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A LUSIN TYPE APPROXIMATION OF BESSEL POTENTIALS AND BESOV FUNCTIONS BY SMOOTH FUNCTIONS

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0. Introduction.

Lusin's theorem is a simple and important result in classical analysis. One way of stating the theorem is for measurable functions f on an interval. For every $\varepsilon > 0$ there is a continuous function g such that $f(x) = g(x)$ except on a set of measure less than ε. F. C. Liu proved the following, v. [6]. If f is a Sobolev function, $f \in W_p^l$, then for every $\varepsilon > 0$ there is a $g \in C^l$ such that if E is the set for which $f(x) \neq g(x)$ then the Sobolev norm of $f - g$, restricted to E, is less than ε. In [7], J. H. Michael and W. P. Ziemer obtained refinements of these results, see also [13], Chapter 3.

In this paper we will prove that if a function f is a Bessel potential or a Besov function, then f can be approximated by smooth functions both in appropriate norm and capacity. This is again a refinement of the results of Liu and Michael-Ziemer.

The main result is stated for Bessel potentials in section 2 and for Besov functions in section 4. Proofs are given in section 3.

1. Notation and definitions.

Let $R^n, n \geqq 1$, denote the n-dimensional Euclidean space. Let Ω denote an arbitrary open subset of R^n. Let $L^p(\Omega)$, $1 \leqq p < \infty$, denote the usual Lebesgue space of measurable functions on Ω and let $\|\cdot\|_p$ denote the norm. The space of Bessel potentials $L^p_\alpha(R^n)$ is defined by

$$L^p_\alpha(R^n) = \{G_\alpha * g : g \in L^p(R^n)\}.$$

Here G_α denotes the Bessel kernel of order $\alpha > 0$, v. [8]. The norm in $L^p_\alpha(R^n)$ is given by $\|G_\alpha * g\|_{\alpha,p} = \|g\|_p$. For $0 < \alpha < \infty$ and $1 < p < \infty$ we define, for a compact set $K \subset R^n$, the Bessel capacity

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$$B_{\alpha,p}(K) = \inf \|f\|_{\alpha,p}^p$$

where the infimum is taken over all functions $f \in C_0^\infty(R^n)$ such that $f \geqq 1$ on K. C_0^∞ denotes the infinitely differentiable functions on R^n with compact support, v. [2] and [9].

For $0 < \alpha < 1$ the Besov space $\Lambda_\alpha^{p,q}(R^n)$ consists of all functions f for which the norm is finite, i.e.

$$\|f\|_{\alpha,p,q} = \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}} dt \right\}^{1/q} < \infty.$$

For $1 \leqq \alpha < 2$ the first difference is replaced by the second difference. For $\alpha > 1$ the space $\Lambda_\alpha^{p,q}(R^n)$ consists of those functions f for which

$$\|f\|_{\alpha,p,q} = \|f\|_p + \sum_{j=1}^n \|D^j f\|_{\alpha-1,p,q} < \infty.$$

Here the derivatives are taken in the sense of distributions, v. [8], Chapter 5.

The Besov space $B_\alpha^{p,q}(E)$ is defined as in [5] for a set $E \subset R^n$, with sequences of families $\{f_v^{(j)}\}_{|j| \leqq [\alpha]}$, $v = 0, 1, 2, \ldots$, where $f_v^{(j)} \in L^p(\mu)$, approximating $f^{(j)}$ in norm. Let E be an arbitrary Borel set and let m_d denote the d-dimensional Hausdorff measure. The set E is a d-set if for any closed ball $B(x,r)$

$$c_1 r^d \leqq m_d(B(x,r) \cap E) \leqq c_2 r^d, \; x \in E, \; r \leqq 1$$

for constants $c_1, c_2 > 0$.

In [5] it is shown that

(1) $$\Lambda_\alpha^{p,q}(R^n)|_E = B_\alpha^{p,q}(E) \text{ when } E \text{ is an } n\text{-set, i.e. } d = n$$

and that

(2) $$\Lambda_\alpha^{p,q}(R^n) = B_\alpha^{p,q}(R^n).$$

For $0 < \alpha < \infty$, $1 < p < \infty$ and $1 < q \leqq \infty$ the Besov capacity for a compact set K is defined by

$$A_{\alpha,p,q}(K) = \begin{cases} \inf \|f\|_{\alpha,p,q}^p & \text{for} \quad p \leqq q \\ \inf \|f\|_{\alpha,p,q}^q & \text{for} \quad p > q \end{cases}$$

where the infimum is taken over all $f \in C_0^\infty(R^n)$ such that $f \geqq 1$ on K, v. [10]. The extension to all sets is made by

$$A_{\alpha,p,q}(E) = \sup_{K \subset E} A_{\alpha,p,q}(K).$$

If the function $\psi \in \Lambda_\gamma^{p,q}(R^n)$, $0 < \gamma < \alpha$, $1 < p < \infty$, then $f = G_{\alpha-\gamma} * \psi$ is well defined, i.e.

$$\int G_{\alpha-\gamma}(x-y)\,|\psi(y)|\,dy < \infty$$

$A_{\alpha,p,q}$ a.e. The corresponding result is true for Bessel potentials.

Differentiability in the L^p-sense will be used. A function f has an L^p-derivative of order j at x_0, $1 \leqq p < \infty$, if there is a polynomial P_{x_0} of degree less than or equal to j such that

$$\left(r^{-n} \int_{|y| \leqq r} |f(x_0 + y) - P_{x_0}(y)|^p \, dy\right)^{1/p} = o(r^j) \tag{4}$$

as $r \to 0$. This is equivalent to f belonging to the Calderon-Zygmund class $t^{j,p}(x_0)$ which is the original concept from [4].

2. Lebesgue points of Bessel potentials and the main approximation result for Bessel potentials.

A Bessel potential $u \in L^p_\alpha(R^n)$ can be defined everywhere except for a set of capacity zero by integral avarages, c.f. [3].

THEOREM A. *Let $\alpha > 0$, $\alpha p < n$, $p > 1$ and let $u \in L^p_\alpha(R^n)$. Then there exists a set E, such that $B_{\alpha,p}(E) = 0$ and*

$$\lim_{\delta \to 0+} \fint_{B(x,\delta)} u(y)\, dy = \bar{u}(x)$$

exists for all $x \in R^n \backslash E$.

Besselpotentials do not have smoothness properties in the classical sense for $\alpha p \leqq n$. But we will show that the function defined quasi almost everywhere with the help of integral averages can be approximated by smooth functions both in norm and capacity. The main result for Bessel potentials can now be stated.

THEOREM 1. *Let $1 < p < \infty$, $0 < j \leqq \alpha$, j integer, and let $(\alpha - j)p < n$. Then, for $u \in L^p_\alpha(R^n)$ and each $\varepsilon > 0$, there exists a function $v \in C^j(R^n)$ such that if*

$$F = \{x : u(x) \neq v(x)\}$$

then

$$B_{\alpha-j,p}(F) < \varepsilon \quad \text{and} \quad \|u - v\|_{j,p} < \varepsilon.$$

When α is a positive integer $L^p_\alpha(R^n) = W^p_\alpha(R^n)$ and as Bessel- and Rieszcapacity are comparable, this result is a generalization of Theorem 3.11.6 in [13]. Theorem 1 is proved in section 3.

3. Proof of Theorem 1, the Bessel case.

The proof depends as in [12] on the Whitney Extension Theorem. We formulate the theorem with Calderon-Zygmund classes.

THEOREM B. *Let $E \subset R^n$ be closed and let $U = \{x: d(x, E) < 1\}$. If $u \in L^p(U)$, $1 \leqq p \leqq \infty$ and $u \in t^{j,1}(x)$ for all $x \in E$ with condition* (4) *holding uniformly on E, then there exists $\bar{u} \in C^j(U)$ such that*

$$D^\beta \bar{u}(x) = D^\beta P_x(x)$$

for $x \in E$, $0 \leqq |\beta| \leqq j$.

We will now show that the hypothesis of the Whitney Extension Theorem is satisfied by $u \in L^p_\alpha(R^n)$.

LEMMA 1. *Let $0 \leqq j \leqq k$, j integer, and let $(k-j)p < n$. Let $u \in L^p_k(R^n)$. Then, for every $\varepsilon > 0$, there exists an open set U with $B_{k-j,p}(U) < \varepsilon$ such that*

$$r^{-j} \fint_{B(x,r)} |u(y) - P_x(y)|\, dy \to 0 \tag{5}$$

uniformly on $R^n \setminus U$ as $r \to 0$, i.e. *$u \in t^{j,1}(x)$ holds uniformly when $x \in R^n \setminus U$. $P_x(y)$ is a polynomial of degree at most j.*

PROOF OF LEMMA 1. The lemma is proved for $j = 1$. Choose $\varepsilon > 0$ arbitrarily. We write $u(x) = (G_k * \psi)(x)$, $\psi \in L^p(R^n)$. Then there exist a constant C and a set E_1, such that $(G_k * |\psi|)(x) \leqq C$ for all $x \in R^n \setminus E_1$, where $B_{\alpha,p}(E_1) < \varepsilon$. Choose C and E_1 such that $(G_k * |\psi|(x) \leqq C$, $(G_{k-1} * |\psi|)(x_0) \leqq C$ and $B_{k-1,p}(E_1) < \varepsilon/2$. Choose a point x outside E_1. Consider the Taylor polynomial

$$P_x(y) = u(x) + \sum_{i=1}^{n} \int_{R^n} \psi(t) \frac{\partial G_k}{\partial x_i}(x-t)\, dt (x_i - y_i).$$

We investigate the L^1-derivative and get

$$\frac{1}{r^{n+1}} \int_{|x-y| \leqq r} |u(y) - P_x(y)|\, dy \leqq$$

$$\leqq \frac{1}{r^{n+1}} \int_{|x-y| \leqq r} dy \int_{|x-t| \leqq 2r} |\psi(t)| \Big| G_k(y-t) - G_k(x-t) -$$

$$- \sum_{i=1}^{n} \frac{\partial G_k}{\partial x_i}(x-t)(x_i - y_i) \Big|\, dt +$$

$$+\frac{1}{r^{n+1}}\int_{|x-y|\leqq r}dy\int_{2r<|x-t|\leqq 1}|\psi(t)|\bigg|G_k(y-t)-G_k(x-t)-$$

$$-\sum_{i=1}^{n}\frac{\partial G_k}{\partial x_i}(x-t)(x_i-y_i)\bigg|dt+$$

$$+\frac{1}{r^{n+1}}\int_{|x-y|\leqq r}dy\int_{|x-t|>1}|\psi(t)|\bigg|G_k(y-t)-G_k(x-t)-$$

$$-\sum_{i=1}^{n}\frac{\partial G_k}{\partial x_i}(x-t)(x_i-y_i)\bigg|dt=J_1+J_2+J_3.$$

Make a dyadic decomposition of the set $\{r: 2r<|x_0-t|\leqq 1\}$ and let $\frac{1}{2^{M+1}}<r\leqq\frac{1}{2^M}$, M a positive integer. If we choose M large enough then

$$J_1\leqq\frac{1}{r^{n+1}}\int_{|x-t|\leqq 2r}|\psi(t)|\int_{|x-y|\leqq r}\left(\frac{c_1}{|y-t|^{n-k}}+\frac{c_2 r}{|x-t|^{n-k+1}}\right)dydt\leqq$$

$$\leqq c\int_{|x-t|\leqq 2^{1-M}}|\psi(t)|\,G_{k-1}(x-t)\,dt<\varepsilon$$

except on a set E_2, $B_{k-1,p}(E_2)<\varepsilon/2$. To see this, we observe that there is a function $\psi_i\in\mathscr{S}$, $\psi_i\to\psi$ in L^p-norm as $i\to\infty$. We use $\mathscr{S}$ for the Schwartz spaces of C^∞ rapidly decreasing functions on R^n. Then

$$\left\{x:\int_{|x-t|\leqq r}|\psi(t)-\psi_i(t)|\frac{1}{|x-t|^{n-k+1}}dt\geqq\varepsilon\right\}\subseteq$$

$$\subseteq\left\{x:\int_{R^n}|\psi(t)-\psi_i(t)|\,G_{k-1}(x-t)\,dt\geqq\varepsilon\right\}=E_2$$

and hence

$$B_{k-1,p}(E_2)=B_{k-1,p}\left(\left\{x:\int_{R^n}\frac{|\psi(t)-\psi_i(t)|}{\varepsilon}G_{k-1}(x-t)\,dt\geqq 1\right\}\right)\leqq$$

$$\leqq\left\|\frac{\psi-\psi_i}{\varepsilon}\right\|_p^p=\frac{1}{\varepsilon^p}\|\psi-\psi_i\|^p<\frac{\varepsilon}{2}$$

if only i is chosen sufficiently large. Thus, there exists an integer M_1, so that for $M\geqq M_1$,

$$J_1 \leqq \int\limits_{|x-t| \leqq 2^{1-M}} \left|\psi(t) - \psi_i(t)\right| \frac{1}{|x-t|^{n-k+1}}\, dt + $$
$$+ \int\limits_{|x-t| \leqq 2^{1-M}} |\psi_i(t)| \frac{1}{|x-t|^{n-k+1}}\, dt < 2\varepsilon$$

as the functions ψ_i are continuous with compact support and therefore bounded.

Using wellknown properties of the Bessel kernel when $2r < |x - t| \leqq 1$ and $|y - t| \leqq 1 + r$ we get for sufficiently small $r > 0$, ($M \geqq M_2$) that

$$\left| G_k(y-t) - G_k(x-t) - \sum_{i=1}^{n} \frac{\partial G_k}{\partial x_i}(x-t)(y_i - x_i) \right| \leqq \frac{c_1}{|x-t|^{n-k+2}} |y-x|^2.$$

Hence,

$$J_2 \leqq cr \int\limits_{2r < |x-t| \leqq 1} |\psi(t)| \frac{1}{|x-t|^{n-k+2}}\, dt.$$

Using the dyadic decomposition and choosing $(M + 1)/2$ in the sum below if M is odd we have

$$J_2 \leqq \frac{c}{2^M} \sum_{m=1}^{M/2} 2^m \int\limits_{\frac{1}{2^m} \leqq |x-t| \leqq \frac{1}{2^{m-1}}} |\psi(t)| \frac{1}{|x-t|^{n-k+1}}\, dt + $$
$$+ \frac{c}{2^M} \sum_{m=M/2+1}^{M} 2^m \int\limits_{\frac{1}{2^m} \leqq |x-t| \leqq \frac{1}{2^{m-1}}} |\psi(t) \frac{1}{|x-t|^{n-k+1}}\, dt \leqq$$
$$\leqq \left(\frac{M/2}{2^{M/2}} \int\limits_{|x-t| \leqq 1} |\psi(t)| \frac{1}{|x-t|^{n-k+1}}\, dt + \int\limits_{|x-t| \leqq \frac{1}{2^{M/2}}} |\psi(t)| \frac{1}{|x-t|^{n-k+1}}\, dt \right)$$
$$\leqq c_1 \left(\frac{M}{2^{M/2}} (G_{k-1} * |\psi|)(x) + \int\limits_{|x-t| \leqq \frac{1}{2^{M/2}}} |\psi(t)|\, G_{k-1}(x-t)\, dt \right) < 2\varepsilon.$$

The last inequality follows as before for all $x \notin E_1 \cup E_2$ and M sufficiently large.

Now consider $|x - t| > 1$ and $|x - y| \leqq r$. Then $|y - t| > 1 - r$. For $|x - t + \theta(y - x)| > 1$ and r small enough we have

$$\left| \frac{\partial G_k}{\partial x_i}((x-t) + \theta(y-x)) \right| \leqq c G_{k-1}(x-t).$$

Hence $J_3 \leqq c_n r G_{k-1}(x - t)$, where the constant only depends on the dimension. Now

$$J_3 \leqq cr \int_{|x-t|>1} |\psi(t)|\, G_{k-1}(x-t)\,dt \leqq c2^{-M}(G_{k-1} * |\psi|)(x) < \varepsilon$$

for $x \notin E_1 \cup E_2$, if M is chosen large enough. Now let U be an open covering of $E_1 \cup E_2$ with $B_{k-1,p}(U) < \varepsilon$.

For $j = 0$ and $j \geqq 2$ we use the same method. This completes the proof of Lemma 1.

THEOREM C. *Let $0 \leqq j \leqq \alpha$, j integer. Let $1 < p < \infty$ and $(\alpha - j)p < n$. Let $u \in L^p_\alpha(R^n)$ and $\varepsilon > 0$. Then there exist an open set $U \subset R^n$ and a C^j function v on R^n, such that*

$$B_{\alpha-j,p}(U) < \varepsilon$$

and

$$D^\beta v(x) = D^\beta u(x)$$

for all $x \in R^n \setminus U$ and $0 \leqq |\beta| \leqq j$.

Theorem C is a direct consequence of Theorem B and Lemma 1. To make the approximation close to u in norm we need the following lemmas.

LEMMA 2. *Let $\alpha > 0$ and let $u \in L^p_\alpha(R^n)$ which vanishes outside a bounded open set U. Let $\delta, \sigma \in (0,1)$ and let*

$$E = \delta U \cap \left\{x\colon \inf_{0<t\leqq\delta} \frac{m(K(x,t) \cap (R^n \setminus U))}{t^n} \geqq \sigma\right\}$$

where $K(x,t)$ denotes the closed cube with center x and side-length t. Let β be a positive real number such that $\beta \leqq \alpha$ and let $\varepsilon > 0$. Then there exists a function $v \in L^p_\beta(R^n)$ and an open set V such that

(i) $\|u - v\|_{\beta,p} < \varepsilon$

(ii) *$E \subset V$ and $v(x) = 0$ when $x \in V \cup (R^n \setminus U)$.*

PROOF OF LEMMA 2. The function v is constructed as in [13]. It is given here for completeness and for the proof of (i). Let $\lambda \in (0,1]$ and let K_λ denote the set of all closed cubes

$$[(i_1 - 1)\lambda, i_1\lambda] \times [(i_2 - 1)\lambda, i_2\lambda] \times \ldots \times [(i_n - 1)\lambda, i_n\lambda]$$

where $i_1, i_2, \ldots, i_n$ are arbitrary integers. Let $\lambda \leqq \dfrac{\delta}{3}$ and let $K_1, \ldots, K_r$ be the cubes of K_λ that intersect E. Let a_i be the center of K_i and let $P_i = K(a_i, 4\lambda)$. Let ζ be a C^∞ function on R^n, such that $0 \leqq \zeta \leqq 1$ and

$$\zeta(x) = \begin{cases} 0, & x \in K(0,1) \\ 1, & x \notin K(0,3/2). \end{cases}$$

Define

$$u_\lambda(x) = u(x) \prod_{i=1}^{r} \zeta\left(\frac{x - a_i}{2\lambda}\right).$$

Then $u_\lambda(x) = 0$ when $d(x,E) \leqq \frac{1}{2}\lambda$, so that, for any sufficiently small λ, we can define v by $v = u_\lambda$ and find an open set V satisfying (ii).

It remains to prove that $\|u - u_\lambda\|_{\alpha,p} \to 0$ as $\lambda \to 0+$. We use the norm $\|f\|_{\alpha,p} = \|f\|_p + \|D_\alpha(f)\|_p$ for $0 < \alpha < 2$, v. [8], where

$$D_\alpha(f(x)) = \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} dy.$$

We observe that there exists a constant $\tau = \tau(n)$ such that at most τ of the cubes P_j intersect P_i, say $j_1, \ldots, j_\tau$. Then for $x \in P_i$

$$u_\lambda(x) = u(x) \prod_{i=1}^{\tau} \zeta((x - a_{j_i})/2\lambda).$$

Let $\bigcup_{i=1}^{r} P_i = P_\lambda$. Then $U \cap P_\lambda \subset U \cap \{x: d(x, \delta U) < 2\sqrt{n}\lambda\}$ and $m\{U \cap P_\lambda\} \to 0$ as $\lambda \to 0+$. Now

$$\|D_\alpha(u(x) - u_\lambda(x))\|_p^p =$$

$$= \int\limits_{R^n} \left| \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} [(u(x+y) - u_\lambda(x+y)) - (u(x) - u_\lambda(x))] \frac{dy}{|y|^{n+\alpha}} \right|^p dx =$$

$$= \int\limits_{U \cap P_\lambda} \left| \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} (u(x+y) - u(x)) \frac{dy}{|y|^{n+\alpha}} \right|^p dx +$$

$$+ \int\limits_{U \cap P_\lambda} \left| \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} (u_\lambda(x+y) - u_\lambda(x)) \frac{dy}{|y|^{n+\alpha}} \right|^p dx +$$

$$+ \int\limits_{R^n \setminus (U \cap P_\lambda)} \left| \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} \frac{(u(x+y) - u(x)) - (u_\lambda(x+y) - u_\lambda(x))}{|y|^{n+\alpha}} dy \right|^p dx =$$

$$= J_1 + J_2 + J_3.$$

As $u, u_\lambda \in L_\alpha^p(R^n)$, the integrals J_1 and J_2 tend to zero when $\lambda \to 0+$. We observe that $u(x) - u_\lambda(x) = 0$ whenever $x \in R^n \setminus (U \cap P_\lambda)$. Then putting $x + y = x'$ we have

$$J_3^{1/p} \leqq \left(\int\limits_{U \cap P_\lambda} \left| \lim_{\varepsilon \to 0} \int\limits_{|y| \geqq \varepsilon} \frac{u(x') - u(x' - y) - (u_\lambda(x') - u_\lambda(x' - y))\, dy}{|y|^{n+\alpha}} \right|^p dx' \right)^{1/p}$$

$$\leqq \left(\int\limits_{U \cap P_\lambda} \left| \lim_{\varepsilon \to 0} \int\limits_{|y'| \geqq \varepsilon} \frac{u(x' + y') - u(x')}{|y'|^{n+\alpha}} dy' \right|^p dx' \right)^{1/p} +$$

$$+ \left(\int\limits_{U \cap P_\lambda} \left| \lim_{\varepsilon \to 0} \int\limits_{|y'| \geqq \varepsilon} \frac{u_\lambda(x' + y') - u_\lambda(x')}{|y'|^{n+\alpha}} dy \right|^p dx' \right)^{1/p}.$$

In the last two integrals we have put $y' = -y$. As $m(U \backslash P_\lambda) \to 0$ when $\lambda \to 0+$, the integrals tend to zero.

Here we have used the fact that $u_\lambda(x) = u(x) \cdot \prod_{i=1}^{\tau} \zeta\left(\frac{x - a_i}{2\lambda}\right)$ belongs to $L_\alpha^p(R^n)$ as $u \in L_\alpha^p(R^n)$ and $\psi(x) = \prod_{i=1}^{\tau} \zeta\left(\frac{x - a_i}{2\lambda}\right) \in C_0^\infty(R^n)$ where $\psi \in C_0^\infty$ and is zero outside a bounded open set containing U.

For $\alpha > 1$ we can work as usual with the derivatives. This completes the proof of the lemma, as for $\beta \leqq \alpha$ the norms are smaller.

LEMMA 3. *Let k be a nonnegative real number such that $kp < n$. Let U be a bounded non-empty open subset of R^n and F a subset of δU with the property that for each $x \in F$, there is a $t \in (0, 1)$ for which*

$$\frac{m(U \cap B(x, t))}{m(B(x, t))} \geqq \sigma$$

where $\sigma \in (0, 1)$. Then there exists a constant $C = C(n, p, k)$ such that

$$B_{k,p}(U \cup F) \leqq C\sigma^{-p} B_{k,p}(U).$$

For the method of proof, v. Lemma 3.11.4 in [13] with Riesz capacity replaced by Bessel capacity. The proof of Lemma 3.11.6 in [13] can also be modified to give the following.

LEMMA 4. *Let $1 < p < \infty$ and let k and α be real numbers such that $kp < n$ and $0 < k \leqq \alpha$. There exists a constant $C = C(n, p, k, \alpha)$ such that for each bounded non-empty open subset U of R^n, each $u \in L_\alpha^p(R^n)$ which vanishes outside U and every $\varepsilon > 0$ there exists a C^∞ function v on R^n with the properties*

(i) $\|u - v\|_{\beta,p} < \varepsilon, 0 < \beta \leqq \alpha$

(ii) $B_{k,p}(\operatorname{supp} v) \leqq C B_{k,p}(U)$

(iii) $\operatorname{supp} v \subset V = R^n \cap \{x: d(x, U) < \varepsilon\}$.

Now we are ready to prove the main result.

PROOF OF THEOREM 1. We can assume that the set

$$A = R^n \cap \{x: u(x) \neq 0\} \neq \emptyset$$

and that A is bounded, v. [9]. Choose $\varepsilon > 0$ arbitrarily. We show that there exists a C^j function v on R^n such that if $F = \{x: u(x) \neq v(x)\}$ then

$$B_{\alpha-j,p}(F) < \varepsilon \quad \text{and} \quad \|u - v\|_{j,p} < \varepsilon.$$

Let c be the constant of Lemma 4. Let u be defined by its values at Lebesgue points except for a set E with $B_{\alpha,p}(E) = 0$. By Theorem C there exists an open set $U \subseteq R^n$ and a C^j function h on R^n, such that $U \supset E$,

$$B_{\alpha-j,p}(U) < \frac{\varepsilon}{1+c} \tag{1}$$

and

$$h(x) = u(x)$$

for all $x \in R^n \setminus U$. We substitute $\alpha - j$ for k and $u - h$ for u in Lemma 4 and get a C_0^∞ function φ on R^n such that

$$\|(u - h) - \varphi\|_{j,p} < \varepsilon \tag{2}$$

and

$$B_{\alpha-j,p}(\operatorname{supp} \varphi) \leqq cB_{\alpha-j,p}(U).$$

Put $v = h + \varphi$. Then the second part of the theorem follows from (2). Also

$$F \subset [\{x: h(x) \neq u(x)\} \cup \operatorname{supp} \varphi] \subset [U \cup \operatorname{supp} \varphi]$$

so that

$$B_{\alpha-j,p}(F) \leqq (1 + c)B_{\alpha-j,p}(U)$$

and the first part of the theorem follows from (1). This completes the proof of the theorem.

4. Approximation results for Besov functions.

The approximation results shown in section 2 are valid also for Besov functions $f \in B_\alpha^{p,q}(\Omega)$. In the Sobolev case it is meaningful to consider functions defined on open subsets of R^n, v. [7]. This is also true for Besov functions. In order to define the functions except on a set of Besov capacity zero the set Ω has to be an open n-set.

THEOREM 2. *Let $1 < p < \infty$, $1 < q \leqq \infty$, $0 < j \leqq \alpha$, where j integer and let $(\alpha - j)p < n$. Let Ω be an open n-set. Then for $u \in B_\alpha^{p,q}(\Omega)$ and each $\varepsilon > 0$, there exists a function $v \in C^j(\Omega)$ such that if*

$$F = \Omega \cap \{x: u(x) \neq v(x)\}$$

then

$$A_{\alpha-j,p,q}(F) < \varepsilon \quad \textit{and} \quad \|u - v\|_{j,p,q} < \varepsilon.$$

REMARK. The results formulated in Theorem 1 and 2 are also true for Triebel-Lizorkin spaces $F_\alpha^{p,q}(R^n)$, v. [10] for definitions and the corresponding Triebel-Lizorkin capacity.

The proof of Theorem 2 is much the same as the proof of Theorem 1 when $\Omega = R^n$. For $\Omega \subset R^n$ an n-set we have to make some modifications. For such a set Ω, both the function $D^j f$, $|j| \leqq [\alpha]$, in $B_\alpha^{p,q}(\Omega)$ and an arbitrary extension $D^j(\varepsilon f)$ to a function in $B_\alpha^{p,q}(R^n)$, can be strictly defined except on appropriate exceptional sets. For a detailed proof, v. [11].

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