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ON THE BIFURCATION VARIETY OF SOME NON-ISOLATED SINGULARITIES

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Abstract.

We show that for some simple non-isolated singularities $f: (\mathbf{C}^{k+m}, 0) \to (\mathbf{C}, 0)$ with smooth critical set of dimension k, the complement of the bifurcation variety of f is a space of type $K(\pi, 1)$, if $\min(m, k) \leqq 2$.

1. The bifurcation variety.

1.1. Let $k, n, m \in \mathbf{N}$ be positive numbers such that $n = k + m$ and let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic map germ such that the critical set of f is smooth and of dimension k. Then we can choose coordinates $z = (z_1, \ldots, z_n) = (x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_m)$ in $\mathbf{C}^n = \mathbf{C}^k \times \mathbf{C}^m$ such that the singular locus of f becomes $H := \mathbf{C}^k \times \{0\} = \{y = 0\}$. Then $f \in (y)^2$ and, as in [11], [12], the extended tangent space of (the orbit of) f with respect to the action of the group $\mathscr{D}_{(y)}$ is defined by (see also [16], [17] for some notations used here):

$$\tau_e(f) := \left(\frac{\partial f}{\partial x}\right) + (y)\left(\frac{\partial f}{\partial y}\right),$$

while the extended codimension of (the orbit of) f is:

$$c_e(f) := \dim_C \frac{(y)^2}{\tau_e(f)}.$$

We shall suppose that $v := c_e(f) < \infty$. Hence there exists an *I-universal unfolding* for f, in the sense of [11], [12] or [5]. Namely, there exists a map germ

$$F: (\mathbf{C}^n \times \mathbf{C}^v, 0) \to (\mathbf{C} \times \mathbf{C}^v, 0)$$

such that, if $\lambda = (\lambda_1, \ldots, \lambda_v) \in \mathbf{C}^v$ denotes the coordinates in $\mathbf{C}^v$, the following conditions are fulfilled:

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(i) $F(z, \lambda) = (\tilde{f}(z, \lambda), \lambda)$
(ii) $\tilde{f}(z, 0) = f(z)$
(iii) $\tilde{f}(\cdot, \lambda) \in (y)^2$ for any $\lambda \in (\mathbf{C}^\nu, 0)$
(iv) $\tilde{f}$ is a *versal deformation* of f, i.e. the functions

$$\frac{\partial \tilde{f}}{\partial \lambda_i}(\cdot, 0) \quad \text{for} \quad i = 1, \ldots, \nu$$

give rise to a basis of the C-vector space $(y)^2/\tau_e(f)$.

1.2. It is known that for $\lambda \in \mathbf{C}^\nu$ generic and sufficiently small, the deformation $\tilde{f}(\cdot, \lambda)$ of f has only a finite number of singular points of type A_1 outside H and in a neighbourhood of $0 \in \mathbf{C}^n$ (see [14] or Lemma (5.3.2) in [17]). We shall denote by $\sigma + 1$ the maximum number of critical values for the deformation $\tilde{f}(\cdot, \lambda)$: $\mathbf{C}^n \to \mathbf{C}$ of f. Note that this includes 0 as a critical value. We define the *bifurcation variety* of f as being the germ at $0 \in \mathbf{C}^\nu$ of the following set:

$$\mathrm{Bi}f(f) = \{\lambda \in \mathbf{C}^\nu \mid \tilde{f}(\cdot, \lambda): \mathbf{C}^n \to \mathbf{C} \text{ has not } \sigma + 1 \text{ critical values}\}$$

From the universality property of the unfolding F it follows that the germ $\mathrm{Bi}f(f)$ depends only on f, up to an analytic isomorphism, since $\nu = c_e(f)$.

Our aim is to show that the complement of the bifurcation variety of f is a space of type $K(\pi, 1)$, if $\min(m, k) \leqq 2$ and f is a simple non-isolated singularity as in (1.1). For $k = 1$, this result was obtained by V. V. Goryunov, see [6].

1.3. Now we shall give another characterization of $\mathrm{Bi}f(f)$. Firstly, we shall consider the singular locus, Σ, of the unfolding F:

$$\Sigma := \left\{(z, \lambda) \in \mathbf{C}^n \times \mathbf{C}^\nu \,\middle|\, \frac{\partial \tilde{f}}{\partial z_j}(z, \lambda) = 0, \quad j = 1, \ldots, m\right\}.$$

It is clear that $H \times \mathbf{C}^\nu \subseteq \Sigma$. Let Σ' denotes the closure of the complement of $H \times \mathbf{C}^\nu$ in Σ:

$$\Sigma' := \overline{\Sigma \setminus (H \times \mathbf{C}^\nu)}.$$

Let us choose a representative for the germ f, denoted also by f and defined on a neighbourhood Ω of $0 \in \mathbf{C}^n$. Let U be a neighbourhood of $0 \in \mathbf{C}^\nu$. If Ω and U are sufficiently small, then we can assume that Σ' does not intersects $\partial\Omega \times U$. Indeed, for the projection

$$\pi: \mathbf{C}^n \times \mathbf{C}^\nu \to \mathbf{C}^\nu, \quad \pi(z, \lambda) = \lambda,$$

we have:

$$\pi^{-1}(\lambda) \cap \Sigma = \{\text{singular locus of the function } \tilde{f}(\cdot, \lambda)\} \times \{\lambda\} =$$

$$= (H \times \{\lambda\}) \cup (\{\text{a finite set of points non-situated on } H\} \times \{\lambda\}).$$

Σ' corresponds to this set of points, which, for $\lambda \in \mathbf{C}^{\nu}$ generic and sufficiently small, are exactly the points of type A_1 of the deformation $\tilde{f}(\cdot, \lambda)$.

1.4. It follows that if $\Sigma' \neq \emptyset$, then the restriction of the projection

$$\pi\colon \textstyle\sum' \to U$$

is proper and with finite fibres. Now, as in [8], we can find a thin subset Q of U such that the restriction

$$\pi\colon \textstyle\sum' \cap \pi^{-1}(U \backslash Q) \to U \backslash Q$$

is a finite covering with σ sheets. (Recall that σ is the maximum number of A_1 points which can appear in a deformation of f.) For $\lambda \in U\backslash Q$, we put:

$$\pi^{-1}(\lambda) = \{z_1(\lambda), \ldots, z_{\sigma}(\lambda)\} \times \{\lambda\}$$

and we denote by $a_j(\lambda)$ the value of the symmetric function of degree j computed in the point

$$(-\tilde{f}(z_1(\lambda), \lambda), \ldots, -\tilde{f}(z_{\sigma}(\lambda), \lambda))$$

for $j = 1, \ldots, \sigma$. Then $a_j(\lambda)$ is a holomorphic and locally bounded function on $U\backslash Q$. Now, considering the polynomial

$$A(T, \lambda) = T^{\sigma+1} + a_1(\lambda)T^{\sigma} + \cdots + a_{\sigma}(\lambda)T.$$

we can prove, as in [8], the following

THEOREM. *Let f be as in* (1.1) *and let us suppose that* $\operatorname{Bi}f(f) \neq \emptyset$. *Then* $\operatorname{Bi}f(f)$ *is a hypersurface in U; namely,* $\operatorname{Bi}f(f)$ *is exactly the zero set of the discriminant $\delta \in \mathcal{O}_U$ of the polynomial $A(T, \lambda)$.*

1.5. Another description of $\operatorname{Bi}f(f)$ is as follows. Let us denote by P^{σ} the space of monic polynomials of degree σ,

$$P^{\sigma} = \{T^{\sigma} + c_1 T^{\sigma-1} + \cdots + c_{\sigma} \mid c_1, \ldots, c_{\sigma} \in \mathbf{C}\},$$

and let us consider the map

$$\varphi\colon U \to P^{\sigma}, \quad \varphi(\lambda) = T^{\sigma} + a_1(\lambda)T^{\sigma-1} + \cdots + a_{\sigma}(\lambda).$$

Here, the functions $a_j(\lambda)$ are the extensions by continuity of those constructed in (1.4). Then $\varphi(\lambda) = T^{-1}A(T, \lambda)$ and $\operatorname{Bi}f(f) = \varphi^{-1}(\Delta)$, where $\Delta \subseteq P^{\sigma}$ is such that the complement $P^{\sigma}\backslash\Delta$ coincides with the space of polynomials for which $0 \in \mathbf{C}$ is not a root and without multiple roots. Looking at the roots of a polynomial, it is clear that $P^\sigma \backslash \Delta$ coincides with the configuration space $C_\sigma(\mathrm{C}\backslash\{0\})$ of a set of σ unordered points in $\mathrm{C}\backslash\{0\}$; see [7], for the notations not defined here. On the other hand, if $F_\sigma(\mathrm{C}\backslash\{0\})$ is the configuration space of a set of σ ordered points in $\mathrm{C}\backslash\{0\}$, then the group Perm_σ of all permutations of the set $\{1,\ldots,\sigma\}$ acts on $F_\sigma(\mathrm{C}\backslash\{0\})$ and we have

$$C_\sigma(\mathrm{C}\backslash\{0\}) = \frac{F_\sigma(\mathrm{C}\backslash\{0\})}{\mathrm{Perm}_\sigma}.$$

Now it follows that $P^\sigma \backslash \Delta$ is a space of type $K(\pi, 1)$ since $F_\sigma(\mathrm{C}\backslash\{0\})$ has such a property, see [7], the proof of Corollary 2.3, or [4].

2. Some simple germs.

2.1. We shall give now the lists of $\mathscr{D}_{(y)}$-simple germs, supposing that

$$\min(m, k) \leqq 2.$$

We recall first the case of *isolated line singularities*, considered in [13]:

THEOREM (Case: Σ smooth of dimension 1). *If the coordinates in* $(\mathrm{C}^n, 0)$ *are denoted by* $x, y_1, \ldots, y_m$, *then the* $\mathscr{D}_{(y)}$-*simple germs in* $(y)^2$ *are those listed in Table* 1.

TABLE 1: $k = \dim \Sigma = 1$

Name	Normal form	Conditions	v	σ
$A_\infty = D(1,0)$	$y_1^2 + \cdots + y_m^2$	–	0	0
$D_\infty = D(1,1)$	$xy_1^2 + y_2^2 + \cdots + y_m^2$	–	0	0
$J_{k,\infty}$	$y_1^2(y_1 + x^k) + y_2^2 + \cdots + y_m^2$	$k \geqq 2$	$k-1$	$k-1$
$T_{\infty,k,2}$	$y_1^2(y_1^{k-2} + x^2) + y_2^2 + \cdots + y_m^2$	$k \geqq 4$	$k-2$	$k-2$
$Z_{k,\infty}$	$y_1^2(xy_1 + x^{k+2}) + y_2^2 + \cdots + y_m^2$	$k \geqq 1$	$k+2$	$k+2$
$W_{1,\infty}$	$y_1^2(y_1^2 + x^3) + y_2^2 + \cdots + y_m^2$	–	4	4
$T_{\infty,q,r}$	$xy_1y_2 + y_1^q + y_2^r + y_3^2 + \cdots + y_m^2$	$q \geqq r \geqq 3$	$q+r-4$	$q+r-4$
$Q_{k,\infty}$	$xy_2^2 + y_1^3 + x^ky_1^2 + y_3^2 + \cdots + y_m^2$	$k \geqq 4$	$k+1$	$k+1$
$S_{1,\infty}$	$xy_2^2 + y_1^2y_2 + x^2y_1^2 + y_3^2 + \cdots + y_m^2$	–	4	4

The case when the *codimension* of Σ is 1 follows from [2], see for example [6], [15], and is described by the following

2.2. THEOREM (Case: Σ smooth of codimension 1). *If the coordinates in* $(\mathbf{C}^n, 0)$ *are denoted by* $x_1, \ldots, x_k, y$, *then the* $\mathscr{D}_{(y)}$*-simple germs in* $(y)^2$ *are those listed in Table* 2.

TABLE 2: $m = \operatorname{codim} \Sigma = 1$

Name	Normal form	Conditions	ν	σ
YA'	y^2	–	0	0
YD'	$x_1 y^2$	–	0	0
YA_s	$y^2(y + x_1^{s+1} + x_2^2 + \cdots + x_k^2)$	$s \geqq 1$	s	s
YD_s	$y^2(y + x_1^{s-1} + x_1 x_2^2 + x_3^2 + \cdots + x_k^2)$	$s \geqq 4, k \geqq 2$	s	s
YE_6	$y^2(y + x_1^4 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 2$	6	6
YE_7	$y^2(y + x_1^3 x_2 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 2$	7	7
YE_8	$y^2(y + x_1^5 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 2$	8	8
YB_s	$y^2(y^s + x_1^2 + \cdots + x_k^2)$	$s \geqq 2$	s	s
YC_s	$y^2(x_1 y + x_1^s + x_2^2 + \cdots + x_k^2)$	$s \geqq 3$	s	s
YF_4	$y^2(y^2 + x_1^3 + x_2^2 + \cdots + x_k^2)$	–	4	4

Next we consider the case when the *codimension* of Σ is 2. In [16] it is proved the following

2.3. THEOREM (Case: Σ smooth of codimension 2). *If the coordinates in* $(\mathbf{C}^n, 0)$ *are denoted by* $x_1, \ldots, x_k, y_1, y_2$, *then the* $\mathscr{D}_{(y)}$*-simple germs in* $(y)^2$ *are the following ones:*

- *those with corank* 1 *are suspensions of the germs in Table* 2
- *those with corank* 2 *are listed in Table* 3.

TABLE 3: $m = \operatorname{codim}\Sigma = 2$

Name	Normal form	Conditions	v	σ
$D(1,1)$	$x_1y_1^2 + x_2y_2^2 + x_3y_1y_2$	$k \geqq 3$	0	0
I_4	$x_1y_1^2 + x_2y_2^2$	$k = 2$	1	0
IIA_s	$x_1y_1^2 + x_2y_2^2 + y_1y_2(x_3^{s+1} + x_4^2 + \cdots + x_k^2)$	$s \geqq 1, k \geqq 3$	s	0
IID_s	$x_1y_1^2 + x_2y_2^2 + y_1y_2(x_3^{s-1} + x_3x_4^2 + x_5^2 + \cdots + x_k^2)$	$s \geqq 4, k \geqq 4$	s	0
IIE_6	$x_1y_1^2 + x_2y_2^2 + y_1y_2(x_3^4 + x_4^3 + x_5^2 + \cdots + x_k^2)$	$k \geqq 4$	6	0
IIE_7	$x_1y_1^2 + x_2y_2^2 + y_1y_2(x_3^3x_4 + x_4^3 + x_5^2 + \cdots + x_k^2)$	$k \geqq 4$	7	0
IIE_8	$x_1y_1^2 + x_2y_2^2 + y_1y_2(x_3^5 + x_4^3 + x_5^2 + \cdots + x_k^2)$	$k \geqq 4$	8	0
IIB_s	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2 + x_2^s + x_3^2 + \cdots + x_k^2)$	$s \geqq 2, k \geqq 2$	s	$s-1$
IIC_s	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2 + x_2x_3 + x_3^s + x_4^2 + \cdots + x_k^2)$	$s \geqq 3, k \geqq 3$	s	1
IIF_4	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2 + x_2^2 + x_3^3 + x_4^2 + \cdots + x_k^2)$	$k \geqq 3$	4	2
$II'B_s$	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2^s + x_2^2 + \cdots + x_k^2)$	$s \geqq 2, k \geqq 2$	$s+1$	s
$II'C_s$	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2x_2 + x_2^s + x_3^2 + \cdots + x_k^2)$	$s \geqq 3, k \geqq 2$	$s+1$	s
$II''C_3$	$x_1y_1y_2 + x_3y_1^2 + y_2^2(y_2x_2 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 3$	5	3
$II'F_4$	$x_1y_1y_2 + x_2y_1^2 + y_2^2(y_2^2 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 2$	5	4
$II''F_4$	$x_1y_1y_2 + x_3y_1^2 + y_2^2(y_2^2 + x_2^3 + x_3^2 + \cdots + x_k^2)$	$k \geqq 3$	6	4

REMARK. The names of the germs in Tables 2 and 3 differ from the names given in [15] and [16] and the value of v for the germ $II''C_3$ is not correctly calculated in [16].

2.4. We can obtain also the list of simple germs in I^2 for the case when I defines a smooth germ in $(\mathbf{C}^n, 0)$ of *dimension* 2. Namely, we have the following

THEOREM (Case: Σ smooth of dimension 2). *If the coordinates in* $(\mathbf{C}^n, 0)$ *are denoted by* $x_1, x_2, y_1, \ldots, y_m$, *then the* $\mathscr{D}_{(y)}$*-simple germs in* $(y)^2$ *are those listed in Table* 4.

TABLE 4: $\dim \Sigma = 2$

Name	Normal form	Conditions	ν	σ
$D(2,0)$	$y_1^2 + \cdots + y_m^2$	–	0	0
$D(2,1)$	$x_1 y_1^2 + y_2^2 + \cdots + y_m^2$	–	0	0
TYA_s	$y_1^2(y_1 + x_1^{s+1} + x_2^2) + y_2^2 + \cdots + y_m^2$	$s \geqq 1$	s	s
TYD_s	$y_1^2(y_1 + x_1^{s-1} + x_1 x_2^2) + y_2^2 + \cdots + y_m^2$	$s \geqq 4$	s	s
TYE_6	$y_1^2(y_1 + x_1^4 + x_2^3) + y_2^2 + \cdots + y_m^2$	–	6	6
TYE_7	$y_1^2(y_1 + x_1^3 x_2 + x_2^3) + y_2^2 + \cdots + y_m^2$	–	7	7
TYE_8	$y_1^2(y_1 + x_1^5 + x_2^3) + y_2^2 + \cdots + y_m^2$	–	8	8
TYB_s	$y_1^2(y_1^s + x_1^2 + x_2^2) + y_2^2 + \cdots + y_m^2$	$s \geqq 2$	s	s
TYC_s	$y_1^2(x_1 y_1 + x_1^s + x_2^2) + y_2^2 + \cdots + y_m^2$	$s \geqq 3$	s	s
TYF_4	$y_1^2(y_1^2 + x_1^3 + x_2^2) + y_2^2 + \cdots + y_m^2$	–	4	4
TI_4	$x_1 y_1^2 + x_2 y_2^2 + y_3^2 + \cdots + y_m^2$	–	1	0
$TIIB_s$	$x_1 y_1 y_2 + x_2 y_1^2 + y_2^2(y_2 + x_2^s) + y_3^2 + \cdots + y_m^2$	$s \geqq 2$	s	$s-1$
$TII'B_s$	$x_1 y_1 y_2 + x_2 y_1^2 + y_2^2(y_2^s + x_2^2) + y_3^2 + \cdots + y_m^2$	$s \geqq 2$	$s+1$	s
$TII'C_s$	$x_1 y_1 y_2 + x_2 y_1^2 + y_2^2(y_2 x_2 + x_2^s) + y_3^2 + \cdots + y_m^2$	$s \geqq 3$	$s+1$	s
$TII'F_4$	$x_1 y_1 y_2 + x_2 y_1^2 + y_2^2(y_2^2 + x_2^3) + y_3^2 + \cdots + y_m^2$	–	5	4

PROOF. We use the results in Theorem (2.3) and the fact that if a function $f \in (y_1, \ldots, y_m)^2 \subseteq \mathcal{O}$ has corank $\geqq 3$, then f is not simple. To see this, we have to use suspensions and to observe that, when $m \geqq 3$, for any function $f \in (y_1, \ldots, y_m)^2$ with $j^2(f) = 0$, the number of monomials which can occur in $j^3(f)$ is strictly bigger than the numbers of generators of $\tau_{(y)}(f)$.

REMARK. Theorems (2.1), (2.2), (2.3) and (2.4) gives us the classification of $\mathscr{D}_I$-simple germs for the cases when I defines a smooth germ in $(\mathbf{C}^n, 0)$ and $n \leqq 5$.

2.5. The germs in Table 1 and Table 2 have $\sigma = \nu$, see [6], [11], [12] and [15]. For the germs in Table 3 and Table 4, the value of σ can be *not equal* to the value of ν. We computed it directly, using a versal deformation. Namely, we have the following

PROPOSITION. *A versal deformation $\tilde{f}$ for the germs in Table 2 and Table 3 is given in Table 5 and respectively Table 6.*

TABLE 5

f	v	Versal deformation of f
YA'	0	f
YD'	0	f
YA_s	s	$f + y^2(\lambda_1 + \lambda_2 x_1 + \cdots + \lambda_s x_1^{s-1})$
YD_s	s	$f + y^2(\lambda_1 + \lambda_2 x_1 + \cdots \lambda_{s-1} x_1^{s-2} + \lambda_s x_2)$
YE_6	6	$f + y^2(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_1^2 + \lambda_4 x_2 + \lambda_5 x_1 x_2 + \lambda_6 x_1^2 x_2)$
YE_7	7	$f + y^2(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_1^2 + \lambda_4 x_1^3 + \lambda_5 x_1^4 + \lambda_6 x_2 + \lambda_7 x_1 x_2)$
YE_8	8	$f + y^2(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_1^2 + \lambda_4 x_1^3 + \lambda_5 x_2 + \lambda_6 x_1 x_2 + \lambda_7 x_1^2 x_2 + \lambda_8 x_1^3 x_2)$
YB_s	s	$f + y^2(\lambda_1 + \lambda_2 y + \cdots + \lambda_s y^{s-1})$
YC_s	s	$f + y^2(\lambda_1 + \lambda_2 x_1 + \cdots + \lambda_s x_1^{s-1})$
YF_4	4	$f + y^2(\lambda_1 + \lambda_2 x_1 + \lambda_3 y + \lambda_4 x_1 y)$

TABLE 6

f	v	Versal deformation of f
$D(1,1)$	0	f
I_4	1	$f + \lambda_1 y_1 y_2$
IIA_s	s	$f + y_1 y_2(\lambda_1 + \lambda_2 x_3 + \cdots + \lambda_s x_3^{s-1})$
IID_s	s	$f + y_1 y_2(\lambda_1 + \lambda_2 x_3 + \cdots + \lambda_{s-1} x_3 + \lambda_s x_4)$
IIE_6	6	$f + y_1 y_2(\lambda_1 + \lambda_2 x_3 + \lambda_3 x_3^2 + \lambda_4 x_4 + \lambda_5 x_3 x_4 + \lambda_6 x_3^2 x_4)$
IIE_7	7	$f + y_1 y_2(\lambda_1 + \lambda_2 x_3 + \lambda_3 x_3^2 + \lambda_4 x_3^3 + \lambda_5 x_3^4 + \lambda_6 x_4 + \lambda_7 x_3 x_4)$
IIE_8	8	$f + y_1 y_2(\lambda_1 + \lambda_2 x_3 + \lambda_3 x_3^2 + \lambda_4 x_3^3 + \lambda_5 x_4 + \lambda_6 x_3 x_4 + \lambda_7 x_3^2 x_4 + \lambda_8 x_3^3 x_4)$
IIB_s	s	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \cdots + \lambda_s x_2^{s-1})$
IIC_s	s	$f + \lambda_1 y_2^2 + y_1^2(\lambda_2 + \lambda_3 x_3 + \cdots + \lambda_s x_3^{s-2})$
IIF_4	4	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_2 x_3)$
$II'B_s$	$s+1$	$f + y_2^2(\lambda_1 + \lambda_2 y_2 + \cdots + \lambda_s y_2^{s-1}) + \lambda_{s+1} y_1^2$
$II'C_s$	$s+1$	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \cdots + \lambda_s x_2^{s-1}) + \lambda_{s+1} y_1^2$
$II''C_3$	5	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \lambda_3 x_2^2) + y_1^2(\lambda_4 + \lambda_5 x_2)$
$II'F_4$	5	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \lambda_3 y_2 + \lambda_4 x_2 y_2) + \lambda_5 y_1^2$
$II''F_4$	6	$f + y_2^2(\lambda_1 + \lambda_2 x_2 + \lambda_3 y_2 + \lambda_4 x_2 y_2) + y_1^2(\lambda_5 + \lambda_6 x_2)$

3. Complement of the bifurcation variety.

3.1. In this section we shall prove the following

THEOREM. *Let $f \in (y)^2$ be one of the simple germs in Table 2 or Table 3. Then the complement of the bifurcation diagram of f, $(\mathbf{C}^{\nu}\backslash \mathrm{Bi}\, f(f), 0)$, is a space of type $K(\pi, 1)$.*

3.2. This Theorem is similar to some results obtained by E. J. N. Looijenga [8], O. V. Lyashko [1], [9] and V. V. Goryunov [6]. For the proof, we shall consider first the germs in Table 3. We have the following

PROPOSITION. *The bifurcation variety of the germs in Table 3 is described in Table 7.*

TABLE 7

f	$\mathrm{Bi}\, f(f)$
$D(1,1)$	$\emptyset$
I_4	$\emptyset$
IIA_s	$\emptyset$
IID_s	$\emptyset$
IIE_6	$\emptyset$
IIE_7	$\emptyset$
IIE_8	$\emptyset$
IIB_s	$\mathrm{Bi}\, f(YA_{s-1})$
IIC_s	$\{0\} \times \mathbf{C}^{s-1}$
IIF_4	$\mathrm{Bi}\, f(YA_2) \times \mathbf{C}^2$
$II'B_s$	$\mathrm{Bi}\, f(YB_s) \times \mathbf{C}$
$II'C_s$	$\mathrm{Bi}\, f(YC_s) \times \mathbf{C}$
$II''C_3$	$\mathrm{Bi}\, f(YC_3) \times \mathbf{C}^2$
$II'F_4$	$\mathrm{Bi}\, f(YF_4) \times \mathbf{C}$
$II''F_4$	$\mathrm{Bi}\, f(YF_4) \times \mathbf{C}^2$

PROOF. We shall describe the proof only for $f = IIC_s$ and $f = II''C_3$; the other cases are similar.

Suppose first that $f = IIC_s$. We shall denote by $\tilde{f}$ the versal deformation of f described in Table 6. The critical points of $\tilde{f}$ are the solutions of the following system of equations

$$\frac{\partial\tilde{f}}{\partial x_1} = \ldots = \frac{\partial\tilde{f}}{\partial x_k} = \frac{\partial\tilde{f}}{\partial y_1} = \frac{\partial\tilde{f}}{\partial y_2} = 0.$$

Hence the critical points of $\tilde{f}$ non-situated on H are given by the following conditions:

$$x_1 = \ldots = x_k = y_1 = 0, \quad y_2 \neq 0 \quad \text{and} \quad 3y_2^2 + 2\lambda_1 y_2 = 0.$$

Now it is easy to see that only for $\lambda_1 \neq 0$ the function $\tilde{f}(\cdot,\lambda)$ has exactly two critical values.

Let now $f = II''C_3$ and let $\tilde{f}$ denote the versal deformation of f described in Table 6. As before, we obtain that the critical points of $\tilde{f}$ non-situated on H are given by the following conditions

$$x_1 = x_3 = x_4 = \ldots = x_k = y_1 = 0, \quad y_2 \neq 0 \quad \text{and}$$
$$y_2 + 3x_2^2 + \lambda_2 + 2\lambda_3 x_3 = 0,$$
$$3y_2x_2 + 2x_2^3 + 2\lambda_1 + 2\lambda_2 x_2 + 2\lambda_3 x_2^2 = 0.$$

We have also

$$\tilde{f}(0,x_2,0,\ldots,0,0,y_2,\lambda) = y_2^2(y_2x_2 + x_2^3) + y_2^2(\lambda_1 + \lambda_2 x_2 + \lambda_3 x_2^2),$$

and now it is easy to obtain the result.

3.3. It remains to show that the complement of the bifurcation variety of a germ in Table 2 is a space of type $K(\pi, 1)$. For $k = 1$, this was demonstrated in [6]; in particular this gives us the result for the cases when the versal deformation, listed in Table 5, depends only on y and x_1. For $k \geqq 2$, we have to observe that the proof given by V. V. Goryunov in [6] is still valid; for completness, we repeat here some parts of this proof. Namely, we note, firstly, that the simple germs listed in Table 2 and their versal deformation listed in Table 5 are weighted homogeneous polynomials, with *positive* weights for the coordinates (x, y, λ). Also, we remark again that for the germs in Table 2 we have $\nu = \sigma$.

Next we have to check that the map φ, described in (1.5), is a covering above $P^\sigma \backslash \Delta$. This follows from the following steps:

(STEP 1) $\varphi^{-1}(0) = \{0\}$.

(STEP 2) φ is proper.

(STEP 3) φ is a local diffeomorphism on the complement of $\mathrm{Bi}f(f)$.

The proof of the first step uses the connectivity of the Dynkin dagram associated to the corresponding boundary singularity. Namely, the germs in Table 2 and their versal deformations in Table 5 have the form

$$f(x,y) = y^2 \cdot h(x,y), \quad \tilde{f}(x,y,\lambda) = y^2 \cdot \tilde{h}(x,y,\lambda),$$

where h is the corresponding boundary singularity and $\tilde{h}$ is a versal deformation of the boundary singularity h. The relation

$$\lambda \in \varphi^{-1}(0)$$

means that all the critical values of the function $\tilde{f}(\cdot,\cdot,\lambda)$ are equal to 0. It is easy to see that the critical points of this function, which are not situated on $\{y=0\}$, are exactly the solution of the system

$$(1) \qquad 2\cdot\tilde{h} + y\frac{\partial\tilde{h}}{\partial y} = 0, \quad \frac{\partial\tilde{h}}{\partial x_i} = 0, \quad i = 1,\ldots,k$$

with $y \neq 0$. Now, since $\lambda \in \varphi^{-1}(0)$, it is easy to see that *all* the solutions of the system (1) satisfy the relation $\tilde{h} = 0$, hence are also solutions of the system

$$(2) \qquad y\frac{\partial\tilde{h}}{\partial y} = 0, \quad \frac{\partial\tilde{h}}{\partial x_i} = 0, \quad i = 1,\ldots,k.$$

On the other hand, the systems (1) and (2) have the same number of solutions, see for instance [15]. Hence, if $\lambda \in \varphi^{-1}(0)$, then 0 is also the unique critical value of the function $\tilde{h}(\cdot,\cdot,\lambda)$, considered as a function on a manifold with boundary. And now, from [10], p. 105–106, it follows that $\lambda = 0$.

The second step is a consequence of the first one and of the fact that φ is weighted homogeneous, with *positive* weights for the coordinates in $\mathbf{C}^\nu$.

For the proof of the third step it suffices, by weighted homogeneity, to show that the differential of φ, $\mathscr{D}_\lambda\varphi$, is an isomorphism in any point $\lambda \notin \Sigma$, for λ sufficiently small. This can be done by considering the space

$$\mathbf{C} \times (\mathbf{C}^k \times \mathbf{C}) \times \mathbf{C}^\nu,$$

with coordinates (u, x, y, λ), and in this space looking to the closure

$$N := \overline{\left\{ y \neq 0, \quad u = \tilde{f}(x,y,\lambda), \quad \frac{\partial\tilde{f}}{\partial y} = 0, \quad \frac{\partial\tilde{f}}{\partial x_i} = 0 \quad \text{for} \quad i = 1,\ldots,k \right\}}.$$

We consider also the set

$$N^\lambda := N \cap \{\lambda = \text{const}\}.$$

Let us fix $\lambda \notin \Sigma$. Then all the critical values of the function $\tilde{f}(\cdot,\cdot,\lambda)$ are distinct and we have:

$$N^\lambda = \{p^{(i)} = (u^{(i)}, x^{(i)}, y^{(i)}, \lambda) \in \mathbf{C} \times (\mathbf{C}^k \times \mathbf{C}) \times \mathbf{C}^\sigma \mid i = 1,\ldots,\sigma\}.$$

Now, recall from (1.4) and (1.5) that $\varphi(\lambda)$ is computed with the help of the symmetric functions evaluated in the point

$$(-\tilde{f}(x^{(1)}, y^{(1)}, \lambda), \ldots, -\tilde{f}(x^{(\sigma)}, y^{(\sigma)}, \lambda)) = (-u^{(1)}, \ldots, -u^{(\sigma)}).$$

Since $u^{(1)}, \ldots, u^{(\sigma)}$ are critical values of the function $\tilde{f}(\cdot, \cdot, \lambda)$, it follows that these are distinct numbers. Hence the differential $\mathscr{D}_\lambda\varphi$ is not an isomorphism if and only if the differential of the map

$$\lambda \mapsto (u^{(1)}, \ldots, u^{(\sigma)}) \tag{3}$$

is not an isomorphism; we use here the fact that the symmetric functions give rise to a locally diffeomorphism around the point $(-u^{(1)}, \ldots, -u^{(\sigma)})$.

Suppose that $\mathscr{D}_\lambda\varphi$ is not an isomorphism. Then there exists a tangent vector $d\lambda$ to the point $\lambda \in \mathbf{C}^\nu \backslash \Sigma$ such that $d\lambda \neq 0$ and the differential of the map (3) evaluated at $d\lambda$ is equal to 0.

Since the projection

$$N \ni (u, x, y, \lambda) \mapsto \lambda \in \mathbf{C}^\nu$$

is a covering with σ sheets, outside Σ, it follows that there exist σ tangent vectors $V^{(i)}$ to N, at the points $p^{(i)}$, such that

$$V^{(i)} = (0, (dx)^{(i)}, (dy)^{(i)}, d\lambda),$$

namely the tangent vectors which projects onto $d\lambda$.

The condition of being tangent vectors means that

$$V^{(i)} \in \operatorname{Ker} \left.\begin{pmatrix} -1 & \dfrac{\partial\tilde{f}}{\partial x_1} & \cdots & \dfrac{\partial\tilde{f}}{\partial x_k} & \dfrac{\partial\tilde{f}}{\partial y} & \dfrac{\partial\tilde{f}}{\partial \lambda_1} & \cdots & \dfrac{\partial\tilde{f}}{\partial \lambda_\nu} \\ 0 & \dfrac{\partial^2\tilde{f}}{\partial x_1^2} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial x_1\partial x_k} & \dfrac{\partial^2\tilde{f}}{\partial x_1\partial y} & \dfrac{\partial^2\tilde{f}}{\partial x_1\partial \lambda_1} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial x_1\partial \lambda_\nu} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dfrac{\partial^2\tilde{f}}{\partial x_k\partial x_1} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial x_k^2} & \dfrac{\partial^2\tilde{f}}{\partial x_k\partial y} & \dfrac{\partial^2\tilde{f}}{\partial x_k\partial \lambda_1} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial x_k\partial \lambda_\nu} \\ 0 & \dfrac{\partial^2\tilde{f}}{\partial y\partial x_1} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial y\partial x_k} & \dfrac{\partial^2\tilde{f}}{\partial y^2} & \dfrac{\partial^2\tilde{f}}{\partial y\partial \lambda_1} & \cdots & \dfrac{\partial^2\tilde{f}}{\partial y\partial \lambda_\nu} \end{pmatrix}\right|_{p^{(i)}}$$

Since the singularity of the function $\tilde{f}(\cdot, \cdot, \lambda)$ at the point $(x^{(i)}, y^{(i)})$ is of type A_1, we have

$$\frac{\partial\tilde{f}}{\partial x_1}(p^{(i)}) = \ldots = \frac{\partial\tilde{f}}{\partial x_k}(p^{(i)}) = \frac{\partial\tilde{f}}{\partial y}(p^{(i)}) = 0$$

and

$$\det \left(\begin{array}{cccc} \frac{\partial^2 \tilde{f}}{\partial x_1^2} & \cdots & \frac{\partial^2 \tilde{f}}{\partial x_1 \partial x_k} & \frac{\partial^2 \tilde{f}}{\partial x_1 \partial y} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 \tilde{f}}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 \tilde{f}}{\partial x_k^2} & \frac{\partial^2 \tilde{f}}{\partial x_k \partial y} \\ \frac{\partial^2 \tilde{f}}{\partial y \partial x_1} & \cdots & \frac{\partial^2 \tilde{f}}{\partial y \partial x_k} & \frac{\partial^2 \tilde{f}}{\partial y^2} \end{array} \right) \Bigg|_{p^{(i)}} \neq 0.$$

Hence, the condition that the differential $\mathscr{D}_\lambda \varphi$ is not an isomorphism means that

$$d\lambda \neq 0 \quad \text{and} \quad \left(\frac{\partial \tilde{f}}{\partial \lambda_1}(p^{(i)}), \ldots, \frac{\partial \tilde{f}}{\partial \lambda_\nu}(p^{(i)}) \right) \cdot d\lambda = 0, \quad i = 1, \ldots, \sigma,$$

i.e. means that the matrix

$$\left(\begin{array}{ccc} \frac{\partial \tilde{f}}{\partial \lambda_1}(p^{(1)}) & \cdots & \frac{\partial \tilde{f}}{\partial \lambda_\nu}(p^{(1)}) \\ \vdots & \vdots & \vdots \\ \frac{\partial \tilde{f}}{\partial \lambda_1}(p^{(\sigma)}) & \cdots & \frac{\partial \tilde{f}}{\partial \lambda_\nu}(p^{(\sigma)}) \end{array} \right)$$

is degenerated, and this is equivalent to the fact that the functions

$$e_1 := \frac{\partial \tilde{f}}{\partial \lambda_1}, \ldots, e_\nu := \frac{\partial \tilde{f}}{\partial \lambda_\nu}$$

are linearly dependent on the set N^λ. From Table 5, the functions $e_1, \ldots, e_\nu$ are monomials. Moreover, they give rise to a basis of the C-vector space $\mathcal{O}/\tau_e(f)$. It is easy to see that the monomials

$$e_1 \cdot y^{-2}, \ldots, e_\nu \cdot y^{-2}$$

give rise to a basis of the C-vector space

$$\frac{\mathbf{C}[x_1, \ldots, x_k, y]}{\left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_k}, \ 2h + y\frac{\partial h}{\partial y} \right)}$$

and that

$$N^\lambda = \left\{ (u, x, y, \lambda) \,\middle|\, u = y^2 \tilde{h}, 2\tilde{h} + y\frac{\partial \tilde{h}}{\partial y} = 0, \ \frac{\partial \tilde{h}}{\partial x_i} = 0 \ \text{for} \ i = 1, \ldots, k \right\}$$

Hence, for $\lambda \notin \Sigma$ and sufficiently small, the functions

$$e_1 \cdot y^{-2}, \ldots, e_\nu \cdot y^{-2}$$

give rise to a basis for the space of functions defined on the set N^λ, see for example [3] and recall that $\nu = \sigma$. This is in contradiction with the assumption that $\mathscr{D}_\lambda \varphi$ is not an isomorphism.

3.4. CONCLUSION. *The complement of the bifurcation variety of the simple germs listed in Tables* 1–4 *are spaces of type* $K(\pi, 1)$.

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