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AUSLANDER'S DELTA, THE QUASIHOMOGENEITY OF ISOLATED HYPERSURFACE SINGULARITIES AND THE TATE RESOLUTION OF THE MODULI ALGEBRA

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Abstract.

Let $R = S/(f)$ be a complete isolated hypersurface singularity. We show that R is graded if and only if the Tate resolution of the moduli algebra $R/\overline{j(f)}$ is minimal. This criterion is based on the following general result: if the Tate resolution of a cyclic module of infinite projective dimension over a hypersurface ring is minimal then the δ-invariant of (the completion of) that module is zero.

1. Introduction.

The goal of this paper is to establish yet another characterization of quasihomogeneous hypersurface singularities. The fundamental result of K. Saito asserts that a complete hypersurface sungularity is quasihomogeneous if and only if the defining series belongs to its own jacobian ideal (assuming the base field is **C**). This result was later generalized by Scheja and Wiebe to algebraically closed fields of characteristic zero. Recently this author noticed (see Prop. 1.1 of [6]) that the quasihomogeneity of an isolated hypersurface singularity is equivalent to the vanishing of Auslander's δ-invariant of the moduli algebra of the singularity. The δ-invariant of a module is the rank of the largest free summand in the maximal Cohen-Macauly approximation of that module (see [1] and [5] for details on mCM approximations). Based on the above results, we relate, in this paper, the quasihomogeneity of an isolated hypersurface singularity to the minimality of the Tate resolution of its moduli algebra (see the theorem below).

Recall ([8]) that any cyclic module over a commutative noetherian ring R admits a free resolution with an R-algebra structure on it (see [ibid] for terminology). It can be constructed via the process of adjoining variables that kill cycles and is commonly referred to as the Tate resolution. In general it is not minimal (we now assume that R is local). However Theorem 4 of [8] allows to construct a large class of modules whose Tate resolutions are minimal. Namely suppose $R = S/F$, where the ideal F is generated by a regular sequence, $F \subset \underline{m}_S A$,

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where A is another ideal of S generated by a regular sequence and $\underline{m}_S$ is the maximal ideal of S. Then the Tate resolution of the R-module $R/\overline{A}$ (the overbar denotes reduction modulo F) is minimal. Later it was shown by Gulliksen ([3]) that the Tate resolution of the residue field is always minimal. These results lead to an interesting problem: for which cyclic modules are their Tate resolutions minimal? The proof of our theorem rests on a general result, also proved here (see Prop. 2) which gives a necessary condition for the minimality of the Tate resolution over a hypersurface ring. For a module of infinite projective dimension it is formulated in terms of the δ-invariant of its completion. More precisely, the δ-invariant must vanish.

At first glance this connection between multiplicative structures on infinite resolutions and mCM approximations may seem surprising. However it ought to have been expected. Indeed an mCM approximation (at least over a Gorenstein ring) is a construct that relates the beginning of a minimal projective resolution to its infinite part. But a multiplicative structure on a minimal resolution, by its very definitions, does the same thing. This simple-minded observation explains roughly the idea behind this paper. (It would be very interesting to see whether, more generally, the vanishing of the δ-invariant is a necessary condition for the existence of a multiplicative structure on any infinite resolution).

Our main technical tool is a $k[t]$-module structure on a projective resolution of a module over a hypersurface ring. Its existence was proved in [2]. Its relevance to the δ-invariant was explained in [7]. For the convenience of the reader we quickly recall the basic facts about the operator t (see [2] for further details).

Let S be a commutative regular local ring, $x \in S$ and $R := S/(x)$. If $(\mathbf{F}., \partial)$ is a complex of free R-modules let $(\tilde{\mathbf{F}}., \tilde{\partial})$ denote a lifting of it to S. Since $\partial^2 = 0$, we have that $\tilde{\partial}^2 = x\tilde{t}$, where $\tilde{t}$ is a degree -2 endomorphism of the graded S-module $\tilde{\mathbf{F}}$. Let $t := \tilde{t} \otimes R$. It is a degree -2 endomorphism of the complex $\mathbf{F}.$, which is defined uniquely up to homotopy. The corresponding homotopy class will be denoted by the same letter t. One can easily show that the aforementioned homotopy can be chosen to be an R-module homomorphism (see [7]). Therefore, when $(\mathbf{F}., \partial)$ is minimal (i.e., the entries of ∂ belong to the maximal ideal of R) it makes sense to speak of the surjectivity of t (or of any graded piece $t_i: F_{i+2} \to F_i$ of it). Namely a representative of t is surjective if and only if any other representative is. If $(\mathbf{F}., \partial)$ happens to be a minimal projective resolution of an R-module N then the operator t will be denoted $t(N)$. In this case t is surjective if and only if the induced map $t: \mathrm{Tor}_*^R(N, k) \to \mathrm{Tor}_*^R(N, k)$, where k is the residue field R, is surjective. We also observe that, as is easily seen, the operator t commutes with completion.

Our terminology and notation is borrowed from [4]

2. Main theorem and proof.

Let $R = S/(f)$ be a nonregular hypersurface ring, where $S = k[|X_1, \ldots, X_n|]$ is a formal power series ring over an algebraically closed field k of characteristic zero. Let $j(f) \subset S$ denote the jacobian ideal of f and let $\overline{j(f)}$ be its image in R. We call R an isolated singularity if $j(f)$ is an m_s-primary ideal, where $\underline{m}_S = (X_1, \ldots, X_n)$. This is equivalent to saying that the moduli algebra $R/\overline{j(f)}$ is finite-dimensional over k. The ring R is called (positively) graded if there exists a k-derivation $\delta: R \to R$ and a minimal system of generators $x_1, \ldots, x_n$ of the maximal ideal of R such that $\delta x_i = d_i x_i$, where d_i is a positive integer, $i = 1, \ldots, n$ (for affine algebras this definition is equivalent to the usual one). We can now state the main result.

THEOREM 1. *Under the above assumptions, the hypersurface ring R is graded if and only of the Tate resolution of the moduli algebra $R/\overline{j(f)}$ is minimal.*

For the proof we need to establish the following result.

PROPOSITION 2. *Let S be a commutative regular local ring, $x \in S$, $R = S/(x)$ and $\underline{a}$ an ideal of R.*
(a) *If* p.d. $\underline{a} < \infty$ *then the Tate resolution of $R/\underline{a}$ is minimal if and only if $\underline{a}$ is generated by a regular sequence.*
(b) *If* p.d. $\underline{a} = \infty$ *and the Tate resolution of $R/\underline{a}$ is minimal then $\delta(\widehat{R/\underline{a}}) = 0$, where $\widehat{R/\underline{a}}$ is the completion of $R/\underline{a}$.*

REMARK. By Prop. 4.1 of [7], we have that $[\delta(\widehat{R/\underline{a}}) = 0] \Leftrightarrow [t_0(\widehat{R/\underline{a}})$ is surjective]. Since t commutes with completion and the ring R is local, the matter is equivalent to saying that $t_0(R/\underline{a})$ is surjective. Thus the necessary condition (b) can be reformulated in terms of the module $R/\underline{a}$ itself (rather then its completion).

We shall also recall

PROPOSITION 3 (See Prop 1.1 of [6]). *Under the assumptions of the theorem above, the following are equivalent:*

1. *The moduli algebra $R/\overline{j(f)}$ is Gorenstein*
2. $f \in j(f)$
3. $f \in \underline{m}_S(f)$
4. *The maximal Cohen-Macauly approximation of $R/\overline{j(f)}$ has no free summands.*

PROOF OF THE THEOREM. Suppose the Tate resolution of $R/\overline{j(f)}$ is minimal. If p.d. $R/\overline{j(f)}$ is finite then, by Prop. 2, the moduli algebra $R/\overline{j(f)}$ is Gorenstein, which, in view of Prop. 3 and Saito's famous criterion, implies that R is graded. If p.d. $R/\overline{j(f)}$ is infinite then, by Prop. 2 we have that $\delta(R/\overline{j(f)}) = 0$, which is, by Prop. 3 and Saito's criterion, equivalent to saying that R is graded.

Conversely, suppose that R is graded. By Saito's criterion and Prop. 3, we have that $f \in \underline{m}_S j(f)$, i.e., $f = \sum_{i=1}^{n} c_i(\partial f/\partial x_i)$, where $c_i \in \underline{m}_S$. By Theorem 4 of [8], the algebra $R\langle S, T_1, \ldots, T_n\rangle$ with T_i of degree 1, $i = 1, \ldots, n$ and S of degree 2 and with

$$dT_i = \overline{\partial f/\partial x_i} \text{ and } dS = \sum_{i=1}^{n} \overline{c_i} T_i$$

where $\overline{\partial f/\partial x_i}$ and $\overline{c_i}$ are the images of $\partial f/\partial x_i$ and, respectively, c_i in R, is a free resolution of the R-module $R/\overline{j(f)}$. Since $\overline{c_i} \in \underline{m}_R$, we see that the Tate resolution of $R/\overline{j(f)}$ is minimal. The theorem is proved.

3. Proof of Proposition 2.

We begin with Part (a). If $\underline{a}$ is generated by a regular sequence then the minimal resolution of $R/\underline{a}$ is given by the Koszul complex on this sequence, which is a Tate resolution. Conversely, suppose that the Tate resolution of $R/\underline{a}$ is minimal. Let $T_1, \ldots, T_n$ be all the adjoint variables in degree 1. They correspond to a minimal set of generators of $\underline{a}$. The algebra $R\langle T_1, \ldots, T_n\rangle$ is just the Koszul complex on those variables. This complex has no homology in degree 1, otherwise we would have to adjoin a variable in degree 2, which would make the projective dimension of $R/\underline{a}$ infinite, contrary to the assumption. Therefore the Koszul complex, being acyclic in degree 1, is acyclic everywhere and, as is well known, in this case $\underline{a}$ is generated by a regular sequence. Part (a) is proved.

In the rest of this section we shall prove Part (b). According to the remark after the statement of the theorem, it suffices to show that the degree 0 part $t_0(R/\underline{a})$ of the operator t of R/a is surjective. Now we turn again to the process of adjoining variables to kill cycles. Since p.d. $R/\underline{a} = \infty$, we must adjoin at least one variable U in degree 2 after we have adjoined the degree 1 variables $T_1, \ldots, T_n$ corresponding to a minimal set of generators of $\underline{a}$. The resulting algebra $R\langle U, T_1, \ldots, T_n\rangle$ is a complex

$$\ldots \to X_m \to \ldots \to X_1 \to X_0$$

with the property that $X_i \neq 0$ for all $i \geqq 0$. By [2], minimal resolutions over R become periodic after at most $\dim R + 1$ steps. This implies that no other variable of even degree can ever be adjoined. Otherwise, since all X_i have nonzero ranks, the betti numbers of $R/\underline{a}$ would be unbounded. Thus only odd degree variables can be adjoined to $R\langle U, T_1, \ldots, T_n\rangle$. Moreover, since the betti numbers are bounded, we can only have a finite number of those variables. Let V be the last one adjoined. The Tate resolution can then be written as

$$\begin{array}{ccccccccccccc} & \ldots \to & Y_2 V & \to & Y_1 V & \to & Y_0 V & & & & & & \\ & & & \searrow & & \searrow & & \searrow & & & & & \\ & & \ldots \to & Y_{i+1} & \to & Y_i & \to & Y_{i-1} & \to & \ldots & \to & Y_1 & \to Y_0 \end{array} \tag{3.1}$$

where the lower row is just the result of adjoining all other variables to R. By [2], for any module over a hypersurface ring the operator t is eventually surjective (more precisely, $t_i : F_{i+2} \to F_i$ is the class of the identity map for $i \geqq \dim R + 1$). But then the operator t for the top row is eventually surjective. (Indeed, since the surjectivity does not depend on the choice of a representative, we can choose a lifing of the differential in (3.1) for which all the maps from the bottom row to the top one are still zero). However the top row is just a copy of the bottom row and therefore the operator t for the bottom row is also eventually surjective. We can now repeat this argument with the variable adjoined just before V and, repeating this process of "killing the odd degree variables and resurrecting cycles" we will, at some point, come back to the algebra $R\langle U, T_1, \ldots, T_n\rangle$. According to our argument, the operator t for this complex is eventually surjective.

In order to compute this operator explicitly, we shall now take a closer look at the differential of $R\langle U, T_1, \ldots, T_n\rangle$. Let $a'_1, \ldots, a'_n$ be a minimal generating set of $\underline{a}$ and $a_1, \ldots, a_n$ their liftings to S. Let the variable U correspond to the cycle $u := \Sigma_{i=1}^n \lambda'_i T_i$, where $\lambda'_i \in \underline{m}_R$, $i = 1, \ldots, n$ and let $\lambda_i \in \underline{m}_S$ be a lifting of λ'_i to S, $i = 1, \ldots, n$. Since u is a cycle, we have that $\Sigma_{i=1}^n \lambda'_i a'_i = 0$ and therefore $\Sigma_{i=1}^n \lambda_i a_i = \varepsilon x$ for some $\varepsilon \in S$.

For our purposes, it is convenient to switch to the notation introduced in [6]. More precisely, let X_m denote the degree m part of $X := R\langle U, T_1, \ldots, T_n\rangle$. Then the set

$$\{T_{i_1} T_{i_2} \ldots T_{i_l} U^{(p)} \mid i_1 < i_2 < \ldots < i_l, l + 2p = m\}$$

where l, p, and m are nonnegative integers, is a basis of the free R-module X_m. Let $I : [l] \to [n]$ be an increasing map from the set $[l]$ of integers $\{0, 1, \ldots, l\}$ to the set of integers $[n] = \{0, \ldots, n\}$. Then the aforementioned basis elements can be symbolically written as $T_I U^{(p)}$. Let I_j denote the map $[l-1] \to [n]$ defined, for $q = 1, \ldots, l-1$, by the formulas

$$I_j(q) := I(q), \text{ if } q < j$$

$$I_j(q) := I(q+1), \text{ if } q \geqq j$$

and let CI denote the complement $[n] \backslash Im(I)$ of the image of I in $[n]$. Let also d (respectively, d_{m-1}) denote the differential of X (respectively, its homogeneous part $X_m \to X_{m-1}$). Then

$$d_0 T_i = a'_i, i = 1, \ldots, n$$

$$d_1 U = \Sigma_{i=1}^{n} \lambda'_i T_i$$

and a trivial calculation shows that

$$d_{m-1}(T_I U^{(p)}) = \Sigma_{j\in[l]}(-1)^{j-1} a'_{I(j)} T_{I_j} U^{(p)} + (-1)^l T_I(\Sigma_{k\in[n]} \lambda'_k T_k) U^{(p-1)}$$

$$= \Sigma_{j\in[l]}(-1)^{j-1} a'_{I(j)} T_{I_j} U^{(p)} + (-1)^l \Sigma_{k\in CI} \lambda'_k T_I T_k U^{(p-1)}$$

We now have an obvious lifting of d_{m-1}, denoted $d_{m-1}: \tilde{X}_m \to \tilde{X}_m$, to S:

$$(3.2) \qquad d_{m-1}(T_I U^{(p)}) = \sum_{j\in[l]} (-1)^{j-1} a_{I(j)} T_{I_j} U^{(p)} + (-1)^l \sum_{k\in CI} \lambda_k T_I T_k U^{(p-1)}$$

In order to compute the operator t we need to compute $d_{m-1} \circ d_m$ and "divide" the result by x.

Let r_I be the row of d_{m-1} corresponding to the basis element $T_I U^{(p)}$, where I is an increasing map from $[l]$ to $[n]$ with $l + 2p = m - 1$, and let v_J be the column of d_m corresponding to the basis element $T_J U^{(q)}$, where J is an increasing map from $[l']$ to $[n]$ with $l' + 2q = m + 1$. Thus the entries of r_I are the components of the images of various basis vectors of $\tilde{X}_m$ corresponding to $T_I U^{(p)}$, and the entries of v_J are the components of the image of $T_J U^{(q)}$. To finish the proof we need

LEMMA 1. *In the above notation, we have that $r_I v_J = \delta_{IJ} \varepsilon x$, where $r_I v_J$ is the usual matrix product, ε is an element of S, and δ_{IJ} is Kronecker's delta.*

PROOF OF THE LEMMA. First we identify the basis elements of $\tilde{X}_m$ whose images have nonzero components in row r_I. It follows from (1) that, up to sign, those elements are precisely $\{T_I T_g U^{(p)} \mid g \in CI\}$ and $\{T_{I_f} U^{(p+1)} \mid f \in [l]\}$ (one of the two sets may be empty). Up to sign, the components of their images in row r_I are, respectively, a_g and λ_f.

On the other hand,

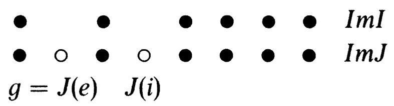
$$d_m(T_J U^{(q)}) = \sum_{j\in[l']} (-1)^{i-1} a_{J(i)} T_{J_i} U^{(q)} + (-1)^{l'} \sum_{h\in CJ} \lambda_h T_J T_h U^{(q-1)}$$

and, therefore, the nonzero entries in v_J correspond to the elements of the two new sets $\{T_{J_i} U^{(q)} \mid i \in [l']\}$ and $\{T_J T_h U^{(q-1)} \mid h \in CJ\}$. If the two pairs of sets have no common elements then $r_I v_J = 0$ and, obviously, $I \neq J$, and we have nothing to prove. Thus we may assume that the two pairs have an element in common. This possibility can be realized in the following ways:

1. $T_I T_g U^{(p)} = \pm T_{J_i} U^{(q)}$
2. $T_I T_g U^{(p)} = \pm T_J T_h U^{(q-1)}$
3. $T_{I_f} U^{(p+1)} = \pm T_{J_i} U^{(q)}$
4. $T_{I_f} U^{(p+1)} = \pm T_J T_h U^{(q-1)}$

We begin with

Case 1) We have that $p = q$ and $\operatorname{Im} J\backslash\{J(i)\} = \operatorname{Im} I \cup \{g\}$. In particular $I \nparallel J$ and $g \in \operatorname{Im} J$, i.e., $g = J(e)$ for some $e \in [l']$. Graphically this situation can be represented as follows:

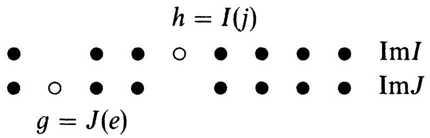


Here the black dots correspond to the elements of $\operatorname{Im} I \cap \operatorname{Im} J$ (we have also assumed that $e < i$). The contribution to $r_I v_J$ from $T_I T_{J(e)} U^{(p)} = \pm T_{J_i} U^{(q)}$ is $(-1)^{e-1} a_{J(e)} (-1)^{i-1} a_{J(i)}$. But we also have that $T_I T_{J(i)} U^{(p)} = \pm T_{J_e} U^{(q)}$ (i.e., we are again in case 1)), and the corresponding contribution is $(-1)^{i-2} a_{J(i)} (-1)^{e-1} a_{J(e)}$ which cancels the previously computed product. Comparing p and q we see that no other case can be realized simultaneously with case 1). Thus in case 1) we have both $r_I v_J = 0$ and $\delta_{IJ} = 0$.

The same argument proves the lemma in Case 4).

Next we turn to Case 2). Now we have that $p = q - 1$ and $\operatorname{Im} I \cup \{g\} = \operatorname{Im} J \cup \{h\}$.

First we want to consider the case $I \nparallel J$. Under our assumptions, $[I \nparallel J] \Leftrightarrow [g \nparallel h]$. It is clear that in this case $l = l'$, $h = I(j)$ and $g = J(e)$ for some $e, j \in [l]$. Graphically, we are in the following situation:



where again the black dots indicate the elements of $\operatorname{Im} I \cap \operatorname{Im} J$. We have also assumed that $g < h$.

Now we want to compute the coefficient of the image, under d_{m-1}, of $\pm T_I T_g U^{(p)}$ by the basis element $T_I U^{(p)}$. To this end we put the elements of $\operatorname{Im} I \cup \{g\}$ in the increasing order and then apply formula (1). Then have that the desired coefficient equals $(-1)^{\varepsilon(g,I)-1} a_g$, where $\varepsilon(g, I)$ is the ordinal number of g in $\operatorname{Im} I \cup \{g\}$. Because of the assumption $g < h$, we have that the ordinal number of g in $\operatorname{Im} I \cup \{g\}$ is the same as the ordinal number of g in J, which equals e. Thus the element of r_I corresponding to $\pm T_I T_g U^{(p)}$ equals $(-1)^{e-1} a_g$. Next we want to compute the coefficient of the image, under d_m, of $T_J U^{(q)}$ by the element $T_J T_h U^{(q-1)}$. This time, to order $\operatorname{Im} J \cup \{h\}$ we first move h to the leftmost position, which contributes the coefficient $(-1)^{l+1}$, then move it to the place corresponding to its ordinal number which is one more then its ordinal number in $\operatorname{Im} I$ (see picture above). The latter move contributes the coefficient $(-1)^{j+1}$. Utilizing formula (1) we obtain the desired coefficient $(-1)^l \lambda_h (-1)^{l+1}(-1)^{j+1} = (-1)^j \lambda_h$. Thus the total contribution to $r_I v_J$ from case 2) is $(-1)^{e-1+j} a_g \lambda_h$.

But we also have that $T_{I_h} U^{(p+1)} = T_{J_g} U^{(q)}$, i.e., there will be a contribution to $r_I v_J$ from case 3). In fact, as is easily seen, case 3) happens exactly when case 2) does. (One can make $\operatorname{Im} I$ and $\operatorname{Im} J$ equal by either adding one element to each or removing one element from each!) Assuming we are in case 3), the coefficient of the image of $T_{I_h} U^{(p+1)}$ by $T_I U^{(p)}$ equals $(-1)^{l-1} \lambda_h (-1)^l (-1)^j = (-1)^{j-1} \lambda_h$. On the other hand, the coefficient of the image of $T_J U^{(q)}$ by $T_{J_g} U^{(q)}$ equals $(-1)^{e-1} a_g$. Thus the total contribution from case 3) equals $(-1)^{e+j} a_g \lambda_h$, which cancels the contribution from case 2). The same proof works in the case $g > h$. Thus we have that $r_I v_J = 0$ whenever $I \neq J$.

We now consider the only remaining possibility in case 2): $I = J$. Now $T_I T_g U^{(p)} = T_J T_g U^{(q-1)}$ for all $g \in CI = CJ$ and $T_{I_f} U^{(p+1)} = T_{J_f} U^{(q)}$ for all $f \in [l]$. The corresponding contributions are $\Sigma_{g \in CI} a_g \lambda_g$ and $\Sigma_{f \in \operatorname{Im} I} a_f \lambda_f$ which add up to $\Sigma_{i=1}^n a_i \lambda_i = \varepsilon x$. This finishes the proof of the lemma.

Returning now to the proof of Proposition 1 we see that, for each component, the operator t for $R\langle U, T_1, \ldots, T_n \rangle$ can be chosen to be $(\delta_{IJ} \varepsilon)$ (as a matrix). Since it is eventually surjective, we have that ε is a unit. Therefore the operator t of $R\langle U, T_1, \ldots, T_n \rangle$ is surjective. But its degree zero part is, by construction, the degree zero part $t_0(R/\underline{a})$ of the operator t to $R/\underline{a}$. This finishes the proof of the theorem.

REMARK. Since ε turned out to be a unit, we have that $x = \varepsilon^{-1} \Sigma_{i=1}^n a_i \lambda_i$, i.e., $x \in \underline{m}_S \underline{a}^c$, where $\underline{a}^c$ is the contraction of $\underline{a}$ to S.

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