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# MULTIPLIERS OF IMPRIMITIVITY BIMODULES AND MORITA EQUIVALENCE OF CROSSED PRODUCTS

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The multiplier algebra $M(A)$ of a $C^*$-algebra $A$ is now recognised as an object of fundamental importance in operator algebras (see, e.g., [4], [9, §3.12]). Here we shall show that every $A$-$B$ imprimitivity bimodule $X$ has a *multiplier bimodule* $M(X)$ with analogous properties. Thus, by definition, $M(X)$ will consist of compatible pairs of maps $m_A: A \to X$, $m_B: B \to X$, representing left and right multiplication by the *multiplier* $m = (m_A, m_B)$; we shall characterise $M(X)$ as the universal $A$-$B$ bimodule $M$ containing $X$ as a submodule satisfying $A \cdot M \subset X$ and $M \cdot B \subset X$, just as $M(A)$ is the universal $C^*$-algebra containing $A$ as an essential ideal; and we shall show that appropriately nondegenerate homomorphisms of $X$ into another multiplier bimodule $M(Y)$ extend to strictly continuous homomorphisms of $M(X)$ into $M(Y)$. We shall also prove that we can simultaneously represent $A$ on a Hilbert space $\mathscr{H}$, $B$ on a Hilbert space $\mathscr{K}$, and $X$ as bounded operators from $\mathscr{K}$ to $\mathscr{H}$; if these representations are faithful, then $M(X)$ is faithfully represented as

$$\{T \in B(\mathscr{K}, \mathscr{H}): aT \in X,\ Tb \in X \text{ for all } a \in A, b \in B\}$$

(cf, for example, [9, 3.12.3]).

We were led to these ideas by our interest in recent work of Baaj-Skandalis [1] and Bui [3] on Morita equivalence of crossed products by coactions. A coaction of a locally compact group $G$ on a $C^*$-algebra $A$ is a nondegenerate homomorphism $\delta_A$ of $A$ into the multiplier algebra $M(A \otimes C_r^*(G))$, satisfying several axioms (e.g. [7, §2.1]); a coaction of $G$ on an imprimitivity bimodule should therefore be, *inter alia*, a map $\delta_X$ of $X$ into $M(X \otimes C_r^*(G))$. Baaj and Skandalis worked in the context of (one-sided) Hilbert modules, and used the space $\mathscr{L}_{B \otimes C_r^*(G)}(B \otimes C_r^*(G),\ X \otimes C_r^*(G))$ of adjointable $B \otimes C_r^*(G)$-linear operators rather than $M(X \otimes C_r^*(G))$. When Bui defined Morita equivalence of systems $(A, \delta_A)$ and $(B, \delta_B)$, he also used $\mathscr{L}_{B \otimes C_r^*(G)}(B \otimes C_r^*(G),\ X \otimes C_r^*(G))$. However, in this situation, one might equally well view $X$ as a left Hilbert $A$-module and use $\mathscr{L}_{A \otimes C_r^*(G)}(A \otimes C_r^*(G), X \otimes C_r^*(G))$, and it is not immediately clear that this gives

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the same theory. Our Proposition 1.3 resolves this problem: for any $A$-$B$ imprimitivity bimodule $X$, $\mathscr{L}_B(B, X) \cong M(X) \cong \mathscr{L}_A(A, X)$.

Both Baaj-Skandalis and Bui proved that Morita equivalent systems $(A, \delta_A)$, $(B, \delta_B)$ have Morita equivalent crossed products, by showing that, if $X$ is an $A$-$B$ imprimitivity bimodule with a compatible coaction $\delta_X$ of $G$, then $X \otimes_B (B \times_{\delta_B} G)$ is an $A \times_{\delta_A} G$-$B \times_{\delta_B} G$ imprimitivity bimodule. This formulation has several disadvantages. First, the symmetry of the situation implies that $(A \times_{\delta_A} G) \otimes_A X$ is also an $A \times_{\delta_B} G$-$B \times_{\delta_B} G$ imprimitivity bimodule, but it is not obviously isomorphic to $X \otimes_B (B \times_{\delta_B} G)$, and, second, it is not clear how to describe the action of $A \times_{\delta_A} G$ on $X \otimes_B (B \times_{\delta_B} G)$. We shall obtain a symmetric version of this theorem, by representing $A$ on $\mathscr{H}$, $B$ on $\mathscr{K}$, and $X \otimes C_r^*(G)$ and its multiplier bimodule as operators in $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$; our $A \times_{\delta_A} G$-$B \times_{\delta_B} G$ imprimitivity bimodule is then given by the space

$$\overline{\mathrm{sp}}\,\{\delta_X(x)(1 \otimes M_f) : x \in X, f \in C_0(G)\} \subset B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G)),$$

with module actions and inner products defined using the corresponding actions of $A \times_{\delta_A} G$ and $B \times_{\delta_B} G$ on $\mathscr{H} \otimes L^2(G)$ and $\mathscr{K} \otimes L^2(G)$. (Of course, this turns out to be isomorphic to both $X \otimes_B (B \times_{\delta_B} G)$ and $(A \times_{\delta_A} G) \otimes_A X$.) We feel that our proof is more transparent because we have available the basic properties of multiplier bimodules, which help justify the complicated manipulations which are always necessary when dealing with coactions. As evidence that our formulation is also convenient, we use it to give a relatively short proof of Bui's main theorem, which is the corresponding Morita equivalence for the twisted systems of [10].

When dealing with Morita equivalence, one always has the choice of using the imprimitivity bimodule ${}_AX_B$ or the linking algebra $L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$, and in an appendix we give a brief review of our theory in the context of linking algebras. We show that the multiplier bimodule $M(X)$ can be naturally identified with the top righthand corner in $M(L)$, and discuss how some of our results might be obtained from the analogous properties of $M(L)$. We have preferred to stick with bimodules in the body of the paper, largely because it is bimodules rather than linking algebras which arise in applications (see, for example, [5, 8, 14]), but also because we feel it gives a clearer picture of what is going on.

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## §1. The multiplier bimodule of an imprimitivity bimodule.

In the following, let $A$ and $B$ be $C^*$-algebras, and suppose that $X$ is a complete $A$-$B$ imprimitivity bimodule. In other words, $X$ is an $A$-$B$ bimodule, a full left

Hilbert $A$-module, a full right Hilbert $B$-module, such that $A$ and $B$ act as bounded operators on $X$, and the $A$- and $B$-valued inner products ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_B$ satisfy the compatibility condition

$$ {}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B \text{ for all } x, y, z \in X. $$

We shall sometimes just say that ${}_AX_B$ is an imprimitivity bimodule. We shall frequently use the canonical identifications of $A$ with the algebra $\mathscr{K}_B(X)$ of compact operators on the Hilbert $B$-module $X$, and that of $M(A)$ with $\mathscr{L}_B(X)$.

DEFINITION 1.1. Suppose that $X$ is an $A$-$B$ imprimitivity bimodule. A *multiplier* of $X$ is a pair $m = (m_A, m_B)$, where $m_A : A \to X$ is $A$-linear, $m_B : B \to X$ is $B$-linear, and

$$ m_A(a) \cdot b = a \cdot m_B(b) \text{ for all } a \in A, b \in B. \tag{1.1} $$

We write $M(X)$ for the set of all multipliers of $X$.

As with double centralizers of a $C^*$-algebra, we should think of $m_A$ as the map $a \mapsto a \cdot m$, $m_B$ as the map $b \mapsto m \cdot b$, and (1.1) as the commuting of the module actions. To make this precise, note that every $x \in X$ can be viewed as a multiplier of $X$ via the actions of $A$ and $B$ on $X$; in other words, $x \mapsto (x_A, x_B)$, where $x_A(a) = a \cdot x$, $x_B(b) = x \cdot b$, is a map of $X$ into $M(X)$, and this is an injection because $a \cdot x = 0$ for all $a$ implies

$$ 0 = \|{}_A\langle {}_A\langle x, x\rangle \cdot x, x\rangle\| = \|{}_A\langle x, x\rangle^* \, {}_A\langle x, x\rangle\| = \|x\|^4. $$

We use this injection to identify $X$ with a subspace of $M(X)$. In particular, this allows us to define actions of $A$ and $B$ on $M(X)$ by $a \cdot m = m_A(a)$, $m \cdot b = m_B(b)$, and then $M(X)$ is an $A$-$B$ bimodule. We therefore refer to $M(X)$ as the *multiplier bimodule* of $X$. It can be characterised alternatively as follows:

PROPOSITION 1.2. *Let $X$ be an $A$-$B$ imprimitivity bimodule. Then $M(X)$ is an $A$-$B$ bimodule which satisfies the following two conditions:*

(1) *$A \cdot M(X) \subseteq X$ and $M(X) \cdot B \subseteq X$.*

(2) *If $M$ is any other $A$-$B$ bimodule which contains $X$ and satisfies (1), then there exists a unique bimodule homomorphism $M \to M(X)$ which is the identity on $X$.*

*Moreover, any $A$-$B$ bimodule which contains $X$ and which satisfies Conditions (1) and (2) is isomorphic (as an $A$-$B$ bimodule) to $M(X)$.*

PROOF. Property (1) follows immediately from the definition of the module actions on $M(X)$. If $M$ is any other $A$-$B$ bimodule which contains $X$ and satisfies (1), then we define $\Phi : M \to M(X)$ by $\Phi(m) = (\Phi(m)_A, \Phi(m)_B)$ with $\Phi(m)_A(a) = a \cdot m$ and $\Phi(m)_B(b) = m \cdot b$, and $\Phi$ is a bimodule homomorphism which is the identity on $X$. Suppose now that $\Psi : M \to M(X)$ is another bimodule homomorphism which fixes $X$. Then $(\Phi(m)_A - \Psi(m)_A)(a) = a\cdot m - \Psi(a\cdot m)_A = 0$, and similarly $\Psi(m)_B = \Phi(m)_B$. Thus $\Psi = \Phi$.

Suppose now that $M$ is an $A$-$B$ bimodule which contains $X$ and which satisfies Conditions (1) and (2). Then there exist unique $X$-fixing bimodule homomorphisms $\Phi: M \to M(X)$ and $\Psi: M(X) \to M$. The uniqueness condition implies that $\Psi \circ \Phi$ is the identity on $M$ and $\Phi \circ \Psi$ is the identity on $M(X)$.

We denote by $\mathscr{L}_B(B, X)$ the set of all $B$-linear operators $T: B \to X$ which are adjointable, i.e., for which there is a $B$-linear map $T^*: X \to B$ such that

$$\langle T(b), x\rangle_B = b^*T^*(x) \text{ for all } b \in B, x \in X.$$

It is shown in [6, p. 4] that every $T \in \mathscr{L}_B(B, X)$ is automatically bounded. There are canonical left and right actions of $A$ and $B$ on $\mathscr{L}_B(B, X)$ given by

$$(a\cdot T)(c) = a\cdot(T(c)) \text{ and } (T\cdot b)(c) = T(bc).$$

Moreover, the left action of $B$ gives a canonical embedding $x \mapsto T_x$ of $X$ into $\mathscr{L}_B(B, X)$, i.e., $T_x(b) = x\cdot b$ for $b \in B$; the adjoint is given by $T_x^*(y) = \langle x, y\rangle_B$. Of course, the actions of $A$ and $B$ on the subspace $X$ of $\mathscr{L}_B(B, X)$ agree with the usual ones, so we can view $X$ as an $A$-$B$ submodule of $\mathscr{L}_B(B, X)$.

Similarly, if $\mathscr{L}_A(A, X)$ denotes the set of all $A$-linear adjointable maps from $A$ to $X$, then we have actions of $A$ and $B$ on $\mathscr{L}_A(A, X)$ and $X$ embeds as a submodule of $\mathscr{L}_A(A, X)$.

PROPOSITION 1.3. *Let $X$ be an $A$-$B$ imprimitivity bimodule. Then $\mathscr{L}_B(B, X)$ and $\mathscr{L}_A(A, X)$ satisfy the conditions of Proposition 1.2. Indeed, the maps $m \mapsto m_B$: $M(X) \to \mathscr{L}_B(B, X)$ and $m \mapsto m_A: M(X) \to \mathscr{L}_A(A, X)$, are bimodule isomorphisms which are the identities on the embedded copies of $X$.*

For the proof of Proposition 1.3 we need:

LEMMA 1.4. *Let $m \in M(X)$. Then $m_B \in \mathscr{L}_B(B, X)$, with $m_B^*(x)$ given by the unique element of $B$ satisfying*

$$m_A({}_A\langle z, x\rangle) = z\cdot(m_B^*(x))^* \text{ for all } z \in X. \tag{1.2}$$

*Similarly, $m_A \in \mathscr{L}_A(A, X)$, with $m_A^*(x)$ characterized by*

$$(m_A^*(x))^*\cdot z = m_B(\langle x, z\rangle_B) \text{ for all } z \in X. \tag{1.3}$$

*Moreover, $m_B^*(a\cdot x) = \langle m_A(a^*), x\rangle_B$ and $m_A^*(x\cdot b) = {}_A\langle x, m_B(b^*)\rangle$.*

PROOF. By symmetry, it is enough to prove the statements for $m_B^*$. We first observe that for each $y \in X$ the map $z \to m_A({}_A\langle z, y\rangle)$ is a compact $A$-linear operator on $X$; to see this, factor $y = a\cdot x$ for $a \in A$ and $x \in X$, and observe that $m_A({}_A\langle z, a\cdot x\rangle) = m_A({}_A\langle z, x\rangle a^*) = {}_A\langle z, x\rangle \cdot m_A(a^*)$. Since $B = \mathscr{K}_A(X)$, it follows that there exists a unique element $c \in B$ such that $z \cdot c = m_A({}_A\langle z, y\rangle)$ for all $z \in X$, and we can indeed define a map $m_B^*: X \to B$ by (1.2). For $b \in B$ and $z \in X$, we have

$$z \cdot (m_B^*(y)^* b) = m_A({}_A\langle z, y\rangle) \cdot b = {}_A\langle z, y\rangle \cdot m_B(b) = z \cdot \langle y, m_B(b)\rangle_B,$$

which implies that $m_B^*$ is an adjoint of $m_B$. Finally,

$$z \cdot (m_B^*(a \cdot x))^* = m_A({}_A\langle z, a \cdot x\rangle) = {}_A\langle z, x\rangle \cdot m_A(a^*) = z \cdot \langle x, m_A(a^*)\rangle_B,$$

and hence $m_B^*(a \cdot x) = \langle m_A(a^*), x\rangle_B$.

PROOF OF PROPOSITION 1.3. We show that $\mathscr{L}_B(B, X)$ satisfies Conditions (1) and (2) of Proposition 1.2. For the first, suppose that $T \in \mathscr{L}_B(B, X)$. Then we immediately have $T \cdot b = T(b) \in X$. To prove that $a \cdot T \in X$ for $a \in A$, note that because the module action of $A$ on $\mathscr{L}_B(B, X)$ is continuous, it is enough to consider $a$ of the form ${}_A\langle x, y\rangle$. But then the equation

$$({}_A\langle x, y\rangle \cdot T)(b) = {}_A\langle x, y\rangle \cdot T(b) = x \cdot \langle y, T(b)\rangle_B = x \cdot (T^*(y))^* b$$

implies that $a \cdot T = x \cdot (T^*(y))^*$ is in $X$.

To verify (2), suppose that $M$ is an $A$-$B$ bimodule containing $X$ such that $A \cdot M \subseteq X$ and $M \cdot B \subseteq X$. We have to show that there is a unique bimodule homomorphism $\Phi: M \to \mathscr{L}_B(B, X)$ such that $\Phi(x) = T_x$ for $x \in X$. The uniqueness is easy: if $\Phi$ and $\Psi$ agree on $X$, then for $m \in M$ we have $\Phi(m)(b) = \Phi(m) \cdot b = \Phi(m \cdot b) = \Psi(m \cdot b) = \Psi(m)(b)$ for all $b$, which implies $\Phi(m) = \Psi(m)$. To construct $\Phi$, note that left and right multiplication by $m \in M$ define a multiplier $(m_A, m_B)$ of $X$. Then Lemma 1.4 shows that $m_B \in \mathscr{L}_B(B, X)$, and we can define $\Phi(m) = m_B$, i.e. $\Phi(m)(b) = m \cdot b$.

Notice that if we started with $M = M(X)$, $\Phi(m)$ is by definition just $m_B$, and since $M(X)$ has the same universal properties, $\Phi$ is an isomorphism. (See the last paragraph in the proof of Proposition 1.2.) Similarly, $m_A$ is an isomorphism of $M(X)$ onto $\mathscr{L}_A(A, X)$.

DEFINITION 1.5. Let ${}_AX_B$ be an imprimitivity bimodule. The *strict topology* on the multiplier bimodule $M(X)$ is the topology generated by the seminorms $m \mapsto \|m \cdot b\| = \|m_B(b)\|$ and $m \mapsto \|a \cdot m\| = \|m_A(a)\|$.

REMARK. Let $(u_i)_{i \in I}$ be an approximate identity of $B$ satisfying $u_i = u_i^*$ and $\|u_i\| = 1$. Because $m_A$ and $m_B$ are bounded linear maps, [6, Lemma 1.1.4] implies that $m \cdot u_i$ converges strictly to $m \in M(X)$. Thus $X$ is strictly dense in $M(X)$.

Furthermore, $M(X)$ is complete with respect to the strict topology. To see this, assume that $(m_i)_{i \in I}$ is a strictly Cauchy net in $M(X)$, so that for all $a \in A$ and $b \in B$, $(a \cdot m_i)_{i \in I}$ and $(m_i \cdot b)_{i \in I}$ are Cauchy nets in $X$ with respect to the norm topology. If we define $a \cdot m = \lim_{i \in I} a \cdot m_i$ and $m \cdot b = \lim_{i \in I} m_i \cdot b$, then $m \in M(X)$ and $(m_i)_{i \in I}$ converges strictly to $m$.

PROPOSITION 1.6. *Let ${}_AX_B$ be an imprimitivity bimodule. The module actions and inner products on $X$ extend to separately strictly continuous pairings*

$$M(A) \times M(X) \to M(X), \quad M(X) \times M(B) \to M(X) \text{ and}$$

$$_{M(A)}\langle \cdot,\cdot \rangle : M(X) \times M(X) \to M(A), \quad \langle \cdot,\cdot \rangle_{M(B)} : M(X) \times M(X) \to M(B).$$

*Then $M(X)$ is a left $M(A)$-Hilbert module, right $M(B)$-Hilbert module satisfying $l \cdot \langle m, n \rangle_{M(B)} = {}_{M(A)}\langle l, m \rangle \cdot n$. However, $M(X)$ is not always an $M(A) - M(B)$ imprimitivity bimodule.*

For the proof we need two lemmas.

LEMMA 1.7. *The isomorphisms $m \mapsto m_A$ and $m \mapsto m_B$ of $M(X)$ onto $\mathscr{L}_A(A, X)$ and $\mathscr{L}_B(B, X)$, respectively, are continuous with respect to the strict topology on $M(X)$ and the $*$-strong topologies on $\mathscr{L}_A(A, X)$ and $\mathscr{L}_B(B, X)$, respectively.*

PROOF. Suppose that $m^i$ converges strictly to $m$ in $M(X)$. Then it follows from the definition of the strict topology that $m_B^i(b)$ converges to $m_B(b)$ in norm for all $b \in B$. To see that $(m_B^i)^*(y)$ converges to $m_B^*(y)$ for all $y$, write $y = a \cdot x$. Then using Lemma 1.4 we have $(m_B^i)^*(a \cdot x) = \langle m_A^i(a^*), x \rangle_B$, and this converges to $\langle m_A(a^*), x \rangle_B = m_B^*(a \cdot x)$ by the continuity of $\langle \cdot,\cdot \rangle_B$.

LEMMA 1.8. *Let ${}_AX_B$ be an imprimitivity bimodule and let $m, n \in M(X)$. Using the canonical identifications of $\mathscr{L}_B(B)$, $\mathscr{L}_A(X)$ with $M(B)$ and $\mathscr{L}_A(A)$, $\mathscr{L}_B(X)$ with $M(A)$, we have*

$$m_B^* \circ n_B = n_A \circ m_A^* \text{ in } M(B) \text{ and } m_B \circ n_B^* = n_A^* \circ m_A \text{ in } M(A).$$

PROOF. By symmetry, it is enough to prove the identity $m_B^* \circ n_B = n_A \circ m_A^*$. Let $x \in X$ and $b \in B$. Then from several applications of Lemma 1.4 and the equation $(n_A \circ m_A^*)^* = m_A \circ n_A^*$ we obtain

$$\begin{aligned} x \cdot (m_B^* \circ n_B(b))^* &= x \cdot (m_B^*(n_B(b)))^* = m_A({}_A\langle x, n_B(b) \rangle) \\ &= m_A(n_A^*(x \cdot b^*)) = x \cdot b^*(m_A \circ n_A^*) \\ &= x \cdot b^*(n_A \circ m_A^*)^* = x \cdot ((n_A \circ m_A^*)b)^*, \end{aligned}$$

which implies $m_B^* \circ n_B = n_A \circ m_A^*$.

PROOF OF PROPOSITION 1.6. Define the pairing $(k, m) \mapsto k \cdot m$ of $M(A) \times M(X)$ into $M(X)$ by

$$(k \cdot m)_A(a) = m_A(ak) \text{ and } (k \cdot m)_B(b) = k \cdot m_B(b).$$

Then $(k \cdot m)_A(a) \cdot b = m_A(ak) \cdot b = ak \cdot m_B(b) = a \cdot (k \cdot m)_B(b)$ for all $a$ and $b$, which implies $k \cdot m \in M(X)$. A quick calculation shows that this pairing extends the left action of $A$ on $X$ to a left action of $M(A)$ on $M(X)$. Assume that $k_i$ converges strictly to $k$ in $M(A)$. By factoring $x = a \cdot y$, we can see that $k_i \cdot x$ converges in norm to $k \cdot x$ for all $x \in X$, and this in turn implies that $k_i \cdot m$ converges strictly to $k \cdot m$ in $M(X)$. Similarly, if $m_i$ converges strictly to $m$, then $k \cdot m_i$ converges strictly to $k \cdot m$. The same arguments show that the pairing $M(X) \times M(B) \to M(X)$ defined by

$$(m \cdot l)_A(a) = m_A(a) \cdot l \text{ and } (m \cdot l)_B(b) = m_B(lb), \quad l \in M(B),$$

is separately strictly continuous and extends the action of $B$ on $X$.

We define the inner products ${}_{M(A)}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{M(B)}$ on $M(X)$ by

$$\begin{aligned} {}_{M(A)}\langle m, n \rangle &= m_B \circ n_B^* \in \mathscr{L}_B(X) = M(A), \text{ and} \\ \langle m, n \rangle_{M(B)} &= m_B^* \circ n_B \in \mathscr{L}_B(B) = M(B). \end{aligned}$$

If $x, y \in X \subseteq M(X)$, then $x_B \circ y_B^*(z) = x_B(\langle y, z \rangle_B) = x \cdot \langle y, z \rangle_B = {}_A\langle x, y \rangle \cdot z$ for all $z \in X$, which implies ${}_{M(A)}\langle x, y \rangle = {}_A\langle x, y \rangle$. Similarly, $x_B^* \circ y_B(b) = \langle x, y \cdot b \rangle_B = \langle x, y \rangle_B b$ implies $\langle x, y \rangle_{M(B)} = \langle x, y \rangle_B$. Thus ${}_{M(A)}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{M(B)}$ do extend the $A$- and $B$-valued inner products on $X$. If $m^i$ converges strictly to $m$, then $\langle m^i, n \rangle_{M(B)} b = (m_B^i)^*(n_B(b))$ converges in norm to $m_B^*(n_B(b)) = \langle m, n \rangle_{M(B)} b$ by Lemma 1.7. On the other hand,

$$(b \langle m^i, n \rangle_{M(B)})^* = ((m_B^i)^* \circ n_B)^* b^* = (n_B^* \circ m_B^i) b^* = n_B^*(m_B^i(b)),$$

which converges to $n_B^*(m_B(b)) = (b \langle m, n \rangle_{M(B)})^*$ by the norm continuity of $n_B^*$. Thus $\langle m^i, n \rangle_{M(B)} \to \langle m, n \rangle_{M(B)}$ strictly, and since

$$\langle m, n \rangle_{M(B)}^* = (m_B^* \circ n_B)^* = n_B^* \circ m_B = \langle n, m \rangle_{M(B)},$$

this implies that $\langle \cdot, \cdot \rangle_{M(B)}$ is separately strictly continuous. The separate strict continuity of ${}_{M(A)}\langle \cdot, \cdot \rangle$ now follows from Lemma 1.8 by applying similar arguments to the expressions $n_A \circ m_A^*$ and $n_A^* \circ m_A$.

Since $m_B^* \circ m_B$ and $m_B \circ m_B^*$ are zero if and only if $m_B = 0$, the inner products are positive definite. By writing the positive element $m_B^* \circ m_B$ as $d^2$ for some $d \in M(B)^+$, and computing the norm of $\|d^2\|$ as $\sup_{\|b\| \leq 1} \|(db)^* db\|$, it is easily checked that $\|m_B^* \circ m_B\| = \|m_B\|_{\mathscr{L}_B(B,X)}^2$. Thus $\|m\|_{M(B)}^2 = \|m_B^* \circ m_B\| = \|m_B\|_{\mathscr{L}_B(B,X)}^2 = \|m_B \circ m_B^*\| = {}_{M(A)}\|m\|^2$. Because $\mathscr{L}_B(B, X)$ is complete in the operator norm, the first two equations imply that $M(X)$ is complete in the norm given by the inner products. That $M(X)$ is a left $M(A)$- and a right $M(B)$-Hilbert module now follows from the separate strict continuity of the operations and the strict density of $X$, $A$ and $B$ in $M(X)$, $M(A)$ and $M(B)$. Similarly, we have $l \cdot \langle m, n \rangle_{M(B)} = {}_{M(A)}\langle l, m \rangle \cdot n$ for $l$, $m$, $n \in M(X)$. For the last assertion, see the remark below.

REMARK. For any $A$-$B$ imprimitivity bimodule $X$, denote by $R(A)$ and $R(B)$ the closed linear spans of $\{{}_{M(A)}\langle m, n \rangle : m, n \in M(X)\}$ and $\{\langle m, n \rangle_{M(B)} : m, n \in M(X)\}$ in $M(A)$ and $M(B)$, respectively. Proposition 1.6 says that $M(X)$ is an $R(A)$-$R(B)$

imprimitivity bimodule having $X$ as an $A$-$B$ imprimitivity submodule. If $A$ or $B$ is unital, then Proposition 1.3 implies that $M(X) = X$, $R(A) = A$ and $R(B) = B$. For example, if we regard a Hilbert space $\mathscr{H}$ as a $K(\mathscr{H})$-C imprimitivity bimodule, then $M(\mathscr{H}) = \mathscr{H}$. Thus $M(X)$ is in general not an $M(A)$-$M(B)$ imprimitivity bimodule. On the other hand, it can be: if we view a $C^*$-algebra $A$ as an $A$-$A$ imprimitivity bimodule, then the multiplier bimodule of $A$ coincides with the multiplier algebra $M(A)$ and we have $R(A) = M(A)$ on either side. In Example 2.5 below we describe an $A$-$B$ imprimitivity bimodule $X$ such that $A \neq R(A) \neq M(A)$.

Recall that for any pair of $C^*$-algebras $B$ and $D$, a homomorphism $\Phi: B \to M(D)$ is called *nondegenerate* if $\Phi(B)D$ is dense in $D$. If $\Phi$ is nondegenerate, then $\Phi$ extends uniquely to a strictly continuous homomorphism $\bar{\Phi}: M(B) \to M(D)$ (e.g. [7, Lemma 1.1]). We now show that a similar result holds for multiplier bimodules.

DEFINITION 1.8. An *imprimitivity-bimodule homomorphism* $\Phi: {}_AX_B \to M({}_CY_D)$ is a triple $\Phi = (\Phi_A, \Phi_X, \Phi_B)$, in which $\Phi_A: A \to M(C)$ and $\Phi_B: B \to M(D)$ are homomorphisms, and $\Phi_X: X \to M(Y)$ is a linear map satisfying the following compatibility conditions:

(1) $\Phi_A({}_A\langle x, y\rangle) = {}_{M(C)}\langle \Phi_X(x), \Phi_X(y)\rangle$, $\Phi_B(\langle x, y\rangle_B) = \langle \Phi_X(x), \Phi_X(y)\rangle_{M(D)}$ for all $x, y \in X$, and

(2) $\Phi_X(a \cdot x \cdot b) = \Phi_A(a) \cdot \Phi_X(x) \cdot \Phi_B(b)$ for all $a \in A$, $x \in X$ and $b \in B$.

We say that $\Phi$ is *nondegenerate* if $\Phi_A$ and $\Phi_B$ are nondegenerate.

PROPOSITION 1.9. *Let* $\Phi = (\Phi_A, \Phi_X, \Phi_B): X \to M(Y)$ *be a nondegenerate imprimitivity-bimodule homomorphism. Then there exists a unique strictly continuous extension* $\bar{\Phi} = (\bar{\Phi}_A, \bar{\Phi}_X, \bar{\Phi}_B): M(X) \to M(Y)$, *and the compatibility conditions of Definition 1.8 are still satisfied.*

PROOF. Let $\bar{\Phi}_A$ and $\bar{\Phi}_B$ denote the unique strictly continuous extensions of $\Phi_A$ and $\Phi_B$ to $M(A)$ and $M(B)$, respectively. By Cohen's factorisation theorem we have $D = \Phi_B(B)D$ and $C = C\Phi_A(A)$. Let $m \in M(X)$, $c\Phi_A(a) \in C$ and $\Phi_B(b)d \in D$. Then we aim to define

$$\bar{\Phi}_X(m)_C(c\Phi_A(a)) = c \cdot \Phi_X(a \cdot m) \quad \text{and} \quad \bar{\Phi}_X(m)_D(\Phi_B(b)d) = \Phi_X(m \cdot b) \cdot d.$$

To see, for example, that $\bar{\Phi}_X(m)_D$ is a well defined map of $D$ into $Y$, observe that the right-hand side is bilinear in $(b, d)$ and $B$-balanced, hence determines a linear map of $B \otimes_B D$ into $Y$. Since $b \otimes d \to \Phi_B(b)d$ is an isomorphism of $B \otimes_B D$ onto $D$, it follows that $\bar{\Phi}_X(m)_D$ is well defined.

Routine calculations show that $\bar{\Phi}_X$ is strictly continuous, which implies that $\bar{\Phi}_X$ satisfies the various compatibility conditions. The uniqueness of the strictly continuous extension follows from the strict density of $X$ in $M(X)$.

EXAMPLE 1.10. Suppose that ${}_AX_B$ is an imprimitivity bimodule, and that $\Phi_B: B \to M(D)$ is a nondegenerate homomorphism for some $C^*$-algebra $D$. Let $C$ be the algebra $\mathcal{K}_D(X \otimes_B D)$ of compact operators on the Hilbert $D$-module $X \otimes_B D$. The left action of $A$ on $X$ defines a nondegenerate homomorphism $\Phi_A: A \to M(C) = \mathcal{L}_D(X \otimes_B D)$. We define $\Phi_X: X \to M(X \otimes_B D) \cong \mathcal{L}_D(D, X \otimes_B D)$ by $\Phi_X(x)_D(d) = x \otimes d$. (This is adjointable because $\Phi_X(x)^*_D(y \otimes e) = \Phi_B(\langle x, y\rangle_B)e$ defines an adjoint.) Some quick calculations show that $(\Phi_A, \Phi_X, \Phi_B): X \to M(X \otimes_B D)$ is an imprimitivity-bimodule homomorphism.

Assume now also that ${}_CY_D$ is an imprimitivity bimodule, and that there are a nondegenerate homomorphism $\Phi_B: B \to M(D)$ and a linear map $\Phi_X: X \to M(Y)$ such that $\Phi_B(\langle x, y\rangle_B) = \langle \Phi_X(x), \Phi_X(y)\rangle_{M(D)}$, $\Phi_X(x \cdot b) = \Phi_X(x)\Phi_B(b)$, and $\Phi_X(X)D$ is dense in $Y$. Then the map $x \otimes d \mapsto \Phi_X(x)d$ extends to an isomorphism $X \otimes_B D \cong Y$ of Hilbert $D$-modules, and it follows that there is a nondegenerate homomorphism $\Phi_A: A \to M(C)$ such that $(\Phi_A, \Phi_X, \Phi_B): X \to M(Y)$ is an imprimitivity-bimodule homomorphism.

## §2. Representations of imprimitivity bimodules.

The main purpose of this section is to show that every imprimitivity bimodule ${}_AX_B$ may be represented faithfully as operators on Hilbert space.

DEFINITION 2.1. A *representation* of an $A$-$B$ imprimitivity bimodule $X$ on the pair of Hilbert spaces $(\mathcal{H}, \mathcal{K})$ is a triple $(\pi_A, \pi_X, \pi_B)$ consisting of nondegenerate representations $\pi_A: A \to B(\mathcal{H})$, $\pi_B: B \to B(\mathcal{K})$, and a linear map $\pi_X: X \to B(\mathcal{K}, \mathcal{H})$ satisfying

(1) $\pi_X(x)^*\pi_X(y) = \pi_B(\langle x, y\rangle_B)$ and $\pi_X(x)\pi_X(y)^* = \pi_A({}_A\langle x, y\rangle)$ for $x, y \in X$.
(2) $\pi_X(a \cdot x \cdot b) = \pi_A(a)\pi_X(x)\pi_B(b)$ for $a \in A$, $x \in X$, $b \in B$.

REMARKS. (1) The representations of ${}_AX_B$ on the Hilbert spaces $(\mathcal{H}, \mathcal{K})$ are the nondegenerate imprimitivity-bimodule homomorphisms of $X$ into $M(K(\mathcal{K}, \mathcal{H}))$, where $K(\mathcal{K}, \mathcal{H})$ is viewed as a $K(\mathcal{H}) - K(\mathcal{K})$ imprimitivity bimodule.

(2) Equation (1) together with the equation $\|T\|^2 = \|T^*T\|$ for $T \in B(\mathcal{K}, \mathcal{H})$ implies that $\pi_X$ is isometric iff either $\pi_A$ or $\pi_B$ is isometric, and hence iff either $\pi_A$ or $\pi_B$ is faithful. If so, we say that $\pi$ is faithful.

(3) The map $\pi_X$ is automatically nondegenerate: $\mathrm{sp}\{\pi_X(x)k : x \in X,\ k \in \mathcal{K}\}$ is dense in $\mathcal{H}$ and $\mathrm{sp}\{\pi_X(x)^*h : x \in X,\ h \in \mathcal{H}\}$ is dense in $\mathcal{K}$. For if $h \in \mathcal{H}$ satisfies $(\pi_X(x)k \mid h) = 0$ for all $x, k$, then

$$(\pi_A({}_A\langle x, y\rangle)g \mid h) = (\pi_X(x)(\pi_X(y)^*g) \mid h) = 0$$

for all $x, y \in X$ and $g \in \mathcal{H}$, which implies $h = 0$ by nondegeneracy of $\pi_A$. Thus $\overline{\mathrm{sp}}\{\pi_X(x)k : x \in X,\ k \in \mathcal{K}\} = \mathcal{H}$; the other part follows similarly from the nondegeneracy of $\pi_B$.

(4) We say two representations $(\pi_A, \pi_X, \pi_B)$, $(\rho_A, \rho_X, \rho_B)$ on $(\mathscr{H}_\pi, \mathscr{K}_\pi)$, $(\mathscr{H}_\rho, \mathscr{K}_\rho)$ are *equivalent* if there are unitary transformations $U: \mathscr{H}_\pi \to \mathscr{H}_\rho$, $V: \mathscr{K}_\pi \to \mathscr{K}_\rho$ such that $\rho_X(x) = U\pi_X(x)V^*$. It follows from equation (2) that we then have $\rho_A = \operatorname{Ad} U \circ \pi_A$, $\rho_B = \operatorname{Ad} V \circ \pi_B$.

(5) If $(\pi_A, \pi_X, \pi_B)$ is a representation of ${}_AX_B$ on $(\mathscr{H}, \mathscr{K})$, and we set $\pi_{\tilde{X}}(\tilde{x}) := \pi_X(x)^*$, then $(\pi_B, \pi_{\tilde{X}}, \pi_A)$ is a representation of ${}_B\tilde{X}_A$. Thus if $\pi$ is faithful, and we use $\pi_A, \pi_B$ to view $A, B$ as acting spatially on $\mathscr{H}, \mathscr{K}$, and $\pi_X$ to view $X$ as a subspace of $B(\mathscr{K}, \mathscr{H})$, then we can identify $\tilde{X}$ with the subspace $X^*$ of $B(\mathscr{H}, \mathscr{K})$, with the module actions given by composition of operators, and (e.g.) ${}_B\langle x^*, y^*\rangle = \langle x, y\rangle_B$.

(6) If ${}_AX_B$ is represented on $(\mathscr{H}, \mathscr{K})$ via $\pi = (\pi_A, \pi_X, \pi_B)$, ${}_BY_C$ is represented on $(\mathscr{K}, \mathscr{L})$ via $\rho = (\rho_B, \rho_Y, \rho_C)$, and $\pi_B = \rho_B$, then ${}_A(X \otimes_B Y)_C$ is represented on $(\mathscr{H}, \mathscr{L})$ via $\sigma = (\pi_A, \sigma_{X\otimes_B Y}, \rho_C)$ where $\sigma_{X\otimes_B Y}(x \otimes y) = \pi_X(x)\rho_Y(y) \in B(\mathscr{L}, \mathscr{H})$. If $\pi$ or $\rho$ is faithful, then Remark (2) implies that $\sigma$ is also faithful.

LEMMA 2.2. *The map $(\pi_A, \pi_X, \pi_B) \mapsto \pi_B$ is a bijection between the equivalence classes of representations of the A-B imprimitivity bimodule X and the equivalence classes of nondegenerate representations of B, with inverse given by $\pi \mapsto (\operatorname{Ind}\pi, \rho, \pi)$, where* Ind $\pi$ *denotes the induced representation of A on $X \otimes_B \mathscr{K}$ via the action of A on X, and $\rho(x)k = x \otimes k \in X \otimes_B \mathscr{K}$.*

PROOF. We first claim that $(\operatorname{Ind}\pi, \rho, \pi)$ is a representation of $X$. For $k \in \mathscr{K}$, we have

$$\begin{aligned}\rho(a\cdot x\cdot b)k &= a\cdot x\cdot b \otimes k = (a\cdot x)\otimes \pi(b)k\\ &= \operatorname{Ind}\pi(a)(x\otimes \pi(b)k) = \operatorname{Ind}\pi(a)\rho(x)\pi(b)k,\end{aligned}$$

which gives (1), and the definition of the inner product on $X \otimes_B \mathscr{K}$ implies immediately that $\rho(x)^*\rho(y) = \pi(\langle x, y\rangle_B)$. A quick calculation shows that $\rho(y)^*(z \otimes k) = \pi(\langle y, z\rangle_B)k$, so

$$\begin{aligned}\rho(x)\rho(y)^*(z\otimes k) &= x\otimes \pi(\langle y, z\rangle_B)k = (x\cdot\langle y, z\rangle_B)\otimes k\\ &= ({}_A\langle x, y\rangle\cdot z)\otimes k = \operatorname{Ind}\pi({}_A\langle x, y\rangle)(z\otimes k),\end{aligned}$$

completing the proof of (2). It is easy to check that $\pi \to (\operatorname{Ind}\pi, \rho, \pi)$ respects unitary equivalence, and $(\pi_A, \pi_X, \pi_B) \mapsto \pi_B$ certainly does, so it remains to check that $(\pi_A, \pi_X, \pi_B) \sim (\operatorname{Ind}\pi_B, \rho, \pi_B)$. The formula $U(x \otimes k) := \pi_X(x)k$ defines an isometry $U$ of $X \otimes_B \mathscr{K}$ into $\mathscr{H}$, which is surjective by Remark (3) above, and hence unitary. Straightforward calculations show that $U \operatorname{Ind}\pi_B = \pi_A U$ and $U\rho = \pi_X$, so $(U, 1)$ implements the required equivalence between $(\pi_A, \pi_X, \pi_B)$ and $(\operatorname{Ind}\pi_B, \rho, \pi_B)$.

COROLLARY 2.3. *Every imprimitivity bimodule has a faithful representation.*

PROOF. If $\pi_B$ is a faithful representation of $B$, then each of $\pi_A$ and $\pi_X$ is faithful by Remark (2) above. So the result follows from the lemma.

PROPOSITION 2.4. *Let $\pi = (\pi_A, \pi_X, \pi_B)$ be a representation of an imprimitivity bimodule ${}_AX_B$ on $(\mathscr{H}, \mathscr{K})$. Then there is a unique strict-to-*-strong continuous map $\bar{\pi}_X: M(X) \to B(\mathscr{K}, \mathscr{H})$ which extends $\pi_X$, and is compatible with the canonical extensions of $\pi_A$, $\pi_B$ to $M(A)$ and $M(B)$, respectively. If $\pi$ is faithful, then $\bar{\pi}_X$ is an isometry of $M(X)$ onto*

$$M_\pi(X) = \{T \in B(\mathscr{K}, \mathscr{H}) : T\pi_B(B) \cup \pi_A(A)T \subseteq \pi_X(X)\}.$$

PROOF. We define $\bar{\pi}_X(m)$ by $\bar{\pi}_X(m)(\pi_A(b)k) = \pi_X(m \cdot b)k$; the argument used in Proposition 1.9 shows that $\bar{\pi}_X(m)$ is a well defined linear map. Easy calculations show that $\bar{\pi}_X(m)^*(\pi_A(a)h) = \pi_X(a \cdot m)^*h$ defines an adjoint for $\bar{\pi}_X(m)$. It is clear that $\bar{\pi}_X$ is strictly continuous with range in $M_\pi(X)$.

*If* $\pi_B$ is faithful, then $\bar{\pi}_B$ is isometric, and

$$\|\bar{\pi}_X(m)\|^2 = \|\bar{\pi}_X(m)^*\bar{\pi}_X(m)\| = \|\bar{\pi}_B(\langle m, m\rangle_{M(B)})\| = \|\langle m, m\rangle_{M(B)}\|.$$

Conversely, because $\pi_X$ is bijective, any $T$ in $M_\pi(X)$ defines a pair of maps $(n_A, n_B)$ via

$$\pi_X(n_A(a)) = \pi_A(a)T, \quad \pi_X(n_B(b)) = T\pi_B(b),$$

and $n = (n_A, n_B)$ is easily seen to be a multiplier $n$ of $X$ satisfying $\bar{\pi}_X(n) = T$.

Note that the extension $\bar{\pi}$ in the proposition is identical to the unique strictly continuous extension of $\pi$ regarded as an imprimitivity bimodule homomorphism from ${}_AX_B$ to $M({}_{K(\mathscr{H})}K(\mathscr{K}, \mathscr{H})_{K(\mathscr{K})})$ (see Proposition 1.9).

EXAMPLE 2.5. If $\mathscr{H}, \mathscr{K}$ are two Hilbert spaces then $X = K(\mathscr{K}, \mathscr{H})$ is a $K(\mathscr{H})$-$K(\mathscr{K})$ imprimitivity bimodule. Of course, the identity id is a faithful representation of $X$ on $(\mathscr{H}, \mathscr{K})$, so Proposition 2.4 gives an isomorphism of $M(X)$ onto $M_{\mathrm{id}}(X)$, which in this case is all of $B(\mathscr{K}, \mathscr{H})$. Hence $M(X) = B(\mathscr{K}, \mathscr{H})$. If one of $\mathscr{H}, \mathscr{K}$ is finite dimensional, then we have $M(X) = X$, which then implies that the ranges $R(A)$ and $R(B)$ of the inner products on $M(X)$ (see the remark following Proposition 1.6) are equal to $A = K(\mathscr{H})$ and $B = K(\mathscr{K})$, respectively. If $\mathscr{H}, \mathscr{K}$ have the same infinite dimension, then $M(X) = B(\mathscr{K}, \mathscr{H})$, $R(A) = B(\mathscr{H})$ and $R(B) = B(\mathscr{K})$. Thus $M(X)$ is a $B(\mathscr{H})$-$B(\mathscr{K})$ imprimitivity bimodule. But if $\mathscr{H}, \mathscr{K}$ have different infinite dimensions – say $\mathscr{H}$ is countable – then $R(A)$ is the ideal of $B(\mathscr{H}) = M(K(\mathscr{H}))$ consisting of all operators $T$ such that $\overline{\text{range } T}$ is separable. So in this case we have $A \neq R(A) \neq M(A)$, but $R(B) = M(B)$.

The following lemma will be needed in the next section. All our tensor products of $C^*$-algebras are completed with respect to the *minimal* (or spatial) norm.

LEMMA 2.6. *Let $\pi = (\pi_A, \pi_X, \pi_B)$ be a faithful representation of ${}_AX_B$ on Hilbert spaces $(\mathscr{H}, \mathscr{K})$, and let $\rho$ be a faithful representation of a $C^*$-algebra $C$ on the Hilbert space $\mathscr{L}$. Then there is a faithful representation $\pi \otimes \rho = (\pi_A \otimes \rho, \pi_X \otimes \rho,$*

*$\pi_B \otimes \rho$) of ${}_{A\otimes C}(X \otimes C)_{B\otimes C}$ on $(\mathscr{H} \otimes \mathscr{L}, \mathscr{K} \otimes \mathscr{L})$, such that $\pi_A \otimes \rho$ and $\pi_B \otimes \delta$ are the usual tensor product representations, and $\pi_X \otimes \rho(x \otimes c) = \pi_X(x) \otimes \rho(c)$.*

PROOF. For any $x \in X$, $c \in C$, there is a bounded operator $\pi_X(x) \otimes \rho(c)$: $\mathscr{K} \otimes \mathscr{L} \to \mathscr{H} \otimes \mathscr{L}$, and hence a linear map $\pi_X \otimes \rho : X \odot C \to B(\mathscr{K} \otimes \mathscr{L}, \mathscr{H} \otimes \mathscr{L})$. A calculation using the equality $\pi_B(\langle x, y\rangle_B) = \pi_X(x)^* \pi_X(y)$ shows that $\|\pi_X \otimes \rho(\Sigma_{i=1}^m x_i \otimes c_i)\|^2 = \|\Sigma_{i=1}^m x_i \otimes c_i\|_{B\otimes C}^2$, and hence $\pi_X \otimes \rho$ extends to an isometric linear map of $X \otimes C$ into $B(\mathscr{K} \otimes \mathscr{L}, \mathscr{H} \otimes \mathscr{L})$. Calculations on elementary tensors show that $\pi_X \otimes \rho$ is compatible with $\pi_A \otimes \rho$ and $\pi_B \otimes \rho$.

## §3. Morita equivalent coactions.

We now give our versions of the theorems of Baaj-Skandalis and Bui. Throughout we shall use the conventions of [7, 10] as regards coactions: thus we use the minimal tensor product and the reduced group $C^*$-algebra. The comultiplication $\delta_G$ on $C_r^*(G)$ is the integrated form of the unitary representation $s \mapsto \lambda_s^G \otimes \lambda_s^G$ of $G$, where $\lambda^G$ denotes the left regular representation of $G$ on $L^2(G)$. In the following $\iota$ always denotes the identity homomorphism between algebras, and $M$ the representation of $C_0(G)$ as multiplication operators on $L^2(G)$. We shall find it convenient to distinguish between the function $w_G : s \to \lambda_s^G$ in $M(G, C_r^*(G)))$ and the unitary operator $W_G = M \otimes \iota(w_G)$ on $L^2(G \times G)$.

DEFINITION 3.1 (cf. [1, 2.2], [3, 2.15]). Suppose that $\delta_A : A \to M(A \otimes C_r^*(G))$ and $\delta_B : B \to M(B \otimes C_r^*(G))$ are coactions of a locally compact group $G$ on $C^*$-algebras $A$ and $B$. A *Morita equivalence* between $(A, \delta_A)$ and $(B, \delta_B)$ is an imprimitivity bimodule ${}_AX_B$ together with a linear map $\delta_X : X \to M(X \otimes C_r^*(G))$ satisfying

(1) $(1_{M(A)} \otimes z) \cdot \delta_X(x)$ and $\delta_X(x) \cdot (1_{M(B)} \otimes z)$ lie in $X \otimes C_r^*(G)$ for all $x \in X$, $z \in C_r^*(G)$;

(2) $\delta_X(a \cdot x \cdot b) = \delta_A(a) \cdot \delta_X(x) \cdot \delta_B(b)$ for all $a \in A$, $x \in X$, $b \in B$;

(3) $\delta_A({}_A\langle x, y\rangle) = {}_{M(A\otimes C_r^*(G))}\langle \delta_X(x), \delta_X(y)\rangle$, and $\delta_B(\langle x, y\rangle_B) = \langle \delta_X(x), \delta_X(y)\rangle_{M(B\otimes C_r^*(G))}$ for all $x, y \in X$;

(4) $(\delta_X \otimes \iota) \circ \delta_X = (\iota \otimes \delta_G) \circ \delta_X$.

(In (1) and (2), we implicitly extended the module actions on the $A \otimes C_r^*(G) - B \otimes C_r^*(G)$ imprimitivity bimodule $X \otimes C_r^*(G)$ to actions of the multiplier algebras on the multiplier bimodule; in (3), we extended the inner products to $M(X \otimes C_r^*(G))$; and in (4), we used the strictly continuous extensions of $\delta_X \otimes \iota$ and $\iota \otimes \delta_G$ to make sense of the compositions.)

REMARKS. We claim that this version of Morita equivalence is indeed an equivalence relation. Obviously, $(A, \delta_A)$ is equivalent to itself: take $(X, \delta_X) = (A, \delta_A)$. If $(X, \delta_X)$ is an equivalence between $(A, \delta_A)$ and $(B, \delta_B)$, we can use a faithful representation of ${}_AX_B$ and part (5) of the remark preceding Lemma 2.2 above to identify ${}_B\tilde{X}_A$ with $X^*$. Then, if we represent $X \otimes C_r^*(G)$ faithfully on $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$ using Lemma 2.6,

$$\tilde{X} \otimes C_r^*(G) = X^* \otimes C_r^*(G) = (X \otimes C_r^*(G))^*,$$

and we can define $\delta_{\tilde{X}}: \tilde{X} \to M(\tilde{X} \otimes C_r^*(G))$ by $\delta_{\tilde{X}}(x^*) = \delta_X(x)^*$. It is routine to check that $(\tilde{X}, \delta_{\tilde{X}})$ is an equivalence between $(B, \delta_B)$ and $(A, \delta_A)$. To see transitivity, suppose that $({}_AX_B, \delta_X)$ and $({}_BY_C, \delta_Y)$ are Morita equivalences. We represent ${}_BY_C$ in $B(\mathscr{L}, \mathscr{K})$, and then use Lemma 2.2 to represent ${}_AX_B$ in $B(\mathscr{K}, \mathscr{H})$ for the same space $\mathscr{K}$. Now $X \otimes_B Y$ is represented in $B(\mathscr{L}, \mathscr{H})$, and $(X \otimes_B Y) \otimes C_r^*(G)$ in $B(\mathscr{L} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$. We define

$$\delta_{X \otimes_B Y}(x \otimes y) := \delta_X(x) \circ \delta_Y(y).$$

Routine calculations, using the realisations on Hilbert space, show that $\delta_{X \otimes_B Y}$ satisfies (1) and (2). For (3), we have

$$
\begin{aligned}
{}_{M(A \otimes C_r^*(G))}\langle \delta_X(x)\delta_Y(y), \delta_X(z)\delta_Y(w)\rangle &= \delta_X(x)\delta_Y(y)\delta_Y(w)^*\delta_X(z)^* \\
&= \delta_X(x)\delta_B(\langle y, w\rangle_B)\delta_X(z)^* = \delta_X(x)(\delta_X(z)\delta_B({}_B\langle w, y\rangle))^* \\
&= \delta_X(x)\delta_X(z \cdot {}_B\langle w, y\rangle)^* = \delta_A({}_A\langle x, z \cdot {}_B\langle w, y\rangle\rangle) \\
&= \delta_A({}_A\langle x \otimes y, z \otimes w\rangle).
\end{aligned}
$$

Finally, we check (4). Since the faithful representation of $X \otimes_B Y$ in $B(\mathscr{L}, \mathscr{H})$ takes $x \otimes y$ to $x \circ y$, we have

$$
\begin{aligned}
(\delta_{X \otimes_B Y} \otimes \iota)(\delta_X(x)\delta_Y(y)) &= (\delta_X \otimes \iota)(\delta_X(x)) \circ (\delta_Y \otimes \iota)(\delta_Y(y)) \\
&= (\iota \otimes \delta_G)(\delta_X(x)) \circ (\iota \otimes \delta_G)(\delta_Y(y)) = (\iota \otimes \delta_G)(\delta_X(x)\delta_Y(y)) \\
&= (\iota \otimes \delta_G)(\delta_{X \otimes_B Y}(x \otimes y)).
\end{aligned}
$$

From now on, we shall represent the imprimitivity bimodule ${}_AX_B$ faithfully on $(\mathscr{H}, \mathscr{K})$ and ${}_{A \otimes C_r^*(G)}(X \otimes C_r^*(G))_{B \otimes C_r^*(G)}$ on $(\mathscr{H} \otimes L^2(G), \mathscr{K} \otimes L^2(G))$ using Lemma 2.6. Thus we may identify the multiplier algebras $M(A \otimes C_r^*(G))$, $M(B \otimes C_r^*(G))$ and the multiplier bimodule $M(X \otimes C_r^*(G))$ with their images in $B(\mathscr{H} \otimes L^2(G))$, $B(\mathscr{K} \otimes L^2(G))$ and $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$, respectively. If $\delta_A$ and $\delta_B$ are coactions of $G$ on $A$ and $B$, respectively, then the crossed products $A \times_{\delta_A} G$ and $B \times_{\delta_B} G$ can be realized in $B(\mathscr{H} \otimes L^2(G))$ and $B(\mathscr{K} \otimes L^2(G))$ as the closed linear spans of $\{\delta_A(a)(1 \otimes M_f); a \in A, f \in C_0(G)\}$ and $\{\delta_B(b)(1 \otimes M_f); b \in B, f \in C_0(G)\}$, respectively [7].

THEOREM 3.2 (Baaj-Skandalis, see [1, 6.9], [3, 2.16]). *Suppose* $({}_AX_B, \delta_X)$ *is* *a Morita equivalence between $(A, \delta_A)$ and $(B, \delta_B)$ such that ${}_AX_B$ is faithfully represented on $(\mathscr{H}, \mathscr{K})$. Then the subspace*

$$X \times_{\delta_X} G := \overline{\text{sp}}\,\{\delta_X(x)(1 \otimes M_f) : x \in X, f \in C_0(G)\} \tag{3.1}$$

*of $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$ is an $A \times_{\delta_A} G$-$B \times_{\delta_B} G$ imprimitivity bimodule. Indeed,*

$$X \times_{\delta_X} G = \overline{\text{sp}}\,\{(1 \otimes M_f)\delta_X(x) : x \in X, f \in C_0(G)\}, \tag{3.2}$$

*the algebras $A \times_{\delta_A} G$ act through their faithful representations on $\mathscr{H} \otimes L^2(G)$, $\mathscr{K} \otimes L^2(G)$, respectively, and the inner products are given in terms of the usual adjoint $*$ : $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G)) \to B(\mathscr{H} \otimes L^2(G), \mathscr{K} \otimes L^2(G))$ by*

$$\langle \eta, \zeta \rangle_{B \times_{\delta_B} G} = \eta^*\zeta, \quad {}_{A \times_{\delta_A} G}\langle \eta, \zeta \rangle = \eta\zeta^*. \tag{3.3}$$

*Further, the maps $\phi : x \otimes c \mapsto \delta_X(x) \cdot c$, $\psi : d \otimes x \mapsto d \cdot \delta_X(x)$ extend to $B \times_{\delta_B} G$- resp. $A \times_{\delta_A} G$-linear Hilbert module isomorphisms of $X \otimes_B (B \times_{\delta_B} G)$ onto $X \times_{\delta_X} G$ and $(A \times_{\delta_A} G) \otimes_A \times X$ onto $X \times_{\delta_X} G$, respectively.*

Proof. We begin by proving that $(1 \otimes M_f)\delta_X(x)$ belongs to $X \times_{\delta_X} G$. The argument is basically that used to prove that $(1 \otimes M_f)\delta_A(a)$ belongs to $\overline{\text{sp}}\,\{\delta_A(a)(1 \otimes M_f)\}$ in [7, p. 759], and this fact will also be used later in the proof. By continuity, we may as well suppose that $f \in A(G)$, the Fourier algebra of $G$, and indeed that the functional $f \in L(G)_*$ is given by $T \mapsto (zTh|k)$ for some $z \in C_r^*(G)$ and $h, k \in L^2(G)$ (in other words, if $g \in A(G)$ is given as a functional by $\langle T, g \rangle = (Th|k)$, then $f = g \cdot z$). Since $f = S_f(W_G)$, $M_f$ is characterised spatially by

$$(M_f\xi|\eta) = ((1 \otimes z)W_G(\xi \otimes h)|\eta \otimes k) \quad \text{for } \xi, \eta \in L^2(G).$$

Notice that the equation $\delta_G(z) = W_G(z \otimes 1)W_G^*$ tensors up, first to $X \otimes C_r^*(G)$, and then by strict continuity to $M(X \otimes C_r^*(G))$, to give $(\iota \otimes \delta_G) \circ \delta_X(x) = (1 \otimes W_G)(\delta_X(x) \otimes 1)(1 \otimes W_G)^*$. Thus, for $\xi \in \mathscr{K} \otimes L^2(G)$ and $\eta \in \mathscr{H} \otimes L^2(G)$, we have

$$\begin{aligned}
((1 \otimes M_f)\delta_X(x)\xi|\eta) &= ((1 \otimes 1 \otimes z)(1 \otimes W_G)(\delta_X(x) \otimes 1)\xi \otimes h|\eta \otimes k)\\
&= ((1 \otimes 1 \otimes z)(\iota \otimes \delta_G) \circ \delta_X(x)(1 \otimes W_G)\xi \otimes h|\eta \otimes k)\\
&= ((1 \otimes 1 \otimes z)(\delta_X \otimes \iota) \circ \delta_X(x)(1 \otimes W_G)\xi \otimes h|\eta \otimes k)\\
&= (\delta_X \otimes \iota((1 \otimes z))\delta_X(x))(1 \otimes W_G)\xi \otimes h|\eta \otimes k).
\end{aligned}$$

Since $(1 \otimes z)\delta_X(x) \in X \otimes C_r^*(G)$, we can approximate it in norm by a sum of the form $\Sigma_i x_i \otimes z_i$ in $X \odot C_r^*(G)$. But then we have

$$\Big((1 \otimes M_f)\delta_X(x)\xi \,|\, \eta\Big) \sim \Big(\sum_i ((\delta_X(x_i) \otimes 1)(1 \otimes 1 \otimes z_i))(1 \otimes W_G)\xi \otimes h \,|\, \eta \otimes k\Big)$$
$$= \Big(\sum_i \delta_X(x_i)(1 \otimes M_{g \cdot z_i})\xi \,|\, \eta\Big),$$

and the approximation is uniform in $\xi, \eta$ of norm $\leqq 1$. Thus

$$(3.4) \qquad (1 \otimes M_f)\delta_X(x) \sim \sum_i \delta_X(x_i)(1 \otimes M_{g \cdot z_i}) \in X \times_{\delta_X} G,$$

as claimed.

By symmetry, this claim establishes (3.2) The claim also gives

$$\delta_A(a)(1 \otimes M_f)\delta_X(x) \sim \delta_A(a)\sum_i \delta_X(x_i)(1 \otimes M_{g \cdot z_i}) = \sum_i \delta_X(a \cdot x_i)(1 \otimes M_{g \cdot z_i});$$

because the elements of the form $\delta_A(a)(1 \otimes M_f)$ span a dense subspace of $A \times_{\delta_A} G$, this implies that the subspace $X \times_{\delta_X} G$ of $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$ is closed under left multiplication by elements of $A \times_{\delta_A} G \subset B(\mathscr{H} \otimes L^2(G))$. Similarly, the calculation in [LPRS, p. 759] implies that $X \times_{\delta_X} G$ is closed under right multiplication by elements of $B \times_{\delta_B} G$. Thus $X \times_{\delta_X} G$ is an $A \times_{\delta_A} G - B \times_{\delta_B} G$ bimodule.

We next note that

$$(3.5) \qquad (\delta_X(x)(1 \otimes M_f))^*(\delta_X(y)(1 \otimes M_g)) = (1 \otimes M_{\bar f})(\delta_X(x)^*\delta_X(y))(1 \otimes M_g)$$
$$= (1 \otimes M_{\bar f})\delta_B(\langle x, y\rangle_B)(1 \otimes M_g),$$

so $\eta^*\zeta$ in (3.3) does lie in $B \times_{\delta_B} G$. Since the pairing is defined in terms of the usual adjoint operation on Hilbert space, there is no difficulty in verifying the algebraic properties of $\langle\cdot,\cdot\rangle_{B \times_{\delta_B} G}$, and since the elements of the form (3.5) span a dense subspace of $B \times_{\delta_B} G$, it follows that $X \times_{\delta_B} G$ is a full Hilbert $B \times_{\delta_B} G$ module. Similarly, using instead the characterisation (3.2) of $X \times_{\delta_X} G$, one can verify that it is a full Hilbert $A \times_{\delta_A} G$ module. The compatibility condition

$${}_{A \times_{\delta_A} G}\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_{B \times_{\delta_B} G}$$

amounts to the associativity of composition of operators, and hence $X \times_{\delta_X} G$ is an imprimitivity bimodule, as asserted.

For the last part, we observe that the map $\phi$ is clearly a $B \times_{\delta_B} G$ module homomorphism with dense range, and hence it is enough to check that $\phi$ is isometric for the $B \times_{\delta_B} G$ norm on $X \odot (B \times_{\delta_B} G)$ and the operator norm on $B(\mathscr{K} \otimes L^2(G), \mathscr{H} \otimes L^2(G))$. But for $\Sigma_i x_i \otimes c_i \in X \odot (B \times_{\delta_B} G)$ we have

$$\begin{aligned}\left\|\phi\Big(\sum_i x_i\otimes c_i\Big)\right\|^2 &= \Big\|\sum_{i,j} c_i^*\delta_X(x_i)^*\delta_X(x_j)c_j\Big\|\\ &= \Big\|\sum_{i,j} c_i^*\,\langle\delta_X(x_i),\delta_X(x_j)\rangle_{M(B\otimes C_r^*(G))}c_j\Big\|\\ &= \Big\|\sum_{i,j} c_i^*\delta_B(\langle x_i,x_j\rangle_B)c_j\Big\|\\ &= \Big\|\sum_{i,j}\langle x_i\otimes c_i, x_j\otimes c_j\rangle_{B\times_{\delta_B}G}\Big\|\\ &= \Big\|\sum_{i,j} x_i\otimes c_i\Big\|^2_{X\otimes_B(B\times_{\delta_B}G)}.\end{aligned}$$

Exactly the same argument, but using (3.2) instead of (3.1), shows that $\psi$ is an isomorphism.

Now suppose $N$ is a closed normal amenable subgroup of $G$, and let $q$ denote the quotient map of $C_r^*(G)$ onto $C_r^*(G/N)$, so that $q(\lambda_s^G)=\lambda_{sN}^{G/N}$ for $s\in G$. For groups $H,K$, let $\sigma_{H,K}:C_r^*(H)\otimes C_r^*(K)\to C_r^*(K)\otimes C_r^*(H)$ denote the flip $w\otimes z\mapsto z\otimes w$. As in [10], a *twisted coaction* of $(G,N)$ on $A$ consists of a pair $(\delta_A,W_A)$, where $\delta_A$ is a coaction of $G$ on $A$ and $W_A$ is a unitary in $M(A\otimes C_r^*(G/N))$ satisfying

(1) $(W_A\otimes 1)\,\imath\otimes\sigma_{G/N,G/N}(W_A\otimes 1)=(\imath\otimes\delta_{G/N})(W_A)$;

(2) $(\imath\otimes q)\circ\delta_A(a)=W_A(a\otimes 1)W_A^*$ for all $a\in A$; and

(3) $\delta_A\otimes\imath(W_A)=\imath\otimes\sigma_{G/N,G}(W_A\otimes 1_{C_r^*(G)})$.

A covariant representation of $(A,G,\delta_A)$ is a pair $(\pi,\mu)$ of nondegenerate representations $\pi:A\to B(\mathscr{H})$, $\mu:C_0(G)\to B(\mathscr{H})$ such that

$$\pi\otimes\imath(\delta_A(a))=\mu\otimes\imath(w_G)(\pi(a)\otimes 1)\mu\otimes\imath(w_G^*),$$

and $(\pi,\mu)$ *preserves the twist* $W_A$ if

$$\mu\otimes\imath(w_{G/N})=\pi\otimes\imath(W_A). \tag{3.6}$$

The *twisted crossed product* $A\times_{\delta_A,W_A}G$ is then the quotient of $A\times_{\delta_A}G$ by the ideal

$$I_{W_A}:=\cap\{\ker\pi\times\mu:(\pi,\mu)\text{ is covariant and preserves }W_A\}.$$

Following Bui [3], two twisted coactions $(\delta_A,W_A)$ and $(\delta_B,W_B)$ of $(G,N)$ are *Morita equivalent* if there exists a Morita equivalence $(X,\delta_X)$ for $(A,\delta_A)$ and $(B,\delta_B)$ such that

$$\iota \otimes q(\delta_X(x)) = W_A \cdot (x \otimes 1) \cdot W_B^* \text{ for all } x \in X.$$

Using faithful representations of $X \otimes C_r^*(G)$ and $X \otimes C_r^*(G/N)$ on $(\mathscr{H} \otimes L^2(G), \mathscr{K} \otimes L^2(G))$ and $(\mathscr{H} \otimes L^2(G/N), \mathscr{K} \otimes L^2(G/N))$, one can check that Morita equivalence of twisted actions is an equivalence relation (cf. the remarks following [3, Definition 3.2] and Definition 3.1).

COROLLARY 3.3 (Bui, cf. [3, 3.3]). *Suppose that $(X, \delta_X)$ is a Morita equivalence between the twisted systems $(A, \delta_A, W_A)$ and $(B, \delta_B, W_B)$. Then there is a submodule $E$ of $X \times_{\delta_X} G$ such that $(X \times_{\delta_X} G)/E$ is an $A \times_{\delta_A, W_A} G$-$B \times_{\delta_B, W_B} G$ imprimitivity bimodule.*

PROOF. We show that a covariant representation $(\pi, \mu)$ of $(B, \delta_B)$ preserves the twist $W_B$ if and only if the corresponding induced representation $(\tau, \nu)$ of $(A, \delta_A)$ preserves $W_A$ – indeed, by symmetry, it is enough to show that if $(\pi, \mu)$ preserves $W_B$, then $(\tau, \nu)$ preserves $W_A$. This implies that the ideals $I_{W_B}$ and $I_{W_A}$ are in Rieffel correspondence, so that we can take $E = I_{W_A} \cdot (X \times_{\delta_X} G) = (X \times_{\delta_X} G) \cdot I_{W_B}$ (cf. [Rief2, §3]). We write $X \times_{\delta_X} G = \overline{\operatorname{sp}}\{\delta_X(x)(1 \otimes M_f)\}$, and understand that the closure is in some Hilbert space realisation like that in Theorem 3.2.

Suppose that $(\pi, \mu) : (B, \delta_B) \to B(\mathscr{H})$ preserves $W_B$ and write $Y := X \times_{\delta_X} G$. The induced representation $(\tau, \nu)$ acts in $Y \otimes_{B \times_{\delta_B} G} \mathscr{H}$ via

$$\tau(a)(\delta_X(x)(1 \otimes M_f) \otimes \xi) = \delta_X(a \cdot x)(1 \otimes M_f) \otimes \xi$$
$$\nu(g)(\delta_X(x)(1 \otimes M_f) \otimes \xi) = (1 \otimes M_g)\delta_X(x)(1 \otimes M_f) \otimes \xi,$$

and the key identity is

$$y \cdot (\delta_B(b)(1 \otimes M_f)) \otimes \xi = y \otimes \pi(b)\mu(f)\xi \tag{3.7}$$

for $y \in Y$. To calculate, we use the isomorphism

$$(Y \otimes_{B \times_{\delta_B} G} \mathscr{H}) \otimes L^2(G/N) = (Y \otimes C_r^*(G/N)) \otimes_{(B \times_{\delta_B} G) \otimes C_r^*(G/N)} (\mathscr{H} \otimes L^2(G/N)).$$

Then for $\gamma := (\delta_X(x)(1 \otimes M_f) \otimes \xi) \otimes zk$ in $(Y \otimes \mathscr{H}) \otimes L^2(G/N)$, we have

$$\begin{aligned}
&\tau \otimes \iota(W_A)(\delta_X(x)(1 \otimes M_f) \otimes \xi) \otimes zk \\
&= \tau \otimes \iota(W_A)(\delta_X(x)(1 \otimes M_f) \otimes z) \otimes (\xi \otimes k) \\
&= \delta_X \otimes \iota(W_A \cdot (x \otimes 1))(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= \delta_X \otimes \iota((\iota \otimes q) \circ \delta_X(x) \cdot W_B)(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= \iota \otimes \iota \otimes q((\delta_X \otimes \iota) \circ \delta_X(x))\delta_B \otimes \iota(W_B)(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= \iota \otimes \iota \otimes q((\iota \otimes \delta_G) \circ \delta_X(x))\delta_B \otimes \iota(W_B)(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= \iota \otimes \iota \otimes q((1 \otimes W_G)(\delta_X(x) \otimes 1)(1 \otimes W_G^*))\delta_B \otimes \iota(W_B)(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= (1 \otimes W_{G/N})(\delta_X(x) \otimes 1)(1 \otimes W_{G/N}^*)\delta_B \otimes \iota(W_B)(1 \otimes M_f \otimes z) \otimes (\xi \otimes k).
\end{aligned}$$

At this stage, we observe that if $\zeta$ is any element of $Y \otimes C_r^*(G/N)$, then equations (3.6) and (3.7) give

$$\begin{aligned}
(\zeta(1 \otimes W_{G/N}^*)\delta_B \otimes \iota(W_B))(1 \otimes M_f \otimes z) \otimes (\xi \otimes k) \\
&= \zeta \otimes (\mu \otimes \iota(w_{G/N}^*)\pi \otimes \iota(W_B)(\mu(f) \otimes z)(\xi \otimes k)) \\
&= \zeta \otimes (\mu(f) \otimes z)(\xi \otimes k) \\
&= \zeta(1 \otimes M_f \otimes z) \otimes (\xi \otimes k).
\end{aligned}$$

While this calculation does not immediately apply to our formula for $\tau \otimes \iota(W_A)(\gamma)$, because $\eta = (1 \otimes W_{G/N})(\delta_X(x) \otimes 1)$ lies in $M(Y \otimes C_r^*(G/N))$ rather than $Y \otimes C_r^*(G/N)$, one can see by considering $(\tau \otimes \iota(W_A)(\gamma)|\alpha)$ for $\alpha$ in $(Y \otimes C_r^*(G/N)) \otimes (\mathscr{H} \otimes L^2(G/N))$ that the same manipulation works. (Since $C_0(G)$ acts nondegenerately on $Y$, we can take $\alpha$ of the form $(1 \otimes M_g \otimes w) \cdot \beta$, move $1 \otimes M_g \otimes w$ across the inner product, and note that $\zeta := (1 \otimes M_g \otimes w)^* \eta$ does lie in $Y \otimes C_r^*(G/N)$.) We deduce that

$$\begin{aligned}
\tau \otimes \iota(W_A)(\gamma) &= ((1 \otimes W_{G/N})(\delta_X(x) \otimes 1)(1 \otimes M_f \otimes z)) \otimes (\xi \otimes k) \\
&= v \otimes \iota(w_{G/N})((\delta_X(x)(1 \otimes M_f) \otimes \xi) \otimes zk),
\end{aligned}$$

and hence that $(\tau, v)$ preserves the twist $W_A$.

**Appendix: Linking algebras.**

In this appendix we shall relate the results of the first two sections to the linking algebra of an imprimitivity bimodule ${}_AX_B$. To start with, recall that the *adjoint* imprimitivity bimodule $\tilde{X}$ is the set $X$ with left $B$-action and right $A$-action defined by $b \cdot \tilde{x} = (x \cdot b^*)^\sim$ and $\tilde{x} \cdot a = (a^* \cdot x)^\sim$, where we write $\tilde{x}$ if we view $x \in X$ as an element of $\tilde{X}$. Equipped with the old $A$- and $B$-valued inner products, $\tilde{X}$ is a $B$-$A$ imprimitivity bimodule. Note that $M(\tilde{X})$ is naturally isomorphic to $M(X)^\sim$. The *linking algebra* $L$ for ${}_AX_B$ is $L = \left\{\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} : a \in A,\ x, y \in X, b \in B\right\}$, with multiplication and involution given by

(A1)
$$\begin{pmatrix} a_1 & x_1 \\ \tilde{y}_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ \tilde{y}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A\langle x_1, y_2\rangle & a_1 \cdot x_2 + x_1 \cdot b_2 \\ \tilde{y}_1 \cdot a_2 + b_1 \cdot \tilde{y}_2 & \langle y_1, x_2\rangle_B + b_1b_2 \end{pmatrix}$$
$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ \tilde{x} & b^* \end{pmatrix}.$$

Each element of $L$ acts as an adjointable operator on $X \oplus B$ via

$$\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix}\begin{pmatrix} z \\ c \end{pmatrix} = \begin{pmatrix} a \cdot z + x \cdot c \\ \langle y, z\rangle_B + bc \end{pmatrix}.$$

Indeed, it is shown in [2, §1] that this action defines an isomorphism between $L$ and $\mathcal{K}_B(X \oplus B)$, and hence $L$ is a $C^*$-algebra. We shall often write $L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$.

PROPOSITION A.1. *Let* $L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$ *be the linking algebra for the imprimitivity bimodule* ${}_AX_B$. *Then* $M(L)$ *can be naturally identified with* $\begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix}$, *where we now use Proposition 1.6 to make sense of the formulas (A1) for the multiplication and involution.*

PROOF. $L$ is an ideal in $\begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix}$, because ${}_{M(A)}\langle x, m\rangle \in A$ and $\langle x, m\rangle_{M(B)} \in B$ for all $x \in X$ and $m \in M(X)$ (to see this, factor $x = a \cdot y$, and then $\langle a \cdot y, m\rangle_{M(B)} = \langle y, a^* \cdot m\rangle_{M(B)} \in B$). Thus every element in $\begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix}$ defines a multiplier of $L$. Conversely, let $p = p_A = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $q = q_B = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$. Then $p + q = 1$ and we have canonical isomorphisms $A \cong pLp$, $B \cong qLq$, $X \cong pLq$ and $\tilde{X} \cong qLp$. If $w \in M(L)$, then $pwp$, $pwq$, $qwp$, and $qwq$ define elements $k$, $m$, $\tilde{n}$ and $l$ in $M(A)$, $M(X)$, $M(\tilde{X})$, and $M(B)$ such that $w = \begin{pmatrix} k & m \\ \tilde{n} & l \end{pmatrix}$.

REMARKS. (1) Proposition A.1 suggests that one could alternatively *define* the multiplier bimodule $M(X)$ to be the corner $pM(L)q$. However, that $M(X)$ then has the universal property of Proposition 1.2 is not immediately obvious, and a proof would involve much the same circle of ideas as Propositions 1.2 and 1.3.

(2) From the linking algebra point of view, the nondegenerate imprimitivity-bimodule homomorphisms are in one-to-one correspondence with certain nondegenerate homomorphisms on $L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$, as follows. Suppose $(\Phi_A, \Phi_X, \Phi_B)$: ${}_AX_B \to M({}_CY_D)$, let $K$ be the linking algebra of $Y$, and define $\Phi_L : L \to M(K)$ by

$$\Phi_L \begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} = \begin{pmatrix} \Phi_A(a) & \Phi_X(x) \\ \Phi_X(y)^{\sim} & \Phi_B(b) \end{pmatrix}.$$

It is easy to check that $\Phi_L$ is then a nondegenerate homomorphism. Conversely, given such a homomorphism $\Phi_L$ satisfying $\bar{\Phi}_L(p_A) = p_C$, $\bar{\Phi}(q_B) = q_D$, the triple

$$\left( p_C \Phi_L \begin{pmatrix} \cdot & 0 \\ 0 & 0 \end{pmatrix} p_C,\ p_C \Phi_L \begin{pmatrix} 0 & \cdot \\ 0 & 0 \end{pmatrix} q_D,\ q_D \Phi_L \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix} q_D \right)$$

is a nondegenerate imprimitivity-bimodule homomorphism. Thus we could alternatively prove Proposition 1.8 by extending $\Phi_L$ to a strictly continuous homomorphism $\bar{\Phi}_L: M(L) \to M(K)$, and applying Proposition A.1 to recover $\bar{\Phi}_X$ as the compression of $\bar{\Phi}_L$ to the top right-hand corner $M(X)$ in $M(L)$.

(3) There is a similar one-to-one correspondence between the representations $\pi = (\pi_A, \pi_X, \pi_B)$ of ${}_AX_B$ on $(\mathscr{H}, \mathscr{K})$ and the representations $\pi_L$ of $L$ on $\mathscr{H} \oplus \mathscr{K}$, given by

$$\pi_L\begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} = \begin{pmatrix} \pi_A(a) & \pi_X(x) \\ \pi_X(y)^* & \pi_B(b) \end{pmatrix}.$$

Thus one could obtain Corollary 2.3 by applying the usual Gelfand-Naimark Theorem to $L$.

(4) Two systems $(A, \delta_A)$, $(B, \delta_B)$ are Morita equivalent if and only if there is an $A$-$B$ imprimitivity bimodule $X$ and a coaction $\delta_L$ of $G$ on the linking algebra $L$ which compresses to the given coactions $\delta_A, \delta_B$ on the corners. (To verify this statement, one has to unravel a few hidden identifications, which are legitimate by Proposition A.1.) One can then identify $A \times_{\delta_A} G$, $B \times_{\delta_B} G$ and $X \times_{\delta_X} G$ with the corresponding corners in the crossed product $L \times_{\delta_L} G$: to do this, represent ${}_AX_B$ on $(\mathscr{H}, \mathscr{K})$, $L \otimes C_r^*(G)$ on $(\mathscr{H} \oplus \mathscr{K}) \otimes L^2(G)$, and use the characterisation of the crossed product as $\overline{\mathrm{sp}}\,\{\delta_L(l)(1 \otimes M_f)\}$. This approach would be closer in spirit to that of Baaj-Skandalis; we have preferred to work with $X$ rather than $L$, partly because it is $X$ rather than $L$ which arises in applications, and partly because it minimises the number of identifications one has to make.

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