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HOLOMORPHIC DIRICHLET SERIES IN SEVERAL VARIABLES*

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1. Introduction.

Entire functions defined by multiple Dirichlet series of the form

$$(1.1)\qquad \sum_{m,n=1}^{\infty} c_{m,n} e^{\lambda_m z_1 + \mu_n z_2},\ z_1, z_2 \in \mathbf{C},$$

(here for the sake of simplicity we write in the case of two variables) with the real (even positive) frequencies (λ_m), (μ_n), have been extensively investigated by many authors. Several problems on orders, types, mean values, topological structures, functionals, operators, bases and so on were considered.

What about non-entire holomorphic functions? The aim of this note is to solve some of the problems concerning holomorphic multiple Dirichlet series in a bounded convex domain in $\mathbf{C}^n$.

It should be noted that the frequencies in (1.1) are taken as all possible combinations of the corresponding sequences $(\lambda_m)_{m=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$. It is clear that the frequencies established in such a way are not invariant under linear transformations and, therefore cannot be applied to a general case of domains in $\mathbf{C}^n$, except particular forms (polycylindrical and complete circular domains).

Before describing briefly the content of our work we make the following note. During the last two decades there has been developed extensively a concept of representing systems of exponents (see, e.g., [5, 8]) which shows, in particular, that every entire function as well as every holomorphic function in a convex domain can be represented in the form of Dirichlet series. Moreover, this representation is not unique. In connection with this we should have a careful look at the problem of functions defined by Dirichlet series since it may happen that, depending on the choice of frequencies, different series can represent the

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same function. More precisely, we should distinguish the class of series from the class of functions they represent.

In our paper we consider a multiple Dirichlet series, with a system of complex frequencies $(\lambda^k)_{k=1}^{\infty}$, $\lambda^k \in \mathbf{C}^n$, which is holomorphic in a bounded convex domain in $\mathbf{C}^n$ (not necessarily containing the origin of coordinates). Section 2 deals with some auxiliary results concerning convergence of multiple Dirichlet series in a space of holomorphic functions, which are essentially necessary and important for the holomorphic case. In section 3 we study a sequence space of the coefficients of multiple Dirichlet series. There we follow the terminology in [6]. In section 4 we introduce a class of holomorphic multiple Dirichlet series in a bounded convex domain, endow it with a topological structure and show that this is a nuclear (F)-space (complete metrizable topological vector space). And then we study linear continuous functionals on this space. Differential operators, also of infinite order, are considered. Some of the results in this section are obtained in the spirit of [4], which investigated multiple power series of entire functions. Section 5 concerns one of the possible subclasses of the class introduced in a previous section. It is shown that this subclass with a suitable choice of norm becomes a commutative Banach algebra. Some properties of this algebra are considered. In particular, all results on entire series of the paper [2] are generalized to the case of holomorphic series mainly in a similar way with corresponding modifications.

As will be seen in this note, the supporting function of a convex domain in $\mathbf{C}^n$ plays an essential role in our discussions.

Also note that a topological structure for a space of so-called analytic Dirichlet transformations was considered earlier in [3]. However, in that paper it is done for a particular case: product of half-planes and, moreover, the frequencies are real and of the form (1.1). The techniques used in [3] are essentially real, one-dimensional and, therefore, do not work for the general domains in $\mathbf{C}^n$ considered in our paper.

A final remark is that the topological structures studied in our note are quite different from those in [3] and [4].

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2. Preliminary and auxiliary results.

We use some basic notation:

$\mathcal{O}(\Omega)$ (Ω being a domain in $\mathbf{C}^n$) denotes the space of holomorphic functions in Ω, with the topology of uniform convergence on compact subsets of Ω.

If $z, \zeta \in \mathbf{C}^n$ then $|z| = (z_1\bar{z}_1 + \ldots + z_1\bar{z}_n)^{1/2}$; $\langle z, \zeta \rangle = z_1\zeta_1 + \ldots + z_n\zeta_n$.

Let Ω be a bounded convex domain in $\mathbf{C}^n$, with the supporting function defined as follows

$$H_\Omega(\zeta) = \sup_{z \in \Omega} \operatorname{Re} \langle z, \zeta \rangle, \ \zeta \in \mathbf{C}^n.$$

For a point $a \in \Omega$ we denote

$$\Omega_t^a = (1-t)a + t\Omega, \ 0 < t < 1, \tag{2.1}$$

and

$$\Omega(a) = \Omega - a = \{z - a : z \in \Omega\}. \tag{2.2}$$

We see that $\Omega_t^a \subset \Omega$ and

$$H_{\Omega_t^a}(\zeta) = (1-t)\operatorname{Re}\langle a, \zeta \rangle + tH_\Omega(\zeta), \ \zeta \in \mathbf{C}^n. \tag{2.3}$$

Also we have

$$H_{\Omega(a)}(\zeta) = H_\Omega(\zeta) - \operatorname{Re}\langle a, \zeta \rangle, \ \zeta \in \mathbf{C}^n. \tag{2.4}$$

Furthermore, since $0 \in \Omega(a)$ it is clear that

$$0 < \alpha_a = \inf_{|\zeta|=1} H_{\Omega(a)}(\zeta) \leqq \beta_a = \sup_{|\zeta|=1} H_{\Omega(a)}(\zeta) < \infty, \tag{2.5}$$

and, therefore

$$\alpha_a|\zeta| \leqq H_{\Omega(a)}(\zeta) \leqq \beta_a|\zeta|, \ \forall \zeta \in \mathbf{C}^n. \tag{2.6}$$

Now let $(\lambda^k)_{k=1}^\infty$ be a sequence of complex vectors in $\mathbf{C}^n$. Consider a multiple Dirichlet series

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}, \ z \in \Omega. \tag{2.7}$$

First we make a characterization of the coefficients of the series (2.7) when it converges for the topology of $\mathcal{O}(\Omega)$.

THEOREM 2.1. *If the multiple Dirichlet series (2.7) converges for the topology of $\mathcal{O}(\Omega)$ and $|\lambda^k| \to \infty$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} \frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leqq 0. \tag{2.8}$$

Conversely, if the coefficients of (2.7) satisfy condition (2.8) and if

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0, \tag{2.9}$$

then the series (2.7) converges absolutely for the topology of $\mathcal{O}(\Omega)$.

PROOF. *Necessity.* Suppose that the series (2.7) converges for the topology of $\mathcal{O}(\Omega)$. Fix a point $a \in \Omega$. Then for any $t \in (0, 1)$ there exists a positive constant $A < \infty$ such that

$$\sup\{|c_k e^{\langle \lambda^k, z\rangle}| : z \in \Omega_t^a,\ k \geqq 1\} \leqq A,$$

which, in view of (2.3), is equivalent to

$$|c_k| e^{(1-t)\mathrm{Re}\langle a, \lambda^k\rangle + tH_\Omega(\lambda^k)} \leqq A,\ \forall k \geqq 1.$$

Combining this inequality, (2.4) and (2.6) gives

$$\begin{aligned}\frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} &\leqq \frac{\log A}{|\lambda^k|} + \frac{(1-t)[H_\Omega(\lambda^k) - \mathrm{Re}\langle a, \lambda^k\rangle]}{|\lambda^k|} \\ &= \frac{\log A}{|\lambda^k|} + \frac{(1-t)H_{\Omega(a)}(\lambda^k)}{|\lambda^k|} \leqq \frac{\log A}{|\lambda^k|} + (1-t)\beta_a,\end{aligned}$$

where $\Omega(a)$ and β_a are defined by (2.2) and (2.5) respectively.

Consequently,

$$\limsup_{k\to\infty} \frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leqq (1-t)\beta_a.$$

Letting t tend to 1, we obtain the inequality (2.8).

Sufficiency. Let conditions (2.8) and (2.9) hold. Take an arbitrary compact subset K of Ω. We fix some point $a \in K$. Then it is clear that $K \subset \Omega_t^a$ for some $t \in (0, 1)$, where Ω_t^a is defined by (2.1). We shall prove that

$$\sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)} < \infty.$$

By (2.8), for $0 < \varepsilon < (1-t)\alpha_a$, where α_a is defined by (2.5), there exists N_1 such that $\forall k > N_1$

$$\frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} < \varepsilon,$$

or

$$|c_k|\, e^{H_\Omega(\lambda^k)} < e^{\varepsilon|\lambda^k|}.$$

Hence, for $k > N_1$

$$|c_k| e^{H_{\Omega_t^a}(\lambda^k)} = |c_k| e^{(1-t)\mathrm{Re}\langle a,\lambda^k\rangle + tH_\Omega(\lambda^k)}$$
$$< e^{(t-1)[H_\Omega(\lambda^k) - \mathrm{Re}\langle a,\lambda^k\rangle] + \varepsilon|\lambda^k|} < e^{(t-1)\alpha_a|\lambda^k| + \varepsilon|\lambda^k|}.$$

By (2.9), there exists N_2 such that $\forall k > N_2$

$$[(1-t)\alpha_a - \varepsilon]\,|\lambda^k| > 2\log k,$$

or

$$e^{(t-1)\alpha_a|\lambda^k| + \varepsilon|\lambda^k|} < \frac{1}{k^2}.$$

Therefore, $\forall k > \max(N_1, N_2)$

$$|c_k|\, e^{(1-t)\mathrm{Re}\langle a,\lambda^k\rangle + tH_\Omega(\lambda^k)} < \frac{1}{k^2}.$$

So, we get that

$$\sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)} < \infty,$$

which means that the series (2.7) converges absolutely for the topology of $\mathcal{O}(\Omega)$.

COROLLARY 2.2. *If (2.9) holds, then the series (2.7) converges for the topology of $\mathcal{O}(\Omega)$ if and only if it converges absolutely for the topology of $\mathcal{O}(\Omega)$.*

REMARKS 2.3. 1) For the case when Ω contains the origin of coordinates Theorem 2.1, with $a = 0$, is a generalization to n variables of the corresponding result in [5] and is proved analogously.

2) A similar result for a polycylindrical domain containing the origin of coordinates was also obtained earlier in [9].

3. The sequence space $\Lambda(\Omega)$ of Dirichlet coefficients.

We denote by $\Lambda(\Omega)$ the set of sequences (c_k) satisfying condition (2.8) and call it, following Köthe [6], a sequence space. We shall study some properties of this space.

First note that whenever $\Lambda(\Omega)$ contains (c_k) it also contains (d_k) with $|d_k| \leqq |c_k|$ for $k = 1, 2, \ldots$. So $\Lambda(\Omega)$ is normal.

Denote by $\Lambda^\alpha(\Omega)$ the Köthe dual of $\Lambda(\Omega)$, i.e.

$$\Lambda^\alpha(\Omega) = \left\{(u_k) : \sum_{k=1}^{\infty} c_k u_k \text{ converges absolutely for all } (c_k) \in \Lambda(\Omega)\right\}.$$

Also we introduce the following set

$$\Lambda_0(\Omega) = \left\{(u_k) : \sum_{k=1}^{\infty} c_k u_k \text{ converges for all } (c_k) \in \Lambda(\Omega)\right\}.$$

It is obvious that $\Lambda^\alpha(\Omega) \subset \Lambda_0(\Omega)$. We shall see that the condition (2.9) is sufficient for the equality $\Lambda^\alpha(\Omega) = \Lambda_0(\Omega)$.

We make a characterization of the Köthe dual (cf. Theorem 2.1).

THEOREM 3.1. *If $(u_k) \in \Lambda_0(\Omega)$, then the following condition holds*

$$\text{(3.1)} \qquad \limsup_{k\to\infty} \frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} < 1 \textit{ for some (for all) } a \in \Omega.$$

Conversely, if the sequence (u_k) satisfies condition (3.1) and, in addition, the sequence (λ^k) satisfies condition (2.9), then $(u_k) \in \Lambda^\alpha(\Omega)$.

PROOF. First of all we note that in view of (2.4) — (2.6) we have

$$H_\Omega(\zeta) - \operatorname{Re}\langle a, \zeta\rangle = H_{\Omega(a)}(\zeta) \geqq |\zeta|\alpha_a > 0, \forall \zeta \in \mathbf{C}^n, \zeta \neq 0.$$

Necessity. Suppose that (3.1) is not true. Then

$$\limsup_{k\to\infty} \frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} \geqq 1,$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, for a sequence $(\varepsilon_p)_{p=1}^{\infty} \downarrow 0$ there exists an increasing sequence $(k_p)_{p=1}^{\infty}$ of positive numbers such that

$$\frac{\log|u_{k_p}| - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}{H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle} \geqq 1 - \varepsilon_p, \forall p \geqq 1,$$

which is equivalent to

$$\log(1/|u_{k_p}|) \leqq (1 - \varepsilon_p)[\operatorname{Re}\langle a, \lambda^{k_p}\rangle - H_\Omega(\lambda^{k_p})] - \operatorname{Re}\langle a, \lambda^{k_p}\rangle.$$

Define a sequence (c_k) as follows

$$c_k = \begin{cases} 1/|u_k|, \text{ if } k = k_p, p = 1, 2, \ldots, \\ 0, \quad \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{k\to\infty}\frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leqq \limsup_{p\to\infty}\frac{\log(1/|u_{k_p}|) + H_\Omega(\lambda^{k_p})}{|\lambda^{k_p}|}$$

$$\leqq \limsup_{p\to\infty}\frac{\varepsilon_p[H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle]}{|\lambda^{k_p}|}$$

$$= \limsup_{p\to\infty}\frac{\varepsilon_p H_{\Omega(a)}(\lambda^{k_p})}{|\lambda^{k_p}|} = 0,$$

which means that (c_k) is in $\Lambda(\Omega)$.

However, since $|c_k u_k| = 1$ for $k = k_p$ ($p = 1, 2, \ldots$) it follows that $c_k u_k$ does not tend to 0 as $k \to \infty$. So, the series $\sum_{k=1}^{\infty} c_k u_k$ does not converge. We get a contradiction.

Sufficiency. Assume that there exists a constant Q such that (3.1) holds, i.e.

$$\limsup_{k\to\infty}\frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} = Q < 1, \text{ for some (for all) } a \in \Omega,$$

and also the condition (2.9) is satisfied.

Then for $\varepsilon > 0$ (satisfying $Q + \varepsilon < 1$) there exists N such that $\forall k > N$

$$\frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} \leqq Q + \varepsilon,$$

or, equivalently,

$$|u_k| \leqq e^{(Q+\varepsilon)H_\Omega(\lambda^k) + [1-(Q+\varepsilon)]\operatorname{Re}\langle a, \lambda^k\rangle}.$$

On the other hand, as we have already seen from the proof of the sufficiency part in Theorem 2.1, with condition (2.9) for every sequence (c_k) from $\Lambda(\Omega)$ the series

$$\sum_{k=1}^{\infty} |c_k| e^{(1-t)\operatorname{Re}\langle a, \lambda^k\rangle + tH_\Omega(\lambda^k)}$$

converges for any $t \in (0, 1)$.

Consequently, the series $\sum_{k=1}^{\infty} c_k u_k$ converges absolutely. This completes the proof.

COROLLARY 3.2. *If (2.9) holds, then $(u_k) \in \Lambda_0(\Omega)$ if and only if $(u_k) \in \Lambda^\alpha(\Omega)$*, i.e. $\Lambda^\alpha(\Omega) = \Lambda_0(\Omega)$. *In this case these sequence spaces can be defined as follows*

$$\Lambda_0(\Omega) = \Lambda^\alpha(\Omega) = \{(u_k) \text{ satisfying condition (3.1)}\}.$$

It is clear that $\Lambda(\Omega) \subset \Lambda^{\alpha\alpha}(\Omega)$. We shall prove that with condition (2.9) the inverse inclusion is true.

THEOREM 3.3. *Suppose that condition (2.9) holds. Then the sequence space* $\Lambda(\Omega)$ *is perfect*, i.e. $\Lambda^{\alpha\alpha}(\Omega) = \Lambda(\Omega)$.

PROOF. We follow the scheme of the proof of the necessity part in Theorem 3.1. Suppose that $(c_k) \notin \Lambda(\Omega)$. This means that

$$\limsup_{k\to\infty} \frac{\log|c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} > 0,$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, there exists $Q > 0$ such that for a sequence $(\varepsilon_p)_{p=1}^\infty \downarrow 0$ there exists an increasing sequence $(k_p)_{p=1}^\infty$ of positive numbers such that

$$\frac{\log|c_{k_p}| + H_\Omega(\lambda^{k_p})}{|\lambda^{k_p}|} \geqq Q - \varepsilon_p, \ \forall p \geqq 1,$$

which is equivalent to

$$\log(1/|c_{k_p}|) \leqq H_\Omega(\lambda^{k_p}) - (Q - \varepsilon_p)|\lambda^{k_p}|.$$

Define a sequence (u_k) as follows

$$u_k = \begin{cases} 1/|c_k|, & \text{if } k = k_p, p = 1,2,\ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have, for any $a \in \Omega$,

$$\frac{\log|u_{k_p}| - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}{H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle} = \frac{\log(1/|c_{k_p}|) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}{H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}$$

$$\leqq \frac{H_\Omega(\lambda^{k_p}) - (Q - \varepsilon_p)|\lambda^{k_p}| - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}{H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle} = 1 - (Q - \varepsilon_p)\frac{|\lambda^{k_p}|}{H_{\Omega(a)}(\lambda^{k_p})}$$

$$\leqq 1 - \frac{Q}{\beta_a} + \varepsilon_p \frac{|\lambda^{k_p}|}{H_{\Omega(a)}(\lambda^{k_p})},$$

where β_a is defined by (2.5).

Consequently, taking into account that $\left\{\frac{|\lambda^{k_p}|}{H_{\Omega(a)}(\lambda^{k_p})}\right\}_{p=1}^\infty$ is a bounded set we have

$$\limsup_{k\to\infty} \frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} \leqq \limsup_{p\to\infty} \frac{\log(1/|c_{k_p}|) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}{H_\Omega(\lambda^{k_p}) - \operatorname{Re}\langle a, \lambda^{k_p}\rangle}$$

$$\leqq 1 - \frac{Q}{\beta_a} < 1,$$

which means, by virtue of Corollary 3.2, that (u_k) is in $\Lambda^\alpha(\Omega)$.

However, since $|c_k u_k| = 1$ for $k = k_p$ $(p = 1,2,\ldots)$, we get that $c_k u_k$ does not tend to 0 as $k \to \infty$. So, the series $\sum_{k=1}^{\infty} c_k u_k$ does not converge. Hence, $(c_k) \notin \Lambda^{\alpha\alpha}(\Omega)$. The proof is complete.

From now on a bounded convex domain Ω in $\mathbf{C}^n$ with the supporting function $H_\Omega(\zeta)$ and sequence $(\lambda^k)_{k=1}^{\infty}$ satisfying condition (2.9) are considered to be given.

4. The class $E(\Omega)$ of holomorphic multiple Dirichlet series.

Now we introduce a class $E(\Omega)$ of multiple Dirichlet series of the form

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}, \; z \in \Omega, \tag{4.1}$$

where the sequence (c_k) of the coefficients belongs to $\Lambda(\Omega)$. For different elements of this class of course only the c_k's change.

From previous sections we see that for each sequence $(c_k) \in \Lambda(\Omega)$ the series (4.1) is well defined. To denote (4.1) we use the notation $c \asymp (c_k) \in E(\Omega)$. Then we can check that the following algebraic operations (the usual vector addition and scalar multiplication) are well defined in $E(\Omega)$

$$c + d \asymp (c_k + d_k),$$

$$\lambda c \asymp (\lambda c_k),$$

where $c \asymp (c_k)$ and $d \asymp (d_k) \in E(\Omega)$.

It is easy to verify that $E(\Omega)$ is a vector space. A question naturally arises: is it possible to endow $E(\Omega)$ with some topological structure? For the answer we first define the following function

$$\|c\|_E = \sup_{k \geqq 1} \{|c_k|^{1/H_{\Omega(a)}(\lambda^k)}\}, \tag{4.2}$$

where $c \asymp (c_k) \in E(\Omega)$ and a is an arbitrary point of Ω.

This function (4.2) is finite for $c \asymp (c_k) \in E(\Omega)$. Indeed, since $(c_k) \in \Lambda(\Omega)$ for $\varepsilon > 0$ there exists N such that $\forall k > N$

$$\frac{\log |c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leqq \varepsilon.$$

Then, $\forall k > N$

$$\frac{\log |c_k|}{H_{\Omega(a)}(\lambda^k)} \leqq \frac{\varepsilon|\lambda^k| - H_\Omega(\lambda^k)}{H_{\Omega(a)}(\lambda^k)} = \frac{\varepsilon|\lambda^k| - \operatorname{Re}\langle a, \lambda^k \rangle}{H_{\Omega(a)}(\lambda^k)} - 1$$

$$\leqq \frac{(\varepsilon + |a|)|\lambda^k|}{H_{\Omega(a)}(\lambda^k)} - 1 \leqq \frac{\varepsilon + |a|}{\alpha_a} - 1,$$

where α_a is defined by (2.5). This proves our claim.

Now we fix some point $a \in \Omega$. We prove the following

THEOREM 4.1. *$E(\Omega)$ is a complete metrizable, non-locally bounded space,* i.e. *a non-normable* (F)*-space, where the invariant metric on $E(\Omega)$ is given by* $\rho(c,d) = \|c-d\|_E = \sup_{k\geqq 1}\{|c_k - d_k|^{1/H_{\Omega(a)}(\lambda^k)}\}$, $\forall c \asymp (c_k)$, $d \asymp (d_k) \in E(\Omega)$.

PROOF. It is clear that ρ is an invariant metric on $E(\Omega)$.

For the completeness we consider a Cauchy sequence $(c^{(m)} \asymp (c_k^{(m)}))_{m=1}^{\infty}$ in $(E(\Omega),\rho)$. For $\varepsilon > 0$ given there exists N such that $\rho(c^{(m)}, c^{(m+p)}) < \varepsilon$ for all $m \geqq N$, $p \geqq 1$, or

$$|c_k^{(m)} - c_k^{(m+p)}|^{1/H_{\Omega(a)}(\lambda^k)} < \varepsilon, \forall m \geqq N, \forall p \geqq 1, \tag{4.3}$$

which shows that $(c_k^{(m)})_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbf{C}$ for every $k \in \mathbf{N}$. Therefore, $c_k = \lim_{m\to\infty} c_k^{(m)}$, $k \in \mathbf{N}$, exists.

Letting p go to ∞ in (4.3), we obtain

$$|c_k^{(m)} - c_k|^{1/H_{\Omega(a)}(\lambda^k)} \leqq \varepsilon, \forall m \geqq N. \tag{4.4}$$

Define formally a series

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle}, z \in \Omega. \tag{4.5}$$

Let t be an arbitrary number in $(0,1)$. Take $\varepsilon > 0$ such that $(\varepsilon e^t)^{\alpha_a} e^{|a|} < 1$, where α_a is defined by (2.5). For this ε, as shown above, (4.4) holds for some $m \geqq 1$. Then

$$\sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)} \leqq \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_{\Omega_t^a}(\lambda^k)} + \sum_{k=1}^{\infty} |c_k^{(m)} - c_k| e^{H_{\Omega_t^a}(\lambda^k)}$$

$$\leqq \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_{\Omega_t^a}(\lambda^k)} + \sum_{k=1}^{\infty} e^{H_{\Omega_t^a}(\lambda^k)} \varepsilon^{H_{\Omega(a)}(\lambda^k)}.$$

The first series in the right-hand side of the last inequality converges, since $c^{(m)} \asymp (c_k^{(m)}) \in E(\Omega)$. Concerning the second one we note that

$$\varepsilon^{H_{\Omega(a)}(\lambda^k)} e^{H_{\Omega_t^a}(\lambda^k)} = \varepsilon^{H_{\Omega(a)}(\lambda^k)} e^{tH_{\Omega(a)}(\lambda^k) + \mathrm{Re}\langle a, \lambda^k\rangle}$$

$$= (\varepsilon e^t)^{H_{\Omega(a)}(\lambda^k)} e^{\mathrm{Re}\langle a,\lambda^k\rangle} \leqq (\varepsilon e^t)^{\alpha_a |\lambda^k|} e^{|a||\lambda^k|} = [(\varepsilon e^t)^{\alpha_a} e^{|a|}]^{|\lambda^k|}.$$

Since (λ^k) satisfies condition (2.9) and $(\varepsilon e^t)^{\alpha_a} e^{|a|} < 1$ it is easily seen that the series $\sum_{k=1}^{\infty} [(\varepsilon e^t)^{\alpha_a} e^{|a|}]^{|\lambda^k|}$ converges.

So, we proved that the series $\sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)}$ converges, for every $t \in (0,1)$. This means, by virtue of Theorem 2.1, that series (4.5) is well defined and $c \asymp (c_k) \in E(\Omega)$.

Furthermore, (4.4) yields that $(c^{(m)})_{m=1}^{\infty}$ converges to c in $(E(\Omega),\rho)$. The completeness is proved.

For the last assertion of the theorem let us consider an arbitrary neighbourhood U of zero in the space $E(\Omega)$. Then there exists a positive number ε such that

$$\{c \in E(\Omega): \rho(c, 0) = \|c\|_E < \varepsilon\} \subset U.$$

For every $m \in \mathbb{N}$ we define formally a series

$$\sum_{k=1}^{\infty} c_k^{(m)} e^{\langle \lambda^k, z \rangle}, z \in \Omega, \tag{4.6}$$

where

$$c_k^{(m)} = \begin{cases} (\varepsilon e^{-1})^{H_{\Omega(a)}(\lambda^m)}, & \text{if } k = m, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(c_k^{(m)})_{k=1}^{\infty} \in \Lambda(\Omega)$ the series (4.6) is well defined and $c^{(m)} \cong (c_k^{(m)}) \in E(\Omega)$. It is clear that $c^{(m)} \in U$, $\forall m \in \mathbb{N}$. Choose a sequence $(\varepsilon_m)_{m=1}^{\infty}$ as follows

$$\varepsilon_m = e^{-H_{\Omega(a)}(\lambda^m)}, m = 1, 2, \ldots.$$

We have

$$\begin{aligned} \rho(\varepsilon_m c^{(m)}, 0) &= \|\varepsilon_m c^{(m)}\|_E \\ &= \left[\varepsilon^{H_{\Omega(a)}(\lambda^m)} e^{-2H_{\Omega(a)}(\lambda^m)}\right]^{1/H_{\Omega(a)}(\lambda^m)} = \varepsilon e^{-2}, \end{aligned}$$

which shows that $\varepsilon_m c^{(m)}$ does not tend to 0 as $m \to \infty$.

Therefore, no neighbourhood U of zero in $E(\Omega)$ is bounded with repect to the metric ρ. The theorem is proved.

REMARK 4.2. Since $(E(\Omega), \rho)$ is a metrizable space its metric, as is well-known [6], can always be defined by a so-called (F)-norm [6] (or total paranorm [10]). We can verify that the function (4.2) is, in fact, an (F)-norm (a total paranorm).

REMARK 4.3. 1) By virtue of Theorem 2.1, the sum of the series (4.1) is a holomorphic function in Ω. Therefore, we can define a mapping $\sigma: E(\Omega) \to \mathcal{O}(\Omega)$ acting by the rule: each series is mapped to the function which its sum represents. In general, $\sigma(E(\Omega)) \subset \mathcal{O}(\Omega)$. However, it should be noted that for certain sequences $(\lambda^k)_{k=1}^{\infty}$ the mapping σ can be surjective, i.e. the equality $\sigma(E(\Omega)) = \mathcal{O}(\Omega)$ holds. The choice of the sequence (λ^k) in this case can be realized in different ways. This is so if and only if the system $(e^{\langle \lambda^k, z \rangle})_{k=1}^{\infty}$ is an absolutely representing system in the space $\mathcal{O}(\Omega)$ (see, e.g., [8, 9]). Then, as noted in the introduction, the mapping σ is not injective.

2) Furthermore, it is easy to see that the mapping σ is not injective if and only if there exists a sequence $(c_k) \in \Lambda(\Omega)$, not all zero, such that

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle} = 0, \forall z \in \Omega,$$

and the series converges (absolutely) for the topology of $\mathcal{O}(\Omega)$, or equivalently, if and only if the system $(e^{\langle \lambda^k, z\rangle})_{k=1}^{\infty}$ admits a non-trivial expansion of zero in the space $\mathcal{O}(\Omega)$ (see, e.g., [9]). Moreover, this system of exponents is not necessarily an absolutely representing system in $\mathcal{O}(\Omega)$. So, a non-injective mapping σ may be non-surjective.

With the help of Theorem 3.1 we are able to characterize linear continuous functionals on $E(\Omega)$.

THEOREM 4.4. *Let $c \asymp (c_k)$ be in $E(\Omega)$. Then every linear continuous functional F from the dual space $E(\Omega)^*$ has the form*

$$F(c) = \sum_{k=1}^{\infty} c_k u_k, \tag{4.7}$$

where (u_k) satisfies condition (3.1), i.e. *is in $\Lambda^{\alpha}(\Omega)$.*

PROOF. We recall that condition (3.1) looks as follows

$$\limsup_{k\to\infty} \frac{\log|u_k| - \operatorname{Re}\langle a, \lambda^k\rangle}{H_{\Omega}(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle} < 1.$$

Let $F \in E(\Omega)^*$. For every $k \geqq 1$ define formally a series

$$\sum_{j=1}^{\infty} c_j^{(k)} e^{\langle \lambda^j, z\rangle}, z \in \Omega, \tag{4.8}$$

where

$$c_j^{(k)} = \begin{cases} 1, \text{ if } j = k, \\ 0, \text{ otherwise.} \end{cases}$$

It is clear that the series (4.8) is well defined and $a^{(k)} \asymp (c_j^{(k)}) \in E(\Omega)$ for every $k \geqq 1$.

If $c \asymp (c_k) \in E(\Omega)$, then we have

$$F(c) = F\left(\lim_{N\to\infty} \sum_{k=1}^{N} c_k e^{\langle \lambda^k, z\rangle}\right) = \lim_{N\to\infty} F\left(\sum_{k=1}^{N} c_k e^{\langle \lambda^k, z\rangle}\right)$$
$$= \sum_{k=1}^{\infty} c_k F(a^{(k)}) = \sum_{k=1}^{\infty} c_k u_k,$$

where $u_k = F(a^{(k)})$, $k = 1, 2, \ldots$.

In view of Theorem 3.1, (u_k) satisfies condition (3.1).

Conversely, suppose that (u_k) satisfies condition (3.1). Using again Theorem 3.1 gives that the formula (4.7) defines a linear functional on $E(\Omega)$.

The continuity of F remains to be proved. Let $(c^{(m)} \asymp (c_k^{(m)}))_{m=1}^{\infty}$ be a sequence in

$E(\Omega)$ such that $c^{(m)} \to 0$ in $(E(\Omega), \rho)$. We have to prove that $F(c^{(m)}) \to 0$ as $m \to \infty$.

We prove this fact under a more general assumption than (3.1), namely

$$\limsup_{k \to \infty} \frac{\log |u_k| - \operatorname{Re} \langle a, \lambda^k \rangle}{H_\Omega(\lambda^k) - \operatorname{Re} \langle a, \lambda^k \rangle} < \infty.$$

In this case there exists $T \in (0, +\infty)$ such that the following inequality holds

$$\frac{\log |u_k| - \operatorname{Re} \langle a, \lambda^k \rangle}{H_\Omega(\lambda^k) - \operatorname{Re} \langle a, \lambda^k \rangle} \leqq T, \forall k \geqq 1,$$

which is equivalent to

$$|u_k| \leqq e^{T H_{\Omega(a)}(\lambda^k) + \operatorname{Re}\langle a, \lambda^k \rangle}, \ \forall k \geqq 1.$$

From this place following the last part in the proof of Theorem 4.1 we get that since $c^{(m)} \to 0$ in $(E(\Omega), \rho)$, for a positive number ε (satisfying $(\varepsilon e^T)^{\alpha_a} e^{|a|} < 1$, where α_a is defined by (2.5)), there exists N such that $\forall m \geqq N$

$$\|c^{(m)}\|_E = \sup_{k \geqq 1} \{|c_k^{(m)}|^{1/H_{\Omega(a)}(\lambda^k)}\} < \varepsilon,$$

or

$$|c_k^{(m)}| < \varepsilon^{H_{\Omega(a)}(\lambda^k)}, \ \forall k \geqq 1, \forall m \geqq N.$$

Consequently, $\forall m \geqq N$

$$|F(c^{(m)})| = \left| \sum_{k=1}^{\infty} c_k^{(m)} u_k \right| \leqq \sum_{k=1}^{\infty} |c_k^{(m)} u_k|$$

$$\leqq \sum_{k=1}^{\infty} \varepsilon^{H_{\Omega(a)}(\lambda^k)} e^{T H_{\Omega(a)}(\lambda^k) + \operatorname{Re}\langle a, \lambda^k \rangle} = \sum_{k=1}^{\infty} (\varepsilon e^T)^{H_{\Omega(a)}(\lambda^k)} e^{\operatorname{Re}\langle a, \lambda^k \rangle}$$

$$\leqq \sum_{k=1}^{\infty} [(\varepsilon e^T)^{\alpha_a} e^{|a|}]^{|\lambda^k|}.$$

As noted above, since $(\varepsilon e^T)^{\alpha_a} e^{|a|} < 1$ the last series converges and, therefore, it tends to 0 as ε tends to 0. This means that F is continuous. The proof is complete.

We write $\lambda^k = (\lambda_1^k, \dots, \lambda_n^k)$, $k = 1, 2, \dots$. By virtue of Theorem 2.1 we can define differential operators with respect to every variable z_j $(j = 1, \dots, n)$ as follows

$$\frac{\partial}{\partial z_j} c \asymp (c_k \lambda_j^k), j = 1, \dots, n,$$

where $c \asymp (c_k) \in E(\Omega)$.

It is easy to prove the following

THEOREM 4.5. *The operators* $\dfrac{\partial}{\partial z_j}$, $(j = 1, \ldots, n)$, *are continuous in* $E(\Omega)$.

Furthermore, we shall show that $E(\Omega)$ is invariant under certain differential operators of infinite order.

Let

$$L(\zeta) = \sum_{\|v\|=0}^{\infty} a_v \zeta_1^{v_1} \ldots \zeta_n^{v_n}, \ (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n, \tag{4.9}$$

be an entire function in $\mathbf{C}^n$ such that

$$\lim_{\|v\| \to \infty} \sqrt[\|v\|]{|a_v| \, v!} = 0, \tag{4.10}$$

where $\|v\| = v_1 + \ldots + v_n$, $v! = v_1! \ldots v_n!$. This function generates a linear differential operator of infinite order with constant coefficients

$$\mathscr{L}(D) = \sum_{\|v\|=0}^{\infty} a_v D^v, \tag{4.11}$$

where $D^v = \dfrac{\partial^{\|v\|}}{\partial^{v_1} \ldots \partial^{v_n}}$. As is well-known, for each function $f \in \mathcal{O}(\Omega)$ and each compact subset F of Ω we always have

$$\sum_{\|v\|=0}^{\infty} |a_v| \sup_{z \in F} |D^v f(z)| < +\infty.$$

Now let $c \between (c_k) \in E(\Omega)$. We have

$$\begin{aligned} \mathscr{L}(D)\left(\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}\right) &= \sum_{\|v\|=0}^{\infty} a_v D^v \left(\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}\right) \\ &= \sum_{\|v\|=0}^{\infty} a_v \left(\sum_{k=1}^{\infty} c_k (\lambda^k)^v e^{\langle \lambda^k, z \rangle}\right), \ z \in \Omega. \end{aligned}$$

It is easy to verify that the order of summation in the last double series can be reversed. Then we get

$$\begin{aligned} \mathscr{L}(D)\left(\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}\right) &= \sum_{k=1}^{\infty} c_k \left(\sum_{\|v\|=0}^{\infty} a_v (\lambda^k)^v\right) e^{\langle \lambda^k, z \rangle} \\ &= \sum_{k=1}^{\infty} c_k L(\lambda^k) e^{\langle \lambda^k, z \rangle}, \ z \in \Omega. \end{aligned}$$

Suppose that the following conditions holds

$$\limsup_{k\to\infty} \frac{\log|L(\lambda^k)|}{|\lambda^k|} \leqq 0. \tag{4.12}$$

Applying Theorem 2.1 we obtain

THEOREM 4.6. *Let L be an entire function (4.9) satisfying conditions (4.10) and (4.12). Then $E(\Omega)$ is invariant under a differential operator of infinite order with constant coefficients $\mathscr{L}(D)$ of the form (4.11). More precisely, if $c \asymp (c_k) \in E(\Omega)$ then $\mathscr{L}(D)c \asymp (c_k L(\lambda^k)) \in E(\Omega)$.*

For further study we recall some notation and definitions from [6].

Let P be a class of sequences $x = (x_k) \geqq 0$ (this means $x_k \geqq 0$ for every k) with the following properties:

i) to every k there exists $x \in P$ such that $x_k \neq 0$;

ii) if $x, y \in P$, then there exists $z \in P$ such that $x_k \leqq Cz_k$, $y_k \leqq Cz_k$ for some C and all $k = 1, 2, \ldots$

For such a P let $\lambda(P)$ be a set of all sequences $c = (c_k)$ such that $p_x(c) = \sum_{k=1}^{\infty} x_k|c_k| < \infty$ for all $x \in P$. The p_x, being seminorms on $\lambda(P)$, define the topology of $\lambda(P)$. Such a $\lambda(P)$ is a complete locally convex space.

DEFINITION 4.7. The sequence space $\lambda(P)$ is said to be *nuclear* if to every $x \in P$ there exists $y \in P$ and $r = (r_k) \geqq 0$ in l_1 such that $x_k \leqq y_k r_k$ for all $k = 1, 2, \ldots$

Or equivalently: to every $x \in P$ there exists $y \in P$ such that $\sum_{k=1}^{\infty} x_k/y_k < \infty$ (if $y_k = 0$, then x_k must be 0 and x_k/y_k is omitted).

Returning to our case, as a class P we consider the following set

$$P = \{x_t = (e^{H_{\Omega_t^a}(\lambda^k)})_{k=1}^{\infty} : t \in (0, 1)\}.$$

It is easy to verify that such a set P has the two properties i) and ii) mentioned above. Further, we can prove that in this case the corresponding sequence space $\lambda(P)$ is nuclear. Indeed, let $x_t = (\exp(H_{\Omega_t^a}(\lambda^k)))_{k=1}^{\infty} \in P$. Consider $x_s = (\exp(H_{\Omega_s^a}(\lambda^k)))_{k=1}^{\infty} \in P$, where $s > t$.

Since $(t - s)H_{\Omega(a)}(\lambda^k) \leqq (t - s)\alpha_a|\lambda^k|$, where α_a is defined by (2.5), we get

$$\sum_{k=1}^{\infty} e^{H_{\Omega_t^a}(\lambda^k) - H_{\Omega_s^a}(\lambda^k)} = \sum_{k=1}^{\infty} e^{(t-s)H_{\Omega(a)}(\lambda^k)}$$

$$\leqq \sum_{k=1}^{\infty} e^{(t-s)\alpha_a|\lambda^k|} < \infty \text{ (in view of the fact that } e^{(t-s)\alpha_a} < 1).$$

Our claim is proved.

By virtue of Theorem 2.1, we can check the following

LEMMA 4.8. *$c \asymp (c_k)$ is in $E(\Omega)$ if and only if (c_k) belongs to $\lambda(P) = \lambda((e^{H_{\Omega_t^a}(\lambda^k)}))$.*

Hence, we obtain

THEOREM 4.9. *The space $E(\Omega)$ is nuclear.*

We now consider the possibility for $E(\Omega)$ to be an algebra. For this purpose we introduce the following operation as a multiplication in $E(\Omega)$

$$c \odot d \asymp (c_k d_k e^{H_\Omega(\lambda^k)}),$$

where $c \asymp (c_k)$ and $d \asymp (d_k) \in E(\Omega)$.

This operation is well defined. Indeed, in view of Theorem 2.1 a sequence $(c_k d_k e^{H_\Omega(\lambda^k)})_{k=1}^{\infty}$ is in $\Lambda(\Omega)$, which means that $c \odot d \asymp (c_k d_k e^{H_\Omega(\lambda^k)}) \in E(\Omega)$. We easily obtain the following

THEOREM 4.10. *$E(\Omega)$ is a commutative non-normable algebra.*

5. The Banach algebra $B(\Omega)$.

Consider a subclass $B(\Omega)$ of the class $E(\Omega)$ which consists of multiple Dirichlet series of the form (4.1) satisfying the following condition

$$\sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} < \infty. \tag{5.1}$$

It is obvious that $B(\Omega)$ is non-trivial: it contains all elements $a^{(k)} \asymp (c_j^{(k)}) \in E(\Omega)$, $k \in \mathbf{N}$, where

$$c_j^{(k)} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

(see (4.8)), as well as any finite linear combination of them.

Also note that since (5.1) holds, $\sigma(B(\Omega)) \subset H^\infty(\Omega)$, the space of bounded holomorphic functions in Ω, where σ is the mapping defined in Remark 4.3.

We can check that besides the usual algebraic operations the multiplication $\odot$ induced from $E(\Omega)$ is also well defined in $B(\Omega)$. Indeed, for $c \asymp (c_k)$ and $d \asymp (d_k) \in B(\Omega)$ we have

$$\sum_{k=1}^{\infty} |c_k d_k e^{H_\Omega(\lambda^k)}|\, e^{H_\Omega(\lambda^k)} = \sum_{k=1}^{\infty} (|c_k| e^{H_\Omega(\lambda^k)})(|d_k| e^{H_\Omega(\lambda^k)})$$
$$\leqq \sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} \cdot \sum_{k=1}^{\infty} |d_k| e^{H_\Omega(\lambda^k)} < \infty.$$

This proves our claim.

By virtue of (5.1) for each $c \asymp (c_k) \in B(\Omega)$ we can define

$$\|c\| = \sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)}. \tag{5.2}$$

It is clear that $\|\cdot\|$ is a norm in $B(\Omega)$. As for the space $E(\Omega)$ we can verify that $B(\Omega)$ is complete with respect to the norm (5.2). Indeed, consider a Cauchy sequence $(c^{(m)} \simeq (c_k^{(m)}))_{m=1}^{\infty}$ in $B(\Omega)$. Then $\forall \varepsilon > 0 \, \exists N \, \forall m \geqq N \, \forall p \geqq 1 : \|c^{(m)} - c^{(m+p)}\| < \varepsilon$, or

$$\sum_{k=1}^{\infty} |c_k^{(m)} - c_k^{(m+p)}| e^{H_\Omega(\lambda^k)} < \varepsilon, \tag{5.3}$$

which shows that $(c_k^{(m)})_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbf{C}$ for every $k \in \mathbf{N}$ and, therefore, $c_k = \lim_{m\to\infty} c_k^{(m)}$, $k \in \mathbf{N}$, exists.

Letting p go to ∞ in (5.3), we obtain

$$\sum_{k=1}^{\infty} |c_k^{(m)} - c_k| e^{H_\Omega(\lambda^k)} \leqq \varepsilon, \; \forall m \geqq N. \tag{5.4}$$

First show that $(c_k)_{k=1}^{\infty}$ is in $\Lambda(\Omega)$.

Define formally a series

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle}, \; z \in \Omega. \tag{5.5}$$

Let t be an arbitrary number in (0,1). In view of (5.4) we have

$$\begin{aligned} \sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)} &\leqq \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_{\Omega_t^a}(\lambda^k)} + \sum_{k=1}^{\infty} |c_k^{(m)} - c_k| e^{H_{\Omega_t^a}(\lambda^k)} \\ &\leqq \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_{\Omega_t^a}(\lambda^k)} + \varepsilon \sum_{k=1}^{\infty} e^{H_{\Omega_t^a}(\lambda^k) - H_\Omega(\lambda^k)} \end{aligned}$$

The first series in the right-hand side of the last inequality converges, since $c^{(m)} \simeq (c_k^{(m)}) \in E(\Omega)$. Concerning the second one we note that

$$H_{\Omega_t^a}(\lambda^k) - H_\Omega(\lambda^k) = (t-1)[H_\Omega(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle] \leqq (t-1)\alpha_a|\lambda^k|,$$

where α_a is defined by (2.5).

Since (λ^k) satisfies condition (2.9) and $e^{(t-1)\alpha_a} < 1$, as noted before, the series $\sum_{k=1}^{\infty} e^{(t-1)\alpha_a|\lambda^k|}$ converges.

So, we proved that the series $\sum_{k=1}^{\infty} |c_k| e^{H_{\Omega_t^a}(\lambda^k)}$ converges, for every $t \in (0,1)$. This means, by virtue of Theorem 2.1, that the series (5.5) is well defined and $c \simeq (c_k) \in E(\Omega)$.

Furthermore, we have

$$\sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} \leqq \sum_{k=1}^{\infty} |c_k^{(m)} - c_k| e^{H_\Omega(\lambda^k)} + \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_\Omega(\lambda^k)} < \infty,$$

which shows that

$$c \asymp (c_k) \in B(\Omega). \tag{5.6}$$

Combining (5.4) and (5.6) yields that $(c^{(m)})_{m=1}^{\infty}$ converges to c in $B(\Omega)$.
Also it is already proved above that

$$\|c \odot d\| \leqq \|c\| \, \|d\|, \forall c, d \in B(\Omega).$$

So we obtain

THEOREM 5.1. *$B(\Omega)$ is a commutative Banach algebra.*

Further, for each $k \in \mathbf{N}$ we define a series

$$\sum_{j=1}^{\infty} c_j^{(k)} e^{\langle \lambda^j, z \rangle}, \; z \in \Omega,$$

where

$$c_j^{(k)} = \begin{cases} e^{-H_\Omega(\lambda^k)}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that for every $k \in \mathbf{N}$

$$b^{(k)} \asymp (c_j^{(k)}) \in B(\Omega) \text{ and } \|b^{(k)}\| = 1. \tag{5.7}$$

These elements $b^{(k)}$ are useful for the discussions that follow.

As in section 4 we can give a complete description of the dual $B(\Omega)^*$ to $B(\Omega)$.

THEOREM 5.2. *Let $c \asymp (c_k) \in B(\Omega)$. Then every linear continuous functional F from $B(\Omega)^*$ has the form*

$$F(c) = \sum_{k=1}^{\infty} c_k m_k e^{H_\Omega(\lambda^k)}, \tag{5.8}$$

where $(m_k) \in l_\infty$, the space of bounded sequences.

PROOF. Let $F \in B(\Omega)^*$. For $c \asymp (c_k) \in B(\Omega)$ we have

$$F(c) = F\left(\lim_{N \to \infty} \sum_{k=1}^{N} c_k e^{\langle \lambda^k, z \rangle}\right) = F\left(\lim_{N \to \infty} \sum_{k=1}^{N} c_k e^{H_\Omega(\lambda^k)} e^{\langle \lambda^k, z \rangle - H_\Omega(\lambda^k)}\right)$$
$$= \sum_{k=1}^{\infty} c_k e^{H_\Omega(\lambda^k)} F(b^{(k)}) = \sum_{k=1}^{\infty} c_k e^{H_\Omega(\lambda^k)} m_k,$$

where $m_k = F(b^{(k)})$ and

$$|m_k| = |F(b^{(k)})| \leqq C \|b^{(k)}\| = C.$$

Conversely, if $(m_k) \in l_\infty$, then the formula (5.8) defines a linear functional on $B(\Omega)$. Furthermore,

$$|F(c)| = \left| \sum_{k=1}^{\infty} c_k m_k e^{H_\Omega(\lambda^k)} \right| \leqq \sum_{k=1}^{\infty} |c_k m_k| e^{H_\Omega(\lambda^k)}$$

$$\leqq \sup_k |m_k| \sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} \leqq C \|c\|,$$

which shows the continuity of the considering linear functional F. The theorem is proved.

As is well-known, $B(\Omega)^*$ is also a Banach space with the norm defined as

$$\|F\| = \sup_{\|c\| \leqq 1} |F(c)|.$$

Note, on the one hand, that

$$(5.9) \qquad |m_k| = |F(b^{(k)})| \leqq \|F\| \, \|b^{(k)}\| = \|F\|, \ \forall k \in \mathbf{N}.$$

On the other hand, in the proof of Theorem 5.2 we have already got that

$$|F(c)| \leqq \|c\| \sup_k |m_k|.$$

From this estimate it follows that

$$(5.10) \qquad \|F\| = \sup_{\|c\| \leqq 1} |F(c)| \leqq \sup_k |m_k|.$$

Combining (5.9) and (5.10) gives the following alternative expression for the norm $\| \cdot \|$

THEOREM 5.3. *The norm in the space $B(\Omega)^*$ can be defined as follows*

$$\|F\| = \sup_k |m_k|,$$

where $m_k = F(b^{(k)})$, $k \in \mathbf{N}$.

For further study for a function $c \leftrightarrow (c_k) \in B(\Omega)$ we denote formally

$$(5.11) \qquad \sum_{k=1}^{\infty} c_k e^{\langle \lambda, s \rangle - H_\Omega(\lambda^k)} e^{\langle \lambda^k, z \rangle}, s, z \in \Omega,$$

LEMMA 5.4. *For every $s \in \Omega$, $c_{(s)} \leftrightarrow (c_k e^{\langle \lambda^k, s \rangle - H_\Omega(\lambda^k)})$ belongs to $B(\Omega)$.*

PROOF. Using Theorem 2.1 we have an estimate

$$\limsup_{k\to\infty} \frac{\log |c_k e^{\langle \lambda^k, s\rangle - H_\Omega(\lambda^k)}| + H_\Omega(\lambda^k)}{|\lambda^k|}$$

$$= \limsup_{k\to\infty} \frac{\log |c_k| + \operatorname{Re}\langle \lambda^k, s\rangle}{|\lambda^k|} \leqq \limsup_{k\to\infty} \frac{\log |c_k| + H_\Omega(\lambda^k)}{|\lambda^k|} \leqq 0.$$

This shows that series (5.11) is well defined and

$$c_{(s)} \asymp (c_k e^{\langle \lambda^k, s\rangle - H_\Omega(\lambda^k)}) \in E(\Omega).$$

Furthermore,

$$\|c_{(s)}\| = \sum_{k=1}^{\infty} |c_k e^{\langle \lambda^k, s\rangle - H_\Omega(\lambda^k)}| e^{H_\Omega(\lambda^k)}$$

$$= \sum_{k=1}^{\infty} |c_k| e^{\operatorname{Re}\langle \lambda^k, s\rangle} \leqq \sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} = \|c\| < \infty.$$

The lemma is proved.

DEFINITION 5.5. We say that an element $c \asymp (c_k) \in B(\Omega)$ *is essential if* $c_k \neq 0$, $\forall k \in \mathbb{N}$.

In particular, $c^{(\nu)} \asymp (c_k^{(\nu)})$, with $c_k^{(\nu)} = e^{-(\beta_a + |a| + \nu)|\lambda^k|}$, $k \in \mathbb{N}$, where β_a is defined by (2.5) and $\nu > 0$, are essential elements of $B(\Omega)$ for each $\nu > 0$. Indeed, since

$$H_\Omega(\lambda^k) = H_{\Omega(a)}(\lambda^k) - \operatorname{Re}\langle a, \lambda^k\rangle \leqq \beta_a |\lambda^k| + |a|\,|\lambda^k|,$$

by virtue of Theorem 2.1, we have

$$\limsup_{k\to\infty} \frac{\log |c_k^{(\nu)}| + H_\Omega(\lambda^k)}{|\lambda^k|}$$

$$\leqq \limsup_{k\to\infty} \frac{\log |e^{-(\beta_a + |a| + \nu)|\lambda^k|}| + (\beta_a + |a|)|\lambda^k|}{|\lambda^k|} = -\nu < 0,$$

which shows that such a sequence $(c_k^{(\nu)})_{k=1}^{\infty}$ belongs to $\Lambda(\Omega)$, i.e. $c^{(\nu)} \in E(\Omega)$. Furthermore,

$$\|c^{(\nu)}\| = \sum_{k=1}^{\infty} |c_k^{(\nu)}| e^{H_\Omega(\lambda^k)} = \sum_{k=1}^{\infty} e^{-(\beta_a + |a| + \nu)|\lambda^k|} e^{H_\Omega(\lambda^k)} \leqq \sum_{k=1}^{\infty} e^{-\nu|\lambda^k|} < \infty.$$

This proves our claim.

We recall that a subset T of a topological vector space X is said to be total in X if each continuous linear form F on X satisfying $F(T) = 0$ necessarily satisfies $F(X) = 0$. This subset T is called a fundamental in X if the smallest closed vector subspace of X containing T is X itself.

As is well-known, a subset T of a locally convex topological vector space X is total in X if and only if it is fundamental in X.

Note that a discrete subset which is total in X is also called a complete system in X, thanks to the Banach criterion [1].

We shall prove that there exist total sets in $B(\Omega)$.

THEOREM 5.6. *Let $S \subset \Omega$ be a set of uniqueness of $B(\Omega)$ and $c \asymp (c_k)$ be an essential element of $B(\Omega)$. Then the set*

$$X(c) = \{c_{(s)} : s \in S\}$$

is a total set in $B(\Omega)$.

PROOF. Let $F \in B(\Omega)^*$. Suppose that $F(X(c)) = 0$. By Theorem 5.2

$$\sum_{k=1}^{\infty} c_k m_k e^{\langle \lambda^k, s \rangle - H_\Omega(\lambda^k)} e^{H_\Omega(\lambda^k)} = 0, \forall s \in S,$$

or

$$\sum_{k=1}^{\infty} c_k m_k e^{\langle \lambda^k, s \rangle} = 0, \forall s \in S.$$

Furthermore, using Theorem 2.1 we see that

$$\sum_{k=1}^{\infty} c_k m_k e^{\langle \lambda^k, z \rangle}, z \in \Omega,$$

is well defined and, morover, this series belongs to $B(\Omega)$.

Since S is the set of uniqueness of $B(\Omega)$ the last equality implies that $c_k m_k = 0$, $\forall k \in \mathbf{N}$. Since the element c is essential it follows that $m_k = 0, \forall k \in \mathbf{N}$. The theorem is proved.

The zero divisor concept is one of the standard subjects in algebra. This is in a natural way extended to normed algebras and called a topological zero divisor. We recall, first of all, this notion (see, e.g., [7])

DEFINITION 5.7. An element x of a normed algebra A is called a *topological zero divisor* if there exists a sequence (y_k), $y_k \in A$ such that $\|y_k\| = 1$, $\forall k$ and $\lim \|x \odot y_k\| = \lim \|y_k \odot x\| = 0$.

There was proved a number a number of necessary and sufficient conditions for an element in a Banach algebra to be a topological zero divisor, as well as interesting results on this subject which are related to moduli of integrity, closed ideals and continuous inverses of continuous linear transforms, etc. Some connections between topological zero divisors and the notion of regularity (or inversion) and quasi-regularity (or quasi-inversion) have also been studied. We refer the readers to [7] for more details.

For the completeness of exposition we give some examples taken from [7] of topological zeros divisors in some concrete Banach algebras.

As the first example of topological zero divisors we look at the commutative Banach algebra $C([0, 1])$. It is shown that $f \in C([0,1])$ is a topological zero divisor if and only if there exists some s, $0 \leqq s \leqq 1$, for which $f(s) = 0$. Also f is a topological zero divisor if and only if f is not invertible.

This result, with the aid of Urysohn's Lemma, can be carried over to a more general context. Namely, if X is a compact Hausdorff topological space and $f \in C(X)$, the algebra of all continuous complex-valued functions of X that are bounded, that the following are equivalent: (i) f is a topological zero divisor; (ii) f is not invertible; (iii) there exists some $s \in X$ such that $f(s) = 0$.

For the second example we consider the commutative Banach algebra $L_1(\Gamma)$, where Γ is the compact Abelian group under multiplication of complex numbers of absolute value one, i.e. $\Gamma = \{\zeta \in \mathbf{C} : |\zeta| = 1\} = \{e^{it} : -\pi \leqq t \leqq \pi\}$. The convolution of two elements $f, g \in L_1(\Gamma)$ naturally has the following form

$$f * g\,(e^{is}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i(s-t)}) g(e^{it}) dt.$$

It is proved that every element of $L_1(\Gamma)$ is a topological zero divisor.

Returning to our case we prove the following

THEOREM 5.8. *Every element of $B(\Omega)$ is a topological zero divisor in $B(\Omega)$.*

PROOF. We mention from (5.7) that $b^{(k)} \asymp (c_j^{(k)})$, where

$$c_j^{(k)} = \begin{cases} e^{-H_\Omega(\lambda^k)}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases}$$

belong to $B(\Omega)$ and $\|b^{(k)}\| = 1$ for each $k \in \mathbf{N}$.

Now let $c \asymp (c_j)$ be an element in $B(\Omega)$. It is clear that

$$c \odot b^{(k)} = b^{(k)} \odot c \asymp (c_j c_j^{(k)} e^{H_\Omega(\lambda^k)}).$$

Then we have

$$\|c \odot b^{(k)}\| = \|b^{(k)} \odot c\| = |c_k| e^{H_\Omega(\lambda^k)} \to 0 \text{ as } k \to \infty,$$

since $\|c\| = \sum_{k=1}^{\infty} |c_k| e^{H_\Omega(\lambda^k)} < \infty$. This completes the proof of the theorem.

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