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A NOTE ON A PAPER BY MIYAZAKI

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Abstract.

Miyazaki answers in [1] the following question from Watanabe: Suppose A is a Stanley-Reisner ring, i.e. a ring of type $k[X_1,\ldots,X_n]/I$, k a field and I generated by squarefree monomials, such that I is generated by monomials of degree 2. Is there a minimal free resolution of A such that the components of the matrices representing the maps have degree at most 2? The answer to this question is no, Miyazaki gives an example where there is an element of degree 3 in one matrix. We will show that there is in fact no bound for the degrees.

The example.

Our example will be a mixture of a complete intersection and a ring with a linear resolution. Let $B = k[X_1,\ldots,X_{2n+1}]/I$ where $I = (M_1,\ldots,M_n)$ and $M_i = X_{2i-1}X_{2i}$ (a complete intersection) and let $C = k[X_1,\ldots,X_{2n+1}]/J$ where $J = (N_1,\ldots,N_n)$ and $N_i = X_{2i-1}X_{2n+1}$ (a ring with linear resolution). Finally let $A = k[X_1,\ldots,X_{2n+1}]/(I + J)$ (our example). We will show that the last matrix of a $k[X_1,\ldots,X_{2n+1}]$-resolution of A will always have an element of degree n.

E $$0 \to E_n \to \cdots \to E_1 \to B \to 0$$

be the Taylor resolution of B and let

F $$0 - \mathrm{F}_n \to \cdots \to \mathrm{F}_1 \to \mathrm{C} \to 0$$

be the Taylor resolution of C. This means that E_i has a $k[X_1,\ldots,X_{2n+1}]$-basis $\{e_I; I \subset \{1,\ldots,n\},\ |I| = i\}$, where e_I has multidegree $\mathrm{LCM}(\{m_i; i \in I\}) = \deg \prod_{i\in I} m_I$ and $d(e_{\{j_1,\ldots,j_i\}}) = \sum_{k=1}^{i} (-1)^{k-1} m_{j_k} e_{\{j_1,\ldots,\hat{j_k},\ldots,j_i\}}$ and that F_i has a $k[X_1,\ldots,X_{2n+1}]$-basis $\{f_I; I \subset \{1,\ldots,n\},\ |I| = i\}$, where f_I has multidegree $\mathrm{LCM}(\{n_i; i \in I\})$ and $d(f_{\{j_1,\ldots,j_i\}}) = \sum_{k=1}^{i} (-1)^{k-1} \frac{n_{j_k}}{X_{2n+1}} f_{\{j_1,\ldots,\hat{j_k},\ldots,j_i\}}$. (The Taylor resolution is described in [1].)

A minimal $k[X_1,\ldots,X_{2n+1}]$-resolution of A has the following form:

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$$\mathbf{H} \quad 0 \to G_n \to E_n \oplus F_n \oplus G_{n-1} \to \cdots \to E_2 \oplus F_2 \oplus G_1 \to E_1 \oplus F_1 \to A \to 0$$

It remains to describe the G_i's and the maps from them. G_i is generated by $\binom{n}{i}$ elements g_I of multidegree $\deg(e_I) + (0,0,\ldots,1)$ where $\{e_I\}$ is a basis for E_i, and $d(g_{\{j_1,\ldots,j_i\}}) = X_{2n+1}e_{\{j_1,\ldots,j_i\}} + \sum_{k=1}^{i}(-1)^{k-1}m_i g_{\{j,\ldots,\hat{j}_k,\ldots,j_i\}} - p_{\{j_1,\ldots,j_i\}}f_{\{j_1,\ldots,j_i\}}$, where $p_{\{j_1,\ldots,j_i\}}$ is the power product of correct multidegree. Thus the last matrix in the resolution has a degree vector $(1,2,2,\ldots,2,n)$. To show that we really get a resolution one can argue like this: It is easy to check that we have a complex which is exact at $E_1 \oplus F_1$. The factor complex $\mathbf{G} = \mathbf{H}/\mathbf{E} \oplus \mathbf{F}$ is just a shifted copy of $\mathbf{E}$ and thus exact. Hence if z is a cycle in H_i, we can find a g in G_i so that $z - d(g) \in E_i \oplus F_i$. But both $\mathbf{E}$ and $\mathbf{F}$ are exact so we are through. The same reasoning as in [1] shows that one can not get rid of the component of degree n of the last matrix.

REFERENCES

1. M. Miyazaki, *On the canonical map to the local cohomology of a Stanley-Reisner ring*, Bull. Kyoto Univ. Ed. Ser. B. 79 (1991), 1–8.

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