPERHARMONIC FUNCTIONS IN A HALFSPACE: AN INVERSE PROBLEM

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1. Introduction.

The problem of the present paper can be stated in the following way: assume that the set where a potential is large is known to be "small". Can we say anything about the Riesz mass?

This is of course far too general. Let us consider $D = \{x \in \mathbf{R}^p : x_1 > 0\}$ where $x = (x_1, \dots, x_p)$ and $p \geqq 2$. We shall discuss superharmonic functions of the form $u = P\nu + G\mu$, where $P\nu$ is the Poisson integral of a measure ν on the (euclidean) boundary ∂D of D and $G\mu$ is the Green potential of a measure μ on D: we assume that these integrals are convergent. It is known that

$$(P\nu(x) + G\mu(x))/x_1 \to 0, \; x \to \infty \text{ in } D, \; x \in D \backslash E,$$

$$(P\nu(x) + G\mu(x))/|x| \to 0, \; x \to \infty \text{ in } D, \; x \in D \backslash F,$$

where the exceptional set E is minimally thin at infinity in D (cf. [13]) and the exceptional set F is rarefied at infinity in D (cf. [7]).

If $\beta \in [0, 1]$ is given, we define S_β to be the class of positive superharmonic functions $u = P\nu + G\mu$ in D which are such that

$$\int_{\partial D} (1 + |y|)^{1-p-\beta} \, d\nu(y) + \int_D (1 + |y|)^{1-p-\beta} y_1 \, d\mu(y) < \infty,$$

(cf. [7, Definition 4.1]).

It follows from the convergence of the integrals in the representation $u = \mathrm{P}\nu + G\mu$ of u that $u \in S_1$. We quote the following result from [7, Theorem 4.7]:

THEOREM A. *Let $\beta \in [0, 1]$ be given and assume that $D \subset \mathbf{R}^p$. If $u \in S_\beta$, there exists a set $E \subset D$ which is rarefied or minimally thin at infinity in D such that*

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$$\lim u(x)/|x|^\beta = 0,\ x \to \infty,\ x \in D\backslash E,$$

$$\lim u(x)/(x_1|x|^{\beta-1}) = 0,\ x \to \infty,\ x \in D\backslash E,$$

respectively.

Conversely, this result is best possible in the sense that if E is rarefied or minimally thin at infinity in D, then there exists $u \in S_\beta$ such that E is contained in the exceptional set for u.

We wish to discuss the following problem: let u be given and assume that the set $\{x \in D : u(x) > |x|^\beta\}$ or the set $\{x \in D : u(x) > x_1|x|^{\beta-1}\}$ is rarefied or minimally thin at infinity in D. Does it follow that $u \in S_\beta$?

The answer is known to be affirmative for rarefied sets in one special case: if $p = 2$, $\beta = 0$ and $u = G\mu$ where μ is a sum of point masses (cf. J. S. Hwang [11, Lemma 1]).

In his proof, J. S. Hwang uses the following fact:

There exists a sequence $\{R_n\}$ such that $\lim_{n\to\infty} R_n/2^n = 1$ and such that the rarefied set does not intersect the circle $\{|x| = R_n\}$ for any n.

REMARK 1.1. This statement is not true for minimally thin sets, neither for rarefied sets in the case $p \geqq 3$. However, we shall prove a weaker statement (cf. Lemmas 4.2 and 4.3) which we can use in the proof in the general case.

2. The main results.

THEOREM 2.1. *Let $\beta \in [0, 1)$ be given and let $u \in S_1$ be of the form $u = Pv + G\mu$. We assume that there exists a set $E_0 \subset D$ which is minimally thin at infinity in D and which is such that $\mu(D\backslash E_0) = 0$. Then the set $\{x \in D : u(x) > x_1|x|^{\beta-1}\}$ is minimally thin at infinity in D if and only if $u \in S_\beta$.*

The analogous result in the rarefied case is different. To explain the difference, we mention that if $\mathcal{O} \subset \mathrm{R}^p$ is an open set such that $\partial D \subset \mathcal{O}$, then $\mathcal{O} \cap D$ is not rarefied at infinity. At the same time, the set $\{x \in D : 0 < x_1 < 1\}$ is minimally thin at infinity in D.

THEOREM 2.2. *Let $\beta \in [0, 1)$ and let $u = Pv + G\mu \in S_1$. Suppose that there exist sets $E_0 \subset D$ and $F_0 \subset \partial D$ which are such that $\mu(D\backslash E_0) = 0$, $v(\partial D\backslash F_0) = 0$, E_0 is rarefied at infinity in D, and furthermore that*

$$\int_{F_0} (1 + |y|)^{1-p}\, dy < \infty.$$

Then the set $\{x \in D : u(x) > |x|^\beta\}$ is rarefied at infinity in D if and only if $u \in S_\beta$.

For examples showing that our conditions on the sets of concentration of the measures are best possible, we refer to Section 7. In the case of general distributions of mass, the conclusions of Theorems 2.1 and 2.2 need no longer be true. As an example, let us take the superharmonic functions $h(x) = x_1 |x|^{\beta - 1}$ and $u = \frac{1}{2}(h + v)$ where $v \in S_\beta$. The set $\{u > h\} = \{v > h\}$ is minimally thin at infinity in D (cf. Theorem A), but the set $\{u > \frac{1}{2}h\} = D$ is not minimally thin at infinity in D and $u \notin S_\beta$.

For a discussion of the general case where there are no restrictions on the supports of the measures, we refer to Section 8.

In the proofs of Theorems 2.1 and 2.2, we consider first two special cases which we state as

THEOREM 2.3. *Let u and β be as in Theorems 2.1 or 2.2 and assume furthermore that the measures v and μ are sums of point masses. We can then conclude that*

i) *the set $\{x \in D : u(x) > x_1 |x|^{\beta - 1}\}$ is minimally thin at infinity if and only if $u \in S_\beta$;*

ii) *the set $\{x \in D : u(x) > |x|^\beta\}$ is rarefied at infinity if and only if $u \in S_\beta$.*

COROLLARY 2.1. *Let u and β be as in Theorem 2.3. Then*

i) *$u \in S_\beta$ if and only if the set $\{x \in D : u(x) > cx_1 |x|^{\beta - 1}\}$ is minimally thin at infinity for some $c > 0$.*

ii) *$u \in S_\beta$ if and only if the set $\{x \in D : u(x) > c|x|^\beta\}$ is rarefied at infinity in D for some $c > 0$.*

REMARK 2.1. In the corollary, we can replace "some" by "every". It follows that if the set $\{x \in D : u(x) > c|x|^\beta\}$ is rarefied at infinity for some $c > 0$, then such sets will be rarefied at infinity for all $c > 0$.

There is a similar remark for minimally thin sets.

REMARK 2.2. It is possible to generalize our results to functions superharmonic in cones in R^p, $p \geqq 2$. The machinery which we need here can be found in Ronkin [14]. The class analogous to our class S_1 is defined by formulas (49) and (50) in [14, p. 68].

The crucial property of potentials of point masses is that for all $c > 0$, there is no Riesz mass in the complements of the sets $\{x \in D : u(x) > cx_1 |x|^{\beta - 1}\}$ and $\{x \in D : u(x) > c|x|^\beta\}$ with respect to $\bar{D}$.

As corollaries, we obtain certain results on the logarithms of moduli of H^∞-functions in the unit disc U. When $p = 2$, we map D conformally onto U. As a normalization we assume that the point at infinity in D is mapped onto the point $z = 1$. This means that we work with the mapping $z = (w - 1)/(w + 1)$, where w denotes the variable in the D-plane.

Since minimal thinness is defined in terms of minimal harmonic functions (cf. [3, p. 397] or [6, Section 0]), this concept is conformally invariant. There is no such analogue for rarefiedness. Therefore, we shall define a set $S \subset U$ to be *rarefiedly thin* (or *rarefied*) *at* $1 \in \partial U$ if the image $w(S)$ is rarefied at infinity in D. We say that a function f has the rarefied fine limit a at 1 if there is a set $S \subset U$ which is rarefied at 1 such that $f(z) \to a$ whenever $z \to 1$ and $z \in U \backslash S$.

It is well known that if $f \in H^\infty(U)$ with $\|f\|_\infty \leqq 1$, we have the representation formula

$$\log|f(z)| = -(G_0\mu_0(z) + P_0\nu_0(z)), \tag{2.1}$$

where $G_0\mu_0$ is a Green potential of a measure μ_0 on U and $P_0\nu_0$ is the Poisson integral of a measure ν_0 on ∂U. Furthermore, μ_0 is the sum of unit masses at points $\{z_n\}$ in D which are such that $\sum(1 - |z_n|) < \infty$ (cf. [5, Ch. 2]).

Mapping U onto the right half-plane with $z = 1$ going to $w = \infty$, we see that

$$-\log|f((w-1)/(w+1))| = P\nu(w) + G\mu(w) + \alpha w_1, \tag{2.2}$$

where $d\nu_0(e^{i\theta}) = |1 - e^{i\theta}|^2\, d\nu(iv)/2$, $\theta \neq 0$ (the boundary correspondence is given by $v = cot(\theta/2)$ and we have $d\mu_0(z) = d\mu(w)$). Thus the superharmonic function defined by (2.2) will be in S_β, $0 \leqq \beta \leqq 1$, if and only if $\alpha = 0$ and

$$\int_{-\pi}^{\pi} (\sin(\theta/2))^{\beta-1}\, d\nu_0(e^{i\theta}) + \sum_n (1 - |z_n|)|1 - z_n|^{\beta-1} < \infty. \tag{2.3}$$

We obtain the following two corollaries.

COROLLARY 2.2. *Let* $\beta \in [0, 1)$ *and let* $f \in H^\infty(U)$ *satisfy* (2.1). *We assume that there exists a set* $F_1 \subset \partial U$ *such that* $\nu_0(\partial U \backslash F_1) = 0$ *and that* $\int_{F_1} d\theta/|\theta| < \infty$. *Then the function* $|1 - z|^\beta \log|f(z)|$ *has the rarefied fine limit* 0 *at* 1 *if and only if* (2.3) *holds.*

REMARK 2.3. The case $\beta = 0$ is due to J. S. Hwang (cf. [11, Theorem 2]) for the case when f is a Blaschke product. In this situation, condition (2.3) is reduced to

$$\sum_n (1 - |z_n|)|1 - z_n|^{-1} < \infty,$$

which is a classical condition considered by Frostman (cf. [4, Theorem 2.13]; cf. also [6, Section 6]).

We say that a function g has the minimal fine limit a at 1 if there exists a minimally thin set $S \subset U$ such that $g(z) \to a$ whenever $z \to 1$ in U and $z \in U \backslash S$.

COROLLARY 2.3. *Let* $\beta \in [0, 1)$ *and let* $f \in H^\infty(U)$ *satisfy* (2.1). *Then the function* $(1 - |z|)^{-1}|1 - z|^{\beta+1} \log|f(z)|$ *has the minimal fine limit* 0 *at* 1 *if and only if* (2.3) *holds.*

These corollaries are immediate consequences of Theorems 2.2 and 2.1.

3. Basic properties of rarefied and minimally thin sets.

Given $E \subset D$, suppose there exists a measure $\lambda = \lambda_E$ whose Green potential is $G\lambda_E = \hat{R}^E_{x_1}$, where $\hat{R}^E_{x_1}$ is the regularized reduced function of x_1 on E with respect to the cone of positive superharmonic functions on D. We call λ_E *the fundamental distribution* on E. We define $\gamma(E) = \int G\lambda_E(x)\,d\lambda_E(x) = \int x_1\,d\lambda_E(x)$ and call $\gamma(E)$ *the Green energy* of E (cf. [13, p. 129], [7, p. 237]).

Suppose that there exist measures v_1 and μ_1 on ∂D and D, respectively, which are such that

$$\hat{R}^E_1(x) = \int_{\partial D} P(x,y)\,dv_1(y) + \int_D G(x,y)\,d\mu_1(y).$$

The Green mass of E is defined as $\lambda'(E) = v_1(\partial D) + \int_D y_1\,d\mu_1(y)$ (cf. [7, p. 239]).

These two set functions are first defined for compact sets and then extended to capacities defined for general sets in a standard way (cf. [7, p. 243]). They are both monotone and countably subadditive. Furthermore, we have $\lambda(E) = \lambda'(E)$ for any set $E \subset D$ (cf. [7, Lemma 2.5]).

Let $E^{(n)}$ be the intersection of E and the half-annulus $\{x \in D \cup \partial D : 4^n \leqq |x| \leqq 4^{n+1}\}$. Then the set E is minimally thin at infinity in D if and only if (cf. [13])

$$\sum_1^\infty \gamma(E^{(n)})4^{-np} < \infty. \tag{3.1}$$

A set E is rarefied at infinity if and only if (cf. [7])

$$\sum_1^\infty \lambda'(E^{(n)})4^{n(1-p)} = \sum_1^\infty \lambda(E^{(n)})4^{n(1-p)} < \infty. \tag{3.2}$$

REMARK 3.1. i) In (3.1) and (3.2) we work with powers of 4 to define our annuli. We could just as well considered annuli $\{a^n \leqq |x| \leqq a^{n+1}\}$ for any $a > 1$: the definitions are independent of what a we would like to use.

ii) The basic definitions and results on rarefied and minimally thin sets do not depend on the general assumption in [7] that the dimension p is at least 3: everything is correct also in the case $p = 2$.

4. Three lemmas.

LEMMA 4.1. *Let u be harmonic and positive in a half-ball $\{x \in \mathsf{R}^p : |x| < 4\} \cap D$ and vanishing on ∂D. If there exists $y \in D$ such that $|y| \leqq 3$ and $u(y) \leqq y_1$, then there exists a constant C'_p such that*

$$u(x) \leqq C'_p x_1, \quad x \in D, \quad |x| \leqq 2. \tag{4.1}$$

The proof is an elementary consequence of Poisson's formula for a half-ball and is omitted.

After having decomposed $u = Pv + G\mu$ into two superharmonic functions, we shall use the following two lemmas to construct spheres with large radii which do not intersect the exceptional sets for these two functions.

In the sequel we denote by c_p constants, which depend only on the dimension p and which may be different from line to line.

Let $X_n = \{x \in \mathrm{R}^p : 4^n \leqq |x| < 4^{n+1}\}$, $n = 1, 2, \ldots$. If $X = \bigcup_1^\infty X_{2n}$, we consider $v_1 = v|_X$, $\mu_1 = \mu|_X$ and $u_1 = Pv_1 + G\mu_1$.

Lemma 4.2. *Assume that* $\beta \in [0, 1)$ *and that the set* $\{x \in D : u(x) > x_1 |x|^{\beta - 1}\}$ *is minimally thin at infinity. Then there exists a constant* $c_p > 1$ *such that*

$$\{x \in D : u_1(x) > c_p x_1 |x|^{\beta - 1}\} \cap \{|x| = 2 \cdot 4^{2n+1}\} = \emptyset$$

for all sufficiently large values of n.

Proof. The set $E = \{x \in D : u_1(x) > x_1 |x|^{\beta - 1}\}$ is minimally thin at infinity in D. Thus we can cover E by balls $\{B_k\}$ such that $\sum (r_k/R_k)^p < \infty$, where r_k is the radius of B_k and R_k is the distance from the centre of B_k to the origin (cf. [8, Corollary 3, p. 397]).

If $\varepsilon > 0$ is given, there exists N such that we have

$$\sum{}' (r_k/R_k)^p < \varepsilon, \tag{4.2}$$

where we sum over all indices k such that $B_k \cap (\bigcup_{n \geqq N} X_n) \neq \emptyset$.

We wish to cover the half-sphere $H_n = D \cap \{|x| = 2 \cdot 4^{2n+1}\}$ by a finite number of balls of radius at least 4^{2n-2}. The balls must be chosen in such a way that u_1 is harmonic in the intersection of the doubled balls and D.

Let us first cover that part of H_n which is near ∂D by finitely many balls $\{B_k'\}$ of radius 4^{2n} and with centres on $\partial D \cap \bar{H}_n$. Let $B \in \{B_k'\}$, let $x_B \in \partial D$ be the centre of B. Furthermore, if $\varepsilon > 0$ and $n > N$, it follows from (4.2) that B is not covered by $\bigcup B_k$. Thus, there exists $y \in B$ such that $u_1(y) \leqq y_1 |y|^{\beta - 1} \leqq y_1 4^{(2n+1)(\beta - 1)}$. According to Lemma 4.1, we have

$$u_1(x) \leqq C_p' x_1 4^{(2n+1)(\beta-1)} \leqq c_p x_1 |x|^{\beta - 1}, \quad x \in B.$$

It follows that

$$u_1(x) < c_p x_1 |x|^{\beta - 1}, \quad x \in \cup B_k'. \tag{4.3}$$

The next step is to cover the rest of H_n by finitely many balls of radius 4^{2n-1} with centres on H_n. The same argument as above shows that for all large n, there exists in each one of these balls a point y with $u_1(y) \leqq y_1 |y|^{\beta - 1}$. Since the doubles of all these balls are contained in X_{2n+1} where u_1 is harmonic, it follows from

Harnack's inequality that we have $u_1(x) \leqq c_p x_1 |x|^{\beta-1}$ in the union of all these balls and thus on H_n. This concludes the proof of Lemma 4.2.

LEMMA 4.3. *Assume that $\beta \in [0, 1)$ and that the set $\{x \in D : u(x) > |x|^\beta\}$ is rarefied at infinity in D. Then there exists a constant c_p such that we have*

$$\{x \in D : u_1(x) \geqq c_p |x|^\beta\} \cap \{|x| = 2 \cdot 4^{2n+1}\} = \emptyset$$

for all sufficiently large values of n.

PROOF. The set $E = \{x \in D : u_1(x) \geqq |x|^\beta\}$ is rarefied at infinity in D (cf. [7, Theorem 4.6]). Thus there exists a covering of E by balls $\{B_k\}$ such that $\sum (r_k/R_k)^{p-1} < \infty$ (cf. [1] and [8, p. 397]; we use the notations as in the proof of Lemma 4.2.). Consequently, we have also $\sum (r_k/R_k)^p < \infty$, and we can use the same method as in the proof of Lemma 4.2.

Let $B \in \{B'_k\}$, and consider the ball $B^* = \{|x - (x_B + 4^{2n-1}(2,0,\ldots,0)| < 4^{2n-1}\} \subset B$, where x_B is the centre of B. It is clear that for sufficiently large n *the ball B^** is not covered by $\bigcup B_k$. Hence, there exists $y \in B^*$ such that $u_1(y) \leqq |y|^\beta \leqq c''_p y_1 |y|^{\beta-1}$. The rest of the proof is the same as in the proof of Lemma 4.2.

REMARK 4.1. When $p = 2$, the argument can be simplified. In this case a set E rarefied at infinity in D can be covered by discs $\{B_k\}$ such that $\sum r_k/R_k < \infty$. It is now easy to see that there exists sequence $\{R_n\}$ increasing to infinity such that $\lim R_n/4^n = 1$ and such that E does not intersect the circle $\{|x| = R_n\}$ for any n.

5. Proof of Theorem 2.3.

With $\{X_n\}$ and X as in Section 4, we also define $X' = \bigcup_0^\infty X_{2n+1}$. We write $u = u_1 + u_2$, where u_1 is the function associated with the measures $\nu_1 = \nu|_X$ and $\mu_1 = \mu|_X$ and $u_2 = u - u_1$: here the corresponding measures are $\nu_2 = \nu|_{X'}$ and $\mu_2 = \mu|_{X'}$.

Let us first prove Theorem 2.3 ii). The two sets $\{u_1(x) > |x|^\beta\}$ and $\{u_2(x) > |x|^\beta\}$ are rarefied at infinity in D. From now on, we discuss $u_1 = P\nu_1 + G\mu_1$: the details in the proof are similar in the discussion of u_2.

Let $c_p > 1$ be the constant in Lemma 4.3. We consider the two rarefied and relatively open sets $E = \{x \in D : P\nu_1(x) > c_p |x|^\beta\}$ and $F = \{x \in D : G\mu_1(x) > c_p |x|^\beta\}$. According to Lemma 4.3, they can for large n be divided into disjoint subsets

$$E_n = E \cap \{2 \cdot 4^{2n-1} \leqq |x| \leqq 2 \cdot 4^{2n+1}\},$$

$$F_n = F \cap \{2 \cdot 4^{2n-1} \leqq |x| \leqq 2 \cdot 4^{2n+1}\},$$

which are such that $P\nu_1(x) = c_p |x|^\beta$ on $\partial E_n \cap D$ and $G\mu_1(x) = c_p |x|^\beta$ on $\partial F_n \cap D$.

Let us first discuss $\{F_n\}$. Since μ_1 is discrete, $G\mu_1$ is infinite on the set of concentration of μ_1 and thus μ_1 is concentrated on the open set F. From Lemma 4.3, we see that F_n is also open. If λ_n is the fundamental distribution on F_n, we have $G\lambda_n(x) = x_1$ on F_n and λ_n is concentrated on $\partial F_n \cap D$. It follows that

$$\int_{F_n} x_1\, d\mu_1(x) = \int_D G\lambda_n(x)\, d\mu_1(x) = \int_{\partial F_n \cap D} G\mu_1(x)\, d\lambda_n(x)$$

$$= \int_{\partial F_n \cap D} c_p\ |x|^\beta\, d\lambda_n(x) \leqq c4^{2n\beta}\lambda_n(\partial F_n \cap D) = c4^{2n\beta}\lambda_n'(D).$$

Let $F_0 = \bigcup_{n=N}^{\infty} \bar{F}_n$, where N is so large that Lemma 4.3 holds for $n \geqq N$. Our estimate above implies that

$$\int_{F_0} x_1(1 + |x|)^{1-p-\beta}\, d\mu_1(x) \leqq c \sum_{n=N}^{\infty} \lambda_n'(D) 4^{2n(1-p)} < \infty. \tag{5.1}$$

The last sum is finite since F_0 is rarefied at infinity in D (cf. (3.2)). Since $G\mu_1$ is harmonic outside F, we have proved that $G\mu_1 \in S_\beta$.

It remains to study $\{E_n\}$. Let λ_n^0 be the fundamental distribution on E_n. Since E_n is relatively open in D, we know that $(\operatorname{supp} \lambda_n^0) \cap D = \partial E_n \cap D$ and that λ_n^0 has no mass on ∂D.

Let $f = P\nu_1$. We claim that $\hat{R}_f^E = f$. To see this, let v be a nonnegative superharmonic function in D which dominates f on E. Our claim will follow if we can prove that v dominates f on D. Using that ν_1 is a sum of point masses, we define f_j to be the Poisson integral of the first j masses. Since v dominates f on E, v dominates the smaller function f_j on the smaller set $A_j = \{x \in D : f_j(x) \geqq c_p |x|^\beta\}$. Furthermore, f_j vanishes on $\partial D \backslash \partial A_j$. Hence $v - f_j$ is a nonnegative superharmonic function on D (cf. [10, p. 232]). Letting $j \to \infty$, we conclude that $v - f$ is also nonnegative on D which finishes this part of the proof.

Let B_E be the set of points on ∂D where E is not minimally thin. Since $\hat{R}_f^E = f$, it follows that $\nu_1(\partial D \backslash B_E) = 0$ (cf. [2, p. 129]). Furthermore, it follows from [7, Lemma 2.3] that

$$P^*\lambda_n^0(x) := c_p \int_D y_1 |x - y|^{-p}\, d\lambda_n^0(y) = 1, \quad x \in B_E \cap \bar{E}_n.$$

Let ν_n be the restriction of ν_1 to $\bar{E}_n$. Using the two facts above and that $(\operatorname{supp} \nu_1) \subset \partial D \cap \bar{E}$, we deduce that $\nu_n(\partial D) = \nu_n(B_E \cap \bar{E}_n)$ and

$$\nu_n(\partial D) \leqq \int_{\partial D} P^*\lambda_n^0(y)\, d\nu_1(y) = \int_D P\nu_1(x)\, d\lambda_n^0(x) \leqq c_p 4^{2n\beta}\lambda_n^0(D).$$

Let $E_0 = \bigcup_{n=N}^{\infty} \bar{E}_n \cap B_E$, where N is so large that Lemma 4.3 holds for $n \geqq N$. Using the estimate of $\nu_n(\partial D)$ and our assumption that E is rarefied at infinity in D, we deduce

$$\int_{E_0} (1 + |y|)^{1-p-\beta} \, d\nu_1(y) \leqq c_p \sum_{n=N}^{\infty} \lambda_n^0(D) 4^{2n(1-p)} < \infty.$$

Hence $P\nu_1 \in S_\beta$ and we have proved Theorem 2.3 ii).

The proof of Theorem 2.3 i) is similar. After having separated the Riesz masses in the even and the odd annuli, we define the sets $E = \{x \in D : P\nu_1(x) > c_p x_1 |x|^{\beta-1}\}$ and $F = \{x \in D : G\mu_1(x) > c_p x_1 |x|^{\beta-1}\}$. We divide the sets into disjoint subsets $\{E_n\}$ and $\{F_n\}$, each one of these associated with an annulus containing $|x| = 4^{2n}$. Then the sets $\{F_n\}$ can be handled in exactly the same way as before.

Let λ_n be the fundamental distribution on E_n. Arguing in the same way as in the proof of Theorem 2.3 ii), we deduce that

$$\nu_n(\partial D) \leqq \int_{\partial D} P^* \lambda_n(y) \, d\nu_1(y) = \int_D P\nu_1(x) \, d\lambda_n(x) \leqq c_p 4^{2n(\beta-1)} \gamma(E_n).$$

Since E is minimally thin at infinity in D, we know that (3.1) holds. It is now easy to prove that $\nu_1 \in S_\beta$.

6. Proof of Theorems 2.1 and 2.2.

In the proof of Theorems 2.1 and 2.2, it is convenient to introduce the kernel

$$K(y, x) = \begin{cases} G(y, x)/y_1, & \text{on } D \times D \\ c_p x_1 |x - y|^{-p}, & \text{on } \partial D \times D, \end{cases}$$

where $c_p = 2(p-2)$, $p > 2$; $c_2 = 2$. We note that if $x \in D$, then $K(\cdot, x)$ is continuous $\bar{D} \backslash \{x\}$. We write our superharmonic function in the form

$$u(x) = P\nu(x) + G\mu(x) = \int_{\bar{D}} K(y, x) \, d\eta(y) = K\eta(x),$$

where

$$\begin{aligned} d\eta(y) &= y_1 \, d\mu(y), \quad && y \in D, \\ d\eta(y) &= d\nu(y), && y \in \partial D \end{aligned}$$

(cf. [7, Section 2]).

We shall say that two positive real valued functions f and g are comparable and write $f \approx g$ if there exist positive constants $A \leqq B$ such that $Ag \leqq f \leqq Bg$.

Let us first prove Theorem 2.1. We assume that the sets $E = \{x \in D : u(x) > x_1 |x|^{\beta-1}\}$ and E_0 are minimally thin at infinity in D. From the subadditivity of the Green energy [7, Lemma 2.1] and the definition of minimal thinness [7, Definition 3.1], it follows that the set $\mathscr{E} := E \cup E_0 \cup \{x \in D : 0 < x_1 < 1\}$ is also minimally thin at infinity in D. Then there exists an open set $\mathcal{O} \subset D$ such that $\mathcal{O} \supset \mathscr{E}$ and $\mathcal{O}$ is minimally thin at infinity in D [7, Lemma 2.1]. For every $y \in \mathcal{O} \cup \partial D$, we select a ball B_y with centre y and radius $r(y) < 1$ such that $B_y \cap D \subset \mathcal{O}$. It is clear that $\mathcal{O} \cup \partial D \subset \cup B_y$. According to Besicovitch-Ahlfors-Landkof's lemma (see [9, p. 2–6; 12, p. 197]), there exists a countable subfamily of balls $\{B_{y_i}\}$ such that $\mathcal{O} \cup \partial D \subset \cup_i B_{y_i}$ and every $y \in \mathcal{O} \cup \partial D$ belongs to at most $N(p)$ balls B_{y_i}, where $N(p)$ depends only on the dimension p. We need the inequality

$$(6.1) \qquad K(y,x) \geqq c_1 K(y_i,x), \quad x \in D \backslash B_{y_i}, \quad y \in \bar{B}_{y_i} \cap \bar{D}, \quad c_1 = c_1(p).$$

The proof will be given at the end of this section.

At every point y_i, we place the point mass $\eta(B_{y_i})$ which is nonnegative (we might have $\eta(B_{y_i}) = 0$). Let η_1 be the sum of all these point masses and let $u_1(x) = K\eta_1(x)$. If $x \in D \backslash \mathcal{O}$, then

$$u(x) = \int_{\bar{D}} K(y,x)\, d\eta(y) \geqq N(p)^{-1} \sum_i \int_{B_{y_i}} K(y,x)\, d\eta(y)$$

$$\geqq N(p)^{-1} c_1 \sum_i K(y_i,x) \eta(B_{y_i}) = c_2 u_1(x),$$

where $c_2 = c_2(p) = c_1/N(p)$. It follows that

$$u_1(x) \leqq u(x)/c_2 \leqq c_2^{-1} x_1 |x|^{\beta-1}, \quad x \in D \backslash \mathcal{O}.$$

Hence

$$\{x \in D : u_1(x) > c_2^{-1} x_1 |x|^{\beta-1}\} \subset \mathcal{O}.$$

Since $\mathcal{O}$ is minimally thin at infinity in D, we can apply Corollary 2.1 to deduce that $u_1 \in S_\beta$ and thus that $u \in S_\beta$. Here we use that by assumption, $\mu(D \backslash \mathcal{O}) = 0$. Therefore, no mass is lost when we go from η to η_1.

In the opposite direction, Theorem 2.1 follows from Theorem A.

To prove Theorem 2.2, we assume that the sets $E = \{x \in D : u(x) > |x|^\beta\}$ and E_0 are rarefied at infinity in D. Since the set $\mathscr{E}_1 = E \cup E_0$ is rarefied at infinity in D, there exists an open set $\mathcal{O}_1 \subset D$ such that $\mathscr{E}_1 \subset \mathcal{O}_1$ and $\mathcal{O}_1$ is also rarefied at infinity in D [7, *Lemma* 2.4]. Let $\mathcal{O}_2$ be a relatively open subset of ∂D such that $F_0 \subset \mathcal{O}_2$ and

$$\int_{\mathcal{O}_2} (1 + |y|)^{1-p} \, dy < \infty.$$

For every $y \in \mathcal{O}_1 \cup \mathcal{O}_2$, we select a ball B_y with centre y and radius $r(y) < 1$ such that $B_y \subset \mathcal{O}_1$, $y \in \mathcal{O}_1$, and $B_y \cap \partial D \subset \mathcal{O}_2$, $y \in \mathcal{O}_2$. Again, we apply the lemma of Besicovitch, Ahlfors and Landkof and obtain a family of balls $\{B_{y_i}\}$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \cup_i B_{y_i}$ and every $y \in \mathcal{O}_1 \cup \mathcal{O}_2$ belongs to at most $N(p)$ balls B_{y_i}. We see that

$$\sum_{y_i \in \mathcal{O}_2} (r(y_i)/|y_i|)^{p-1} \approx \int_{\mathcal{O}_2} (1 + |y|)^{1-p} \, dy < \infty,$$

and that the set $\mathcal{O}_3 = \{(\bigcup B_{y_i}) \cap D : y_i \in \mathcal{O}_2\}$ is rarefied at infinity in D (cf. [7, Theorem 1.1]). Let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_3$. Then $\mathcal{O}$ is rarefied at infinity in D. By assumption, $\mu(D \backslash \mathcal{O}) = \nu(\partial D \backslash \mathcal{O}) = 0$. The rest of the argument is the same one as in the proof of Theorem 2.1.

It remains to prove (6.1). Assuming that $x \in D \backslash B_{y_i}$ and $y \in \bar{B}_{y_i} \cap \bar{D}$, we see that $|y - x| \leqq |y - y_i| + |y_i - x| \leqq 2|y_i - x|$. Let x' be the reflection of x in the plane $\{x_1 = 0\}$. We note that $|x' - y| \leqq 2|x' - y_i|$. If $p > 2$ and $x = (x_1, x_2, \ldots, x_p)$, then

$$K(y, x) \approx x_1 |x' - y|^{-2} |x - y|^{2-p} \geqq c_3(p) x_1 |x' - y_i|^{-2} |x - y_i|^{2-p} \approx K(y_i, x).$$

If $p = 2$ and y_1 is the first coordinate of y, then

$$K(y, x) = \begin{cases} \dfrac{1}{2y_1} \log\left(1 + \dfrac{4x_1 y_1}{|x - y|^2}\right) \geqq \dfrac{1}{2y_1} \log\left(1 + \dfrac{x_1 y_1}{|x - y_i|^2}\right), y_1 > 0, \\ 2x_1 |x - y|^{-2}, y_1 = 0. \end{cases}$$

Let $y_i = (y_{1i}, \ldots, y_{pi})$. If $0 < y_1 < 2r(y_1) = 2r_1$, then

$$\frac{x_1 y_1}{|y_i - x|^2} \leqq \frac{2(|x_1 - y_{1i}| + y_{1i})r}{|y_i - x| r} \leqq 2 \frac{|x - y_i| + (y_1 + r)}{|y_i - x|} \leqq 2\left(1 + \frac{3r}{r}\right) = 8.$$

Hence, for $y_{1i} > 0$,

$$K(y, x) \geqq c_4 x_1 |y_i - x|^{-2} > c_5 K(y_i, x),$$

where c_5 is an absolute constant.

If $y_1 > 2r$, then $y_1 \approx y_{1i}$ and

$$K(y, x) \geqq \frac{1}{2y_1} \log\left(1 + \frac{x_1 y_1}{|x - y_i|^2}\right) \approx K(y_i, x).$$

Since $K(\cdot, x)$ is continuous on $\bar{D}$, (6.1) holds also in the remaining cases when $y_1 = 0$ or $y_{1i} = 0$.

7. Examples.

To show that our results are best possible, we construct examples using the theory in [7, Section 4]. If $s > 1$ is fixed, we define $I_n = \{x \in \bar{D} : s^n \leqq |x| < s^{n+1}\}$.

PROPOSITION 7.1. *Let $\beta \in [0, 1)$ be given and assume that the set E is not rarefied (or minimally thin) at infinity in D. Then there exists a measure η in D such that* $\operatorname{supp} \eta \subset \bar{E}$, $u = K\eta \in S_1$ *and*

$$|x|^{-\beta} u(x) \to 0 \quad (\text{or } x_1^{-1} |x|^{1-\beta} u(x) \to 0) \quad \text{as } x \to \infty \text{ in } D,$$

but $u \notin S_\beta$.

PROOF. Let h_β be the Poisson integral in D with boundary values $|x|^\beta$ and let $r = |x|$. It is easy to see that $h_\beta(x) \approx |x|^\beta$ in D. Thus there exists a positive constant such that $\hat{R}^E_{r\beta} \leqq$ Const. h_β for any set $E \subset \bar{D}$. In particular we must have $\hat{R}^E_{r\beta} \in S_1$ (cf. [7, Remark 4.2]). Let the measure μ_0 be defined by

$$\hat{R}^E_{r\beta} = \int_{\bar{D}} K(y, \cdot)\, d\mu_0(y).$$

Since E is not rarefied at infinity in D, we have

$$\sum_n \mu_0(\bar{I}_n) s^{-n(p+\beta-1)} = \infty$$

(cf. [7, Theorem 4.4]). We find a sequence of positive numbers $\{\varepsilon_n\}$ in $(0, 1)$ tending to zero such that

$$\sum_n \varepsilon_n \mu_0(\bar{I}_n) s^{-n(p+\beta-1)} = \infty.$$

If μ_n is the restriction of μ_0 to I_n, we define $\eta = \sum_{n=1}^{\infty} \varepsilon_n \mu_n$ and $u(x) = K\eta(x)$. Since we know that $\hat{R}^E_{r\beta} \in S_1$, it is clear that we have also $u \in S_1$.

Furthermore, we have

$$u(x) \leqq \sum_1^N \varepsilon_n \int_D K(y, x)\, d\mu_n(y) + \varepsilon \text{ Const. } h_\beta(x)$$

assuming that $\varepsilon_n \leqq \varepsilon$ for $n > N$. Consequently,

$$\limsup |x|^{-\beta} u(x) \leqq \text{Const. } \varepsilon,$$

and we conclude that $|x|^{-\beta} u(x) \to 0$ as $x \to \infty$. It is also clear that $u \notin S_\beta$.

In the minimally thin case, the same argument works, if we replace [7, Theorem 4.4] by [7, Theorem 4.5] and h_β by $x_1 |x|^{\beta-1}$ which is a positive superharmonic function in D.

PROPOSITION 7.2. *Let $F_0 \subset \partial D$ be a closed set such that $\int_{F_0} (1 + |y|)^{1-p} dy = \infty$. Then there exists a measure ν on ∂D with* $\operatorname{supp} \nu \subset F_0$ *such that $u = P\nu \in S_1$, $|x|^{-\beta} u(x) \to 0$ as $|x| \to \infty$ but $u \notin S_\beta$.*

PROOF. Let $T_n = \{x \in \partial D : 2^{n-1} \leqq |x| < 2^n\}$ and let M_n be the $(p-1)$-dimensional measure of $T_n \cap F_0$. Our condition on F_0 can equivalently be written $\sum_n M_n 2^{n(1-p)} = \infty$. Again, we find a sequence $\{\varepsilon_n\}$ in $(0, 1)$ tending to zero such that

$$\sum_n \varepsilon_n M_n 2^{n(1-p)} = \infty. \tag{7.1}$$

We define a measure ν on ∂D as $d\nu = f d\sigma$, where σ is $(p-1)$-dimensional measure on ∂D and

$$f = \sum_n \varepsilon_n 2^{\beta n} \chi_{F_0 \cap T_n}$$

($\chi_{F_0 \cap T_n}$ is the characteristic function of the set $F_0 \cap T_n$). It is clear that $\operatorname{supp} \nu \subset F_0$.

If $u = P\nu$, we note that $u \leqq$ Const. h_β. Hence we have $u \in S_1$ (cf. [7, Remark 4.2]). Arguing as in the proof of Proposition 7.1, we see that if $\varepsilon_n < \varepsilon$ for $n > N$, then

$$u(x) \leqq \sum_1^N \varepsilon_n 2^{\beta n} P\chi_{F_0 \cap T_n}(x) + \varepsilon \,\text{Const.}\, h_\beta(x).$$

It follows that $|x|^{-\beta} u(x) \to 0$ as $x \to \infty$. It is clear from (7.1) that $u \notin S_\beta$.

From Propositions 7.1 and 7.2, we conclude that the conditions on the measures μ and ν in Theorems 2.1 and 2.2 are best possible.

8. Measures with arbitrary sets of concentration.

Let $\beta \in [0, 1)$ be given and let $u \in S_1$ be of the form $u = P\nu + G\mu = K\eta$. We consider sets of the form $E_c = \{x \in D : u(x) > c x_1 |x|^{\beta-1}\}$ (E_c is a set of the same type as the set E used in Section 5; we forget here the notation E_n used in Section 5).

PROPOSITION 8.1. *Assume that the set E_1 is minimally thin at infinity in D. Then $u \in S_{\beta+\varepsilon}$ for each $\varepsilon > 0$.*

PROOF. We consider the superharmonic function $w(x) = \min\{u(x), x_1 |x|^{\beta-1}\}$. We have $w = u$ in $D \backslash E_1$. It is easy to see that $w \leqq h_\beta$ in D. Since $h_\beta \in S_{\beta+\varepsilon}$ for each

$\varepsilon > 0$, we have also $w \in S_{\beta+\varepsilon}$ for each $\varepsilon > 0$ (cf. [7, pp. 250–251]). This means that we can control the size of the mass η in $D \backslash E_1$.

It remains to estimate the size of the remaining part of η. Let μ_0 be the restriction of μ to E_1. If $u_0 = Pv + G\mu_0 = K\eta_0$, it is clear that

$$E' := \{x \in D : u_0(x) > x_1 |x|^{\beta-1}\} \subset E_1.$$

Since E_1 is minimally thin at infinity in D, μ_0 satisfies the assumptions of Theorem 2.1: we know that $\mu_0(D \backslash E_1) = 0$. Applying Theorem 2.1, we see that $u_0 \in S_\beta$. Hence, we can control the size of the mass η also in $\partial D \cup E_1$, and Proposition 8.1 is proved.

In our next result, we consider sets of the type $F_c = \{x \in D : u(x) > c|x|^\beta\}$.

PROPOSITION 8.2. *Assume that the set F_1 is rarefied at infinity in D. Then $u \in S_{\beta+\varepsilon}$ for each $\varepsilon > 0$.*

PROOF. We consider the superharmonic functions $v = \hat{R}^{E_1}_{r\beta}$ and $w = \min(u, v + h_\beta)$. In $D \backslash F_1$ we have $u \leqq |x|^\beta \leqq v + h_\beta$. It follows that $w = u$ in $D \backslash F_1$. Again, $w \leqq \text{Const.}\, h_\beta$ in D with $h_\beta \in S_{\beta+\varepsilon}$ for each $\varepsilon > 0$ and we must have $w \in S_{\beta+\varepsilon}$ for each $\varepsilon > 0$.

Since F_1 is rarefied at infinity in D, we know according to a result of V. Azarin (cf. [8, p. 397]) that F_1 can be covered by a union of balls $\{B_i = B(x^{(i)}, r_i)\}$ such that

$$\sum_1 (r_i/R_i)^{p-1} < \infty, \tag{8.1}$$

where $R_i = |x^{(i)}|$.

Let $\mathcal{O}$ be the orthogonal projection of $\bigcup B_i$ onto ∂D. Let μ_0 be the restriction of μ to F_1 and let v_0 be the restriction of v to $\mathcal{O}$. If $\mu_1 = \mu - \mu_0$ and $v_1 = v - v_0$, we define $u_0 = Pv_0 + G\mu_0$ and $u_1 = Pv_1 + G\mu_1$. Since $u = u_0 + u_1 \leqq u_0 + w$, it remains to study u_0. It is clear that

$$\{x \in D : u_0(x) > |x|^\beta\} \subset F_1,$$

where F_1 is rarefied at infinity in D. Since μ_0 is concentrated on F_1 and v_1 is concentrated on $\mathcal{O}$ which according to (8.1) satisfies the condition

$$\int_{\mathcal{O}} (1 + |x|)^{1-p}\, dx < \infty,$$

we can apply Theorem 2.2 and conclude that $u_0 \in S_\beta$. We have proved Proposition 8.2.

What can we say about the mass η if we assume that the set

$E_c = \{x \in D : u(x) > cx_1 |x|^{\beta-1}\}$ is minimally thin at infinity in D for all $c > 0$? Let $m(\cdot)$ be Lebesgue measure in R^p and let $n(r) = \eta(B(r))$, where $B(r) = \{x \in \mathsf{R}^p : |x| < r\}$.

PROPOSITION 8.3. *Let $u \in S_1$ and assume that E_c is minimally thin at infinity in D for all $c > 0$. Then*

$$n(r)(1 + r)^{1-p-\beta} \to 0, \quad r \to \infty. \tag{8.2}$$

We note that if E_c is minimally thin at infinity in D, then

$$\lim_{r \to \infty} m(E_c \cap B(r))r^{-p} = 0$$

(cf. [8, Corollary 3]). Thus Proposition 8.3 is a direct consequence of

LEMMA 8.1. *Let $u \in S_1$ and assume that*

$$\limsup_{r \to \infty} n(r)(1 + r)^{1-p-\beta} > 0. \tag{8.3}$$

Then there exists a constant $c > 0$ and a set $E \subset D$ such that

i) $u(x) > cx_1 |x|^{\beta-1}$, $x \in E$,

ii) $\limsup_{r \to \infty} m(E \cap B(r))r^{-p} > 0$.

PROOF. Let $\eta_i = n(2^{i+1}) - n(2^i)$. It is easy to see that (8.3) holds if and only if

$$\limsup_{i \to \infty} \eta_i(1 + 2^i)^{1-p-\beta} > 0.$$

Hence there exists a subsequence $\{i_j\}$ and a positive number d such that

$$\eta_{i_j}(1 + 2^{i_j})^{1-p-\beta} > d > 0, \quad j = 1, 2, \ldots.$$

If $x \in X_{i_j} = \{x \in D : 2^{i_j+2} \leqq |x| < 2^{i_j+3}\}$, then

$$u(x) = K\eta(x) \approx \int_{\bar{D}} |x' - y|^{-2}|x - y|^{2-p} x_1 \, d\eta(y) > c' x_1 2^{-pi_j}\eta_{i_j} \geqq cx_1 |x|^{\beta-1},$$

where the positive constants c' and c depend only on p and d. Now define E by $E = \cup_j X_{i_j}$. We have proved Lemma 8.1 and also Proposition 8.3.

Our next example shows that even if the set $\{x \in D : u(x) > cx_1 |x|^{\beta-1}\}$ is bounded for every $c > 0$, we can not prove anything stronger then (8.2).

PROPOSITION 8.4. *Let $\beta \in [0, 1)$ be given and let $n_0(r)$ be a nondecreasing function such that $n_0(r)(1 + r)^{1-p-\beta} \to 0$ as $r \to \infty$. Then there exists a measure μ in D such that $G\mu \in S_1$ and*

i) $\int_{B(r)} y_1 \, d\mu(y) \approx n_0(r)$,

ii) $x_1^{-1} |x|^{1-\beta} G\mu(x) \to 0$ *as* $|x| \to \infty$.

PROOF. Let $V_i = \{x \in D : 2x_1 > |x|, \ 2^{i-1} \leqq |x| < 2^i\}$. Let μ be the measure supported by $\{x \in D : 2x_1 \geqq |x|\}$ which has constant density $d_i = (n_0(2^i) - n_0(2^{i-1}))2^{-i(p+1)}$ on V_i. It is not difficult to verify that all conditions of Proposition 8.4 are satisfied.

There are similar results for rarefied sets. We omit the details.

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