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# ON THE MULTIPLICITY OF AN ANALYTIC OPERATOR-VALUED FUNCTION

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**Abstract.**

A definition of the concept of a multiplicity theory for analytic operator-valued mappings is suggested and an example of such a theory is constructed. It generalizes the notion of multiplicity of a singular point for analytic Fredholm operator functions.

## 1. Restricted multiplicity theories.

Let $Y$ be a complex Banach algebra and let $\mathscr{C}$ be a set of analytic mappings from open subsets of the complex plane $\mathbf{C}$ into $Y$. The domain $D(f)$ of an element $f$ of $\mathscr{C}$ need not be connected and we assume no relation between the values of $f$ on distinct components. This is sometimes expressed by saying that the functions are *locally analytic*. We assume that $\mathscr{C}$ satisfies the conditions:

(a) If $f \in \mathscr{C}$ then the restriction of $f$ to any open subset of $D(f)$ is in $\mathscr{C}$.
(b) If $f, g \in \mathscr{C}$ and $D(f) = D(g)$ then the pointwise product $fg$ is in $\mathscr{C}$.

For each $f \in \mathscr{C}$ we let $\Sigma(f)$ denote the set of points $z \in D(f)$ such that $f(z)$ is not invertible, that is, $\Sigma(f)$ is the singular set of $f$. The regular set of $f$ consists of the points $z$ where $f(z)$ is invertible.

Let $f \in \mathscr{C}$. An open subset $\Omega$ of $D(f)$ will be called admissible for $f$ if it is bounded, its closure lies in $D(f)$ and its boundary is disjoint with $\Sigma(f)$. The pair $(f, \Omega)$ will then be called an admissible pair. If, in addition, $\Omega$ is a Cauchy-domain, that is, its boundary consists of a finite collection of pairwise disjoint rectifiable Jordan curves, then we shall call $(f, \Omega)$ an admissible Cauchy-pair. If $(f, \Omega)$ is an admissible pair then there exists an admissible Cauchy-pair $(f, \Omega')$ such that $\overline{\Omega'} \subset \Omega$ and $\Sigma(f) \cap \Omega \subset \Omega'$. For a reference to a proof see [10] page 289.

By a *restricted multiplicity theory* for $\mathscr{C}$ we shall mean an additive semigroup $M$ together with a mapping $(f, \Omega) \mapsto m(f; \Omega)$ which associates with each admissible pair an element of $M$ in such a way that the following axioms are satisfied:

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(i) If $\Sigma(f) \cap \Omega = \emptyset$ then $m(f; \Omega) = 0$.
(ii) The element $m(f; \Omega)$ depends only on the restriction of $f$ to $\Omega$. (We denote the restriction by $f \mid \Omega$.)
(iii) If $\Omega_1$ and $\Omega_2$ are admissible for $f$ and disjoint then

$$m(f; \Omega_1 \cup \Omega_2) = m(f; \Omega_1) + m(f; \Omega_2)$$

(iv) If $D(f) = D(g)$ and $\Omega$ is admissible for both $f$ and $g$, then

$$m(fg; \Omega) = m(f; \Omega) + m(g; \Omega)$$

The last property is often referred to as the product theorem.

## 2. Examples of restricted multiplicity theories.

For the first non-trivial example of a restricted multiplicity theory we take $Y = \mathsf{C}$, $M = \mathsf{N}$ (the non-negative integers), and let $\mathscr{C}$ be the set of all locally analytic functions. Then we take $m(f; \Omega)$ to be the sum of the multiplicities of the zeros of $f$ in $\Omega$.

We obtain a second example by extending this to the case of $Y = \mathsf{C}^{n \times n}$ (the algebra of $n \times n$ matrices). We take $\mathscr{C}$ to be the set of all locally analytic $Y$-valued mappings and define $m(f; \Omega)$ to be the sum of the multiplicities of the zeros of $\det f$ in $\Omega$.

For a third example we ascend to infinite dimensions and let $Y = L(E, E)$ where $E$ is a Banach space. Quite drastic restrictions have to be made on the mappings involved to define multiplicity by methods analogous to those used in the first and second examples. The purpose of this article is in fact to remove those restrictions. The best previous theory is as follows. Let $\mathscr{C}$ be the set of all analytic mappings whose values are Fredholm operators of index zero. For such a mapping it can be shown that in each connected component of $D$ which contains a regular point of $f$ the singular set is discrete. So we can define $m(f; \Omega)$ to be the sum of the multiplicities (to be defined) of the singular points of $f$ in $\Omega$. The multiplicity of a singular point $z_0$ may be defined as follows. Choose a continuous projection $\pi_0$ with range $\ker f(z_0)$ and define

$$f_1(z) = f(z)((z - z_0)^{-1}\pi_0 + I - \pi_0)$$

Define $f_k$ inductively by choosing a continuous projection $\pi_{k-1}$ with range $\ker f_{k-1}(z_0)$ and setting

$$f_k(z) = f_{k-1}(z)((z - z_0)^{-1}\pi_{k-1} + I - \pi_{k-1})$$

It may be shown [5] that there is an integer $n$ such that $f_n(z_0)$ is invertible. Then the multiplicity of the singular point $z_0$ is the sum $\sum_{k=0}^{n-1} \operatorname{rank} \pi_k$ and this is a finite number. The axiom (iv) is proved in [5] under the name of product theorem.

We may add that the least $n$ such that $f_n(z_0)$ is invertible has been called the ascent of $z_0$ and that in [5] it was incorrectly asserted that ascent was additive for products just like multiplicity. In fact it is only subadditive, as was pointed out by Sarreither [7].

The idea of defining the multiplicity of a singular point of a Fredholm mapping goes back to Keldys, and has been investigated in a number of important papers some of which we shall have occasion to refer to later. See for example references [6], [9], [8], [1] and [2]. The definition used in these papers is different from ours and generalizes the chains of generalized eigenvectors which occur in the construction of Jordan canonical form. It is equivalent to the definition given above. It gives more prominence to vectors in the underlying space $E$ whereas our definition keeps the algebra $Y$ in the foreground.

A fourth and apparently quite different example is obtained by taking $Y$ to be a commutative Banach algebra, $M$ its additive group, and defining

$$m(f;\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega'} f(z)^{-1}Df(z)\,dz$$

where $\Omega'$ is a Cauchy-domain enclosing the same part of the singular set as $\Omega$. Commutativity is needed to get the product theorem. If $Y$ is not commutative we can try the formula

$$m(f;\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega'} \Lambda(f(z)^{-1}Df(z))\,dz$$

where $\Lambda$ is a linear functional which vanishes on commutators. In the case $Y = \mathbf{C}^{n\times n}$ the only such linear functionals are multiples of the trace; so we get essentially the second example. If $E$ is an infinite-dimensional Hilbert space and $Y = L(E, E)$ then the only such $\Lambda$ is zero since every operator is the sum of two commutators [3]. If however we restrict to Fredholm operators of index zero we can use the ordinary trace, and by the results of [1] we again get the third example.

## 3. A non-Fredholm multiplicity theory.

Let $E$ and $F$ be complex Banach spaces and let $f: D \to L(E, E)$ and $g: D \to L(F, F)$ be analytic mappings with the same domain $D$. We say that $f$ and $g$ are *equivalent* if there exist analytic mappings $\phi: D \to L(F, E)$ and $\psi: D \to L(F, E)$, taking invertible values only, such that $f\phi = \psi g$. In each of the examples listed in section 2 the multiplicity is invariant under equivalence.

The second and third examples of section 2 have another property not listed as an axiom in section 1: the multiplicity is invariant under suspension. To explain this concept let $f: D \to L(E, E)$ and $g: D \to L(F, F)$. We say that $g$ is a *suspension* of

$f$ if there exists a Banach space $Z$ such that $F = E \oplus Z$ and $g(z) = f(z) \oplus I_Z$. The concept of suspension has been used before, notably by Gohberg, Kaashoek and Lay, but they call it extension. We prefer the term suspension by analogy with its use in topology to describe extending the domain of a mapping by hanging on to it a trivial mapping in a new dimension.

There is a more general version of suspension; let us use the inappropriate term *weak suspension* for want of anything better. Then $g$ is a weak suspension of $f$ if there exist mappings $\alpha$ and $\beta$ such the diagram

$$\begin{array}{ccccccccc} 0 & \to & E & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & Z & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow I_Z & & \\ 0 & \to & E & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & Z & \to & 0 \end{array}$$

commutes and the rows are exact. Note that in the diagram a morphism from $E$ to $F$ (for example) means an analytic mapping with values in $L(E, F)$. Weak suspension will be investigated in a future paper.

In section I we used the term restricted multiplicity because only one space at a time was involved. When invariance under equivalence and suspension is included as an axiom we shall refer simply to a *multiplicity theory*. More precisely: suppose that we have a category $\mathbf{B}$ of Banach spaces and a class $\mathscr{C}$ of analytic mappings, such that for each $f \in \mathscr{C}$ there is some object $E$ of $\mathbf{B}$ such that $f$ has all its values in $L(E, E)$. In addition to the properties (a) and (b) of section 1 we suppose the following:

(c) If $f \in \mathscr{C}$ and $g$ is equivalent to $f$ then $g \in \mathscr{C}$.

By a *multiplicity theory* for $\mathscr{C}$ we mean a semi-group $M$ and a way of assigning to each admissible pair $(f, \Omega)$, for which $f \in \mathscr{C}$, an element $m(f; \Omega)$ of $M$, such that axioms (i)–(iv) are satisfied, as well the additional axioms:

(v) If $f, g \in \mathscr{C}$ are equivalent then $m(f; \Omega) = m(g; \Omega)$.
(vi) If $f, g \in \mathscr{C}$ are such that $g$ is a suspension of $f$ then $m(f; \Omega) = m(g; \Omega)$.

It should be emphasized that this definition is tentative. It reveals the author's feeling that a multiplicity theory should be defined over a category and an associated class of mappings, that some notion of equivalence and suspension should exist with respect to which multiplicity is invariant.

We proceed to define a multiplicity theory in which $\mathbf{B}$ is the category of all Banach spaces, $\mathscr{C}$ the class of all locally analytic operator-valued mappings, and $M$ is the semi-group of isomorphism classes of Banach spaces. Addition in $M$ is by direct sum of representative spaces. We shall say that $f$ and $g$ are *suspension-equivalent*, or *s-equivalent* for short, if there exist Banach spaces $Z_1$ and $Z_2$ such that the suspensions $f \oplus I_{Z_1}$ and $g \oplus I_{Z_2}$ are equivalent. A straightforward argument shows that s-equivalence is an equivalence relation.

DEFINITION. For each admissible pair $(f, \Omega)$ choose a Banach space $F$ and an operator $T \in L(F, F)$ such that $f \mid \Omega$ is s-equivalent to the mapping $z \mapsto zI_F - T$ on $\Omega$. We define $m(f; \Omega)$ to be the isomorphism class of the range of the projection

$$P = \frac{1}{2\pi i} \int_{\partial \Omega'} (zI_F - T)^{-1}\, dz$$

where $\Omega'$ is a Cauchy-domain such that $\overline{\Omega'} \subset \Omega$ and $\Sigma(f) \cap \Omega' = \Sigma(f) \cap \Omega$.

It might be thought that isomorphism classes of Banach spaces are rather uninteresting from an algebraic point of view. The main interest of this theory lies in the complete irrelevance of dimension and its remarkable constructiveness.

The definition raises two questions the answers to which are needed to remove ambiguity. The first pertains to the existence of at least one pair $F$ and $T$ having the required properties. The second is the usual problem of well-definedness. Given two ways to make the s-equivalence we must show that the outcome is the same for both.

## 4. The GKL-process.

The problem of the existence of a suitable pair $F$ and $T$ is solved by the linearization procedure of [2]. From now on we shall usually denote operator-valued mappings by the upper-case letters $A$, $B$ or $C$.

THEOREM 1 (Gohberg, Kaashoek, Lay). *Let $A : D \to L(E, E)$ be analytic, and let $\Omega$ be a Cauchy-domain admissible for $A$. Let $F$ be the Banach space of all continuous functions from $\partial\Omega$ to $E$ and define the operator $T \in L(F, F)$ by*

$$(Tf)(s) = sf(s) - \frac{1}{2\pi i} \int_{\partial \Omega} (I_E - A(\zeta)) f(\zeta)\, d\zeta$$

*for each $s \in \partial\Omega$. Then $A \mid \Omega$ is s-equivalent to $zI_F - T$ on $\Omega$.*

Actually the authors prove that $A$ has a suspension which is equivalent to $zI_F - T$ on $\Omega$. They assume that 0 is in $\Omega$ but this is seen to be unnecessary by a simple translation argument. It is remarkable that $F$ depends only on $E$ and $\Omega$, but not otherwise on $A$. If $\Omega$ is not a Cauchy-domain we can enlarge it to be a Cauchy-domain enclosing the same singular points before applying the theorem. This theorem is one example of the constructiveness of our theory.

Gohberg, Kaashoek and Lay call a function of the form $zI - T$ which is equivalent to a suspension of $A(z)$ a linearization of $A(z)$. We shall refer to the procedure by which $F$ and $T$ are obtained in theorem 1 as the GKL-process.

Let us now calculate the spectral projection associated with the GKL-process. Let $A : D \to L(E, E)$ be analytic and let $\Omega$ be an admissible Cauchy-domain. As we saw, the GKL-process gives rise to the space $F = C(\partial\Omega, E)$, and on $F$ the operator $T$ given by

$$(Tf)(s) = sf(s) - \frac{1}{2\pi i}\int_{\partial\Omega}(I_E - A(\zeta))f(\zeta)\,d\zeta$$

for each $s \in \partial\Omega$. We shall compute the resolvent of $T$. Let $g \in F$, let $\lambda \in \Omega$ be in the resolvent set of $T$, (by theorem 1 $\lambda$ is a regular point for $A(z)$), and consider the equation

$$\lambda f - Tf = g$$

that is,

$$(\lambda - s)f(s) + \frac{1}{2\pi i}\int_{\partial\Omega}(I_E - A(\zeta))f(\zeta)\,d\zeta = g(s)$$

for all $s \in \partial\Omega$.

Multiplying by $(I_E - A(s))(\lambda - s)^{-1}$ and integrating we obtain

$$\int_{\partial\Omega}(I_E - A(s))f(s)\,ds - (I_E - A(\lambda))\int_{\partial\Omega}(I_E - A(\zeta))f(\zeta)\,d\zeta = \int_{\partial\Omega}\frac{I_E - A(s)}{\lambda - s}g(s)\,ds$$

Hence

$$\int_{\partial\Omega}(I_E - A(\zeta))f(\zeta)\,d\zeta = A(\lambda)^{-1}\int_{\partial\Omega}\frac{I_E - A(s)}{\lambda - s}g(s)\,ds$$

Substituting back we obtain

$$f(s) = -\frac{A(\lambda)^{-1}}{2\pi i(\lambda - s)}\int_{\partial\Omega}\frac{I_E - A(\zeta)}{\lambda - \zeta}g(\zeta)\,d\zeta + \frac{g(s)}{\lambda - s}$$

This formula gives $(\lambda - T)^{-1}$. Now we choose a contour $\Gamma$, the boundary of a Cauchy-domain, inside $\Omega$ and enclosing the spectrum of $T$ in $\Omega$. Integrate the resolvent with respect to $\lambda$ around $\Gamma$. Note that we cannot integrate around $\partial\Omega$ since this contour belongs to the spectrum of $T$ as is shown in [2]. The result is a projection which we shall denote by $\mathrm{pr}_A$. When integrated the second term gives zero, since $\Gamma$ does not enclose $s$, and the result is

$$(1) \qquad (\mathrm{pr}_A g)(s) = \left(\frac{1}{2\pi i}\right)^2\int_\Gamma\int_{\partial\Omega}\frac{A(\lambda)^{-1}(I_E - A(\zeta))}{(s - \lambda)(\lambda - \zeta)}g(\zeta)\,d\zeta\,d\lambda$$

This formula provides an explicit means of calculating multiplicity. If we accept for the moment that the multiplicity is well-defined, then, by theorem 1, we have that $m(A;\Omega)$ is the isomorphism class of ran $\mathrm{pr}_A$. This will be used in the proof of the product theorem.

## 5. Proof that the multiplicity is well-defined.

In order to show that the multiplicity theory of section 3 is well-defined it suffices to solve the following problem. Let $E$ and $F$ be Banach spaces, let $A \in L(E, E)$ and $B \in L(F, F)$ and suppose that $zI_E - A$ and $zI_F - B$ are s-equivalent on the domain $D$. Let $\Omega$ be an admissible Cauchy-domain in $D$. It is then required to show that the projections $(1/2\pi i)\int_{\partial\Omega}(zI_E - A)^{-1}\,dz$ and $(1/2\pi i)\int_{\partial\Omega}(zI_F - B)^{-1}\,dz$ have isomorphic ranges.

In terms of matrices the s-equivalence of $zI_E - A$ and $zI_F - B$ amounts to having

$$\phi(z)\begin{pmatrix} zI_E - A & 0 \\ 0 & I_U \end{pmatrix} = \begin{pmatrix} zI_F - B & 0 \\ 0 & I_V \end{pmatrix}\psi(z)$$

on the domain $D$, where $\phi : D \to L(E \oplus U, F \oplus V)$ and $\psi : D \to L(E \oplus U, F \oplus V)$ are analytic and have invertible values. Let $\Omega$ be an admissible Cauchy-domain in $D$. Choose a bounded domain $D_1$ in $D$ such that $\overline{\Omega} \subset D_1$ and let $z_0 \in D \backslash D_1$. Note that for $z \in D_1$ we have

$$\phi(z)\begin{pmatrix} I_E & 0 \\ 0 & (z - z_0)^{-1}I_U \end{pmatrix}\begin{pmatrix} zI_E - A & 0 \\ 0 & (z - z_0)I_U \end{pmatrix}$$
$$= \begin{pmatrix} zI_F - B & 0 \\ 0 & (z - z_0)I_V \end{pmatrix}\begin{pmatrix} I_F & 0 \\ 0 & (z - z_0)^{-1}I_V \end{pmatrix}\psi(z)$$

Hence $zI_{E\oplus U} - (A \oplus z_0 I_U)$ and $zI_{F\oplus V} - (B \oplus z_0 I_V)$ are equivalent on $D_1$. Moreover the projection

$$\frac{1}{2\pi i}\int_{\partial\Omega}(zI_{E\oplus U} - (A \oplus z_0 I_U))^{-1}\,dz = \left(\frac{1}{2\pi i}\int_{\partial\Omega}(zI_E - A)^{-1}\,dz\right) \oplus 0_U$$

has range isomorphic to that of $(1/2\pi i)\int_{\partial\Omega}(zI_E - A)^{-1}\,dz$.

We may therefore present the problem of well-definedness in the following simplified way. Let $E$ and $F$ be Banach spaces, let $A \in L(E, E)$ and $B \in L(F, F)$. Let $\phi : D \to L(E, F)$ and $\psi : D \to L(E, F)$ be analytic mappings with invertible values, and suppose that

$$\phi(z)(zI_E - A) = (zI_F - B)\psi(z)$$

for all $z \in D$. Let $\Omega$ be an admissible Cauchy-domain. We must show that the projections $P_{A,\Omega} = (1/2\pi i)\int_{\partial\Omega}(zI_E - A)^{-1}\,dz$ and $P_{B,\Omega} = (1/2\pi i)\int_{\partial\Omega}(zI_E - B)^{-1}\,dz$ have isomorphic ranges.

By spectral theory the operator-valued mapping $zI_E - A$ is equivalent to $(zI_{E_1} - A_1) \oplus (zI_{E_2} - A_2)$, where $E_1$ is the range of $P_{A,\Omega}$, and $E_2$ its kernel, $A_1 \in L(E_1, E_1)$ and $A_2 \in L(E_2, E_2)$. Moreover the spectrum of $A_1$ is inside $\Omega$ while the spectrum of $A_2$ is outside $\overline{\Omega}$. Hence $zI_E - A$ is s-equivalent to $zI_{E_1} - A_1$ on $\Omega$. A similar reduction is possible for $zI_F - B$.

The upshot of the previous paragraph is this: we have to show that if $zI_E - A$ and $zI_F - B$ are s-equivalent on a domain $\Omega$ containing the spectra of $A$ and $B$, then $E$ and $F$ are isomorphic Banach spaces. This is a consequence of the following theorem for which a proof may be found in [4].

THEOREM 2 (Kaashoek, van der Mee, Rodman). *Let $zI_E - A$ and $zI_F - B$ be s-equivalent on a domain $\Omega$ containing the spectra of $A$ and $B$. Then $A$ and $B$ are similar operators.*

## 6. The product theorem.

In this section we show that the multiplicity theory of section 3 satisfies the product property: axiom (iv). That it satisfies the other axioms is fairly obvious. We shall then be able to show that it generalizes the multiplicity theory for Fredholm operators.

THEOREM 3. *Let $E$ and $F$ be Banach spaces and let $M : D \to L(E \oplus F, E \oplus F)$ be an analytic mapping whose matrix representation is*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

*where $A : D \to L(E, E)$, $B : D \to L(F, F)$ and $C : D \to L(F, E)$ are analytic. Let $\Omega$ be admissible for both $A$ and $B$. Then*

$$m(M; \Omega) = m(A; \Omega) + m(B; \Omega)$$

PROOF. Apply the GKL-process to $M$; that is, compute the projection $\mathrm{pr}_M$ on the space $C(\partial\Omega, E \oplus F)$; see formula (1) of section 4. We may express it as a matrix

$$\mathrm{pr}_M\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(s) = \int_\Gamma \int_{\partial\Omega} \begin{pmatrix} A(\lambda)^{-1} & -A(\lambda)^{-1}C(\lambda)B(\lambda)^{-1} \\ 0 & B(\lambda)^{-1} \end{pmatrix} \begin{pmatrix} I_E - A(\zeta) & -C(\zeta) \\ 0 & I_F - B(\zeta) \end{pmatrix} \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix} \frac{d\zeta\, d\lambda}{(s-\lambda)(\lambda-\zeta)}$$

whence we have

$$\mathrm{pr}_M = \begin{pmatrix} \mathrm{pr}_A & S \\ 0 & \mathrm{pr}_B \end{pmatrix}$$

where $S$ is an operator from $C(\partial\Omega, F)$ to $C(\partial\Omega, E)$.

We now appeal to the following lemma.

LEMMA 1. *Let $E$ and $F$ be Banach spaces and suppose that $\Pi$ is a projection on $E \oplus F$ with matrix representation*

$$\Pi = \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$$

*Then $P$ and $Q$ are projections, and the range of $\Pi$ is isomorphic to the range of the projection $P \oplus Q$.*

PROOF. From $\Pi^2 = \Pi$ we find that $P^2 = P$, $Q^2 = Q$ and $PR + RQ = R$. Hence

$$\begin{pmatrix} I_E & R \\ 0 & I_F \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} I_E & R \\ 0 & I_F \end{pmatrix} = \begin{pmatrix} P & RQ \\ 0 & Q \end{pmatrix}\begin{pmatrix} I_E & R \\ 0 & I_F \end{pmatrix} = \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \Pi$$

which implies the stated result.

By the lemma, $\operatorname{ran} \operatorname{pr}_M$ is isomorphic to $\operatorname{ran} \operatorname{pr}_A \oplus \operatorname{ran} \operatorname{pr}_B$. But this says that $m(M;\Omega) = m(A;\Omega) + m(B;\Omega)$. This concludes the proof.

COROLLARY 1. *Let $V = \bigoplus_{k=1}^n E_k$ and suppose that $M : D \to L(V, V)$ is analytic, and has upper-triangular matrix representation, with $A_k : D \to L(E_k, E_k)$ on the diagonal. If $\Omega$ is admissible for all $A_k$ then $m(M;\Omega) = \sum_{k=1}^n m(A_k;\Omega)$.*

PROOF. Induction using Theorem 3.

THEOREM 4 (Product theorem). *Let $A : D \to L(E, E)$ and $B : D \to L(E, E)$ be analytic and suppose that $\Omega$ is admissible for both $A$ and $B$. Then*

$$m(AB;\Omega) = m(A;\Omega) + m(B;\Omega).$$

PROOF. We form the suspension $AB \oplus I_E$. Note that

$$\begin{pmatrix} 0 & -I_E \\ I_E & A \end{pmatrix}\begin{pmatrix} AB & 0 \\ 0 & I_E \end{pmatrix}\begin{pmatrix} I_E & 0 \\ -B & I_E \end{pmatrix} = \begin{pmatrix} B & -I_E \\ 0 & A \end{pmatrix}$$

Hence $AB$ is s-equivalent to $\begin{pmatrix} B & -I_E \\ 0 & A \end{pmatrix}$. The result now follows from Theorem 3.

We may now show that the multiplicity theory of section 3 generalizes the multiplicity theory for Fredholm operators defined in section 2. Let $A : D \to L(E, E)$ take its values in the set of Fredholm operators of index zero, and for simplicity assume that $D$ is connected and $A(z)$ is invertible for at least one point in $D$. Let $z_0 \in D$. It is known [5] that in a neighbourhood $\Omega$ of $z_0$ containing no singular point, except possibly $z_0$, one can express $A(z)$ as a product

$$A(z) = A_n(z)(\pi_{n-1}(z - z_0) + I - \pi_{n-1})(\pi_{n-2}(z - z_0) + I - \pi_{n-2})$$
$$\ldots(\pi_0(z - z_0) + I - \pi_0)$$

with finite-rank projections $\pi_0, \ldots, \pi_{n-1}$ and $A_n(z_0)$ invertible. Applying the product theorem we find that $m(A;\Omega)$ is the sum $\sum_{k=0}^{n-1} m(P_k;\Omega)$, where $P_k(z) = (z - z_0)\pi_k + I - \pi_k$. But $P_k(z)$ is a suspension of the function $p_k\colon \mathbf{C} \to L(\operatorname{ran}\pi_k, \operatorname{ran}\pi_k)$, given by $p_k(z) = (z - z_0)I_{\operatorname{ran}\pi_k}$. Clearly $m(P_k;\Omega)$ is the isomorphism class of $\operatorname{ran}\pi_k$ which we identify with the integer $\operatorname{rank}\pi_k$. Hence

$$m(A;\Omega) = \sum_{k=0}^{n-1} \operatorname{rank}\pi_k$$

which is the usual expression for the multiplicity of a singular point.

## 7. Homotopy invariance.

If a continuous curve in $L(E,E)$ contains only projections, then the ranges of these projections belong to the same isomorphism class. This remark enables us to prove the homotopy invariance of multiplicity.

THEOREM 5. *Let the continuous mapping $A\colon [0,1] \times D \to L(E,E)$ be analytic in its second variable. Let $A_t$ denote the mapping $z \mapsto A(t,z)$. Suppose that $\Omega$ is a Cauchy-domain which is admissible for all $A_t$, $0 \leqq t \leqq 1$. Then $m(A_t;\Omega)$ is independent of $t$.*

PROOF. Applying the GKL-process to $A_t$ we find a Banach space $F$ and a continuous curve $T_t$ in $L(F,F)$ such that $A_t$ is suspension equivalent to $zI_F - T_t$ on $\Omega$ for each $t$. Taking a contour $\Gamma$, the boundary of a Cauchy-domain, just inside $\partial\Omega$ and enclosing the singular sets in $\Omega$ of all the functions $A_t$, we find that $m(A_t;\Omega)$ is the isomorphism class of the projection $P_t = (1/2\pi i)\int_\Gamma (zI_F - T_t)^{-1}\,dz$. By the remark preceding the theorem this class is independent of $t$.

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