MATH SCAND 77 (1995), 129–147

GENERALIZED WEYL-VON NEUMANN THEOREMS (II)

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Abstract.

We show that the multiplier algebra $M(A)$ of a σ-unital C^*-algebra with stable rank one and (FU) has real rank zero. We also show that the multiplier algebras $M(A)$ of matroid C^*-algebras and many other C^*-algebras have (FU). Consequently, if u is a unitary in $M(A)$ and $\varepsilon > 0$, there are projections $\{p_n\} \in A$ such that

$$u = \sum_{n=1}^{\infty} \alpha_n p_n + a$$

$u = \sum_{n=1}^{\infty} p_n = 1$, where $|\alpha_n| = 1$, $a \in A$ and $\|a\| < \varepsilon$.

0. Introduction.

Let H be a separable, infinite dimensional Hilbert space, K be the C^*-algebra of compact operators on H and $B(H)$ the C^*-algebra of bounded operators on H. The Weyl-von Neumann theorem says: if T is a self-adjoint operator in $B(H)$ and $\varepsilon > 0$, then there is a diagonizable self-adjoint matrix D in $B(H)$ and a compact operator $k \in K$ such that

$$T = D + k$$

with $\|k\| < \varepsilon$. Let A be a C^*-algebra and $M(A)$ its multiplier algebra ($M(A) = \{m \in A^{**}: ma, \ am \in A, \ \forall a \in A\}$ where A^{**} is the enveloping von-Neumann algebra. So $M(A)$ is the idealizor of A in A^{**}.) We say that the Weyl-von Neumann theorem holds for A and $M(A)$ if for any $T \in M(A)_{\text{s.a.}}$ and $\varepsilon > 0$, there are projections p_n in A and $a \in A$ such that

$$T = \sum_{i=1}^{\infty} \lambda_n p_n + a,$$

where $\sum_{i=1}^{\infty} p_i = 1$, λ_n is a bounded sequence of real numbers and $\|a\| < \varepsilon$. It has been shown ([M] and [Zh 1]) that the Weyl-von Neumann theorem holds for

Received April 25, 1994.

A and $M(A)$ if and only if $M(A)$ has real rank zero. (A C^*-algebra A has real rank zero if the set of self-adjoint elements with finite spectra is dense in $A_{s.a.}$. If A has real rank zero, we will write $\mathrm{RR}(A) = 0$. See [BP]) When is $\mathrm{RR}(M(A)) = 0$? A necessary condition is $\mathrm{RR}(A) = 0$. W^*-algebras and AW*-algebras all have real rank zero. AF-algebras, Bunce-Denddens algebras and all purely infinite simple C^*-algebras have real rank zero (See [BP]). The question whether $\mathrm{RR}(M(A)) = 0$ if A is an AF-algebra was raised formally in [BP]. However, as early as 1974, George A. Elliott raised the same question at Tohoku. It has been shown by L. G. Brown and G. K. Pedersen [BP], S. Zhang [Zh 3, 7, 8] and by N. Higson and M. Rørdam [HR] that the above question has an affirmative answer in the case that A is a matroid C^*-algebra. The author shows recently that $\mathrm{RR}(M(A)) = 0$ for every σ-unital AF-algebra ([Li3]). For more information concerning the generalized Weyl-von Neumann theorem readers are referred to [Zh 1–8] and [Li 3]. One key result we established in [Li 3] is the following:

THEOREM A ([Li 3, 3.2]). *Let A be a σ-unital C^*-algebra. Then $M(A)/A$ has real rank zero if $K_1(B) = 0$ for every hereditary C^*-algebra B of $M(M_n(A))$ which contains $M_n(A)$, where $n = 1, 2, \ldots$.*

We will show in section 2 that every σ-unital C^*-algebra with real rank zero, stable rank one, zero K_1-group and satisfying a certain condition (a) satisfies the conditions in Theorem A. By combining [BP, 3.13 and 3.14] as in [Li 3], we conclude that $\mathrm{RR}(M(A)) = 0$ for these C^*-algebras. We also show, in section 3, that every simple C^*-algebra with real rank zero, stable rank one and satisfying the condition (a) satisfies conditions in Theorem A. Therefore corona algebras of those C^*-algebras have real rank zero. In section 4, we show that the Weyl-von-Neumann theorem for unitaries holds for the multiplier algebras of matroid algebras and other C^*-algebras with real rank zero. Applications of these results to the theory of C^*-algebra extensions will appear elsewhere.

We would like state the following definitions.

DEFINITION 1.1. [Ph 1, 1.2] Let A be a unital C^*-algebra and let $U_0(A)$ be the connected component of the unitary group $U(A)$ of A. The *exponential rank* of A, written $\mathrm{cer}(A)$, is the largest element of the set of symbols $1, 1 + \varepsilon, 2, 2 + \varepsilon, \ldots, \infty$ (with the obvious order) consistent with the following restrictions:

1. $\mathrm{cer}(A) \leqq n$ if every $u \in U_0(A)$, the identity component of the unitary group, is the product $\exp(ih_1)\exp(ih_2)\ldots\exp(ih_n)$ for some $h_1, h_2, \ldots, h_n \in A_{s.a.}$;

2. $\mathrm{cer}(A) \leqq n + \varepsilon$ if every $u \in U_0(A)$ is a norm limit of products of n exponentials as in (1).

For nonunital A, set $\mathrm{cer}(A) = \mathrm{cer}(\tilde{A})$.

DEFINITION 1.2. A unital C^*-algebra A is said to have (FU) (weak (FU)) if the set of unitaries with finite spectra is norm dense in $U(A)$ ($U_0(A)$). For onunital A, we say A has (FU) (weak (FU)), if $\tilde{A}$ has (FU). It is known that W^*-algebras, AW*-algebras, AF-algebras and many other (see [Ph 1]) have (FU). On the other hand, if A has weak (FU), then $\mathrm{RR}(A) = 0$ and $\mathrm{cer}(A) \leqq 1 + \varepsilon$. It is shown in [Ph 1] that the irrational rotation algebras A_θ have weak (FU) for θ in a dense G_δ-set of $[0, 1]\backslash Q$ and that Elliott's C^*-algebras A of inductive limits of basic building blocks have weak (FU). It is shown in [Ph 2] that for every purely infinite simple C^*-algebra A, $\mathrm{cer}(A) \leqq 1 + \varepsilon$. Our results in section 3 show that for matroid C^*-algebras and purely infinite simple C^*-algebras A (and many other C^*-algebras), $\mathrm{cer}(M(A)/A) \leqq 1 + \varepsilon$.

We will use the following notations throughout this paper. K is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. $M_n(A)$ is the $n \times n$ matricies over A. $\mathrm{Her}(a)$ denotes the hereditary C^*-subalgebra generated by element a and $C(A)$ denotes the corona algebra $M(A)/A$.

This work was done while the author visited Mathematical Institute, Copenhagen University and University of Toronto. The author has benefited from conversations or electronic correspondence with Larry Brown, George Elliott, Mikael Rørdam, Gert K. Pedersen, N. C. Phillips and S. Zhang. This work is partially supported by a grant from the Danish Natural Science Reeeeeesearch Council, a grant from the Natural Sciences and Engineering Research Council of Canada and a grant from the National Natural Science Foundation of China. The author would also like to thank George A. Elliott and Gert K. Pedersen for their kind arrangement for these visits.

1. Generalized Weyl-von Neumann theorems for self-adjoint elements.

The main result in this section is Theorem 2.9 which improves our earlier results in [Li 3]. We start with the following lemma.

LEMMA 2.1 ([Zh 10]). *Let A be a C^*-algebra with real rank zero and stable rank one, n be a positive integer. Suppose that*

$$p = \sum_{i=1}^{n} p_i \otimes e_{ii},$$

$$p_1 \leqq p_2 \leqq \cdots \leqq p_n,$$

where the p_i's are projections in A and $\{e_{ij}\}$ is a matrix unit for M_n, the $n \times n$ matrices. Then

$$\mathrm{cer}(pM_n(A)p) \leqq d(n) + \mathrm{cer}(p_nAp_n) + \varepsilon,$$

if $n = 2^{d(n)}$;

$$\mathrm{cer}(pM_n(A)p) \leqq d(n) + 1 + \mathrm{cer}(p_nAp_n) + \varepsilon,$$

if $2^{d(n)} < n < 2^{\{d(n)+1\}}$, *where* $d(n) = \{\ln(n)/\ln 2\}$ *and* $\{k\}$ *is the largest integer smaller or equal to* k. *Moreover, if the unitary group of* p_nAp_n *is connected,* $U(pM_n(A)p)$ *is also connected.*

REMARK 2.2. Our earlier estimate is

$$\operatorname{cer}(pM_n(A)p) \leqq 3(n-1) + \operatorname{cer}(p_nAp_n),$$

which is enough for our purpose in this paper. But since 2.1 is much better, with S. Zhang's permission, we quote it from [Zh 10].

ADDED IN PROOF: It has been shown by the author (Exponential rank of C^*-algebras with real rank zero and the Brown-Pedersen Conjectures, J. Funct. Anal. 114 (1993), 1–11) that $\operatorname{cer}(B) \leqq 1 + \varepsilon$ for every C^*-algebra of real rank zero.

LEMMA 2.3. *Let* A *be a* C^**-algebra with real rank zero. Then the map:* $K_1(I) \to K_1(A)$ *is injective for any ideal* I *of* A.

PROOF. For each n, by [BP, 2.10],

$$\mathrm{RR}(M_n(A)) = \mathrm{RR}(M_n(I)) = \mathrm{RR}(M_n(A/I)) = 0.$$

It follows from [Zh 3, 3.2] that every projection in $M_n(A/I)$ life to a projection in $M_n(A)$.

From the six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0(I) & \to & K_0(A) & \to & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \leftarrow & K_1(A) & \leftarrow & K_1(I), \end{array}$$

we see that the map $K_0(A) \to K_0(A/I)$ is surjective. Hence $K_1(I) \to K_1(A)$ is injective.

LEMMA 2.4. *Let* A *be a* C^**-algebra with real rank zero and* $K_1(A) = 0$. *If* B *is a hereditary* C^**-algebra of* A, *then* $K_1(B) = 0$.

PROOF. We may assume that A is unital. We first consider the case $B = pAp$ for some projection p in A. Let $B_1 = (A \otimes K)^{\sim}$ and $\{1 \otimes e_{ij}\}$ be a matrix unit for $\mathbf{C} \cdot 1 \otimes K$. For an integer n, let w be a unitary in $(\sum_{i=1}^n p \otimes e_{ii})B(\sum_{i=1}^n p \otimes e_{ii})$ and $u = 1 - \sum_{i=1}^n p \otimes e_{ij} + w$. It is enough to show that u is connected to the identity of $(pAp \otimes K)$.

Let A_1 be the C^*-subalgebra of B_1 generated by $\{1, 1 \otimes e_{ij}, i, j = 1, 2, \ldots\}$ and w. Supposethat separable C^*-algebra A_n is constructed. Since B_1 has real rank zero (see [BP]), there is a sequence of projections $\{p_k\}$ such that every self-adjoint element in A_n can be approximated by elements with the form $\sum_{i=1}^m \lambda_i \mathrm{p}_{k_i}$, where $\{\lambda_i\}$ are real numbers and $\{\mathrm{p}_{k_i}\}$ are mutually orthogonal. Suppose that $\{u_k\}$ is a dense sequence of unitaries of A_n. Since $K_1(A) = 0$, each u_k is connected to the identity in B_1. Let $u_{k(1)}, u_{k(2)}, \ldots, u_{k(m)}$ be the unitaries along the path which connectes u_k to 1 such that

$$\|u_k - u_{k(1)}\| < 1, \ \|u_{k(m)} - 1\| < 1$$

and

$$\|u_{k(i)} - u_{k(i+1)}\| < 1,$$

$i = 1, 2, \ldots, m-1$.

Let A_{n+1} be the C^*-subalgebra of B_1 generated by $A_n, p_k, \{u_k, u_{k(1)}, \ldots, u_{k(m)}\}$. Set

$$A_\infty = \left(\bigcup_{n=1}^{\infty} A_n\right)^{-}.$$

By the construction, A_∞ has real rank zero and the unitary group of A_∞ is connected. Let A_0 be the norm closure of

$$\bigcup_{n=1}^{\infty}\left(\sum_{i=1}^{n} 1 \otimes e_{ij}\right) A_\infty \left(\sum_{i=1}^{n} 1 \otimes e_{ij}\right).$$

Then $\tilde{A}_0 \cong A_\infty$. Moreover,

$$(1 \otimes e_{ii}) A_0 (1 \otimes e_{ii})^{\sim} \otimes K \cong A_0.$$

Thus $K_1(A_0) = 0$. Let I be the ideal generated by

$$(p \otimes e_{11} A_0 p \otimes e_{11}) \otimes K.$$

By Lemma 2.3, $K_1(I) = 0$. Since, by [Bn 1],

$$I \otimes K \cong (p \otimes e_{11}) A_0 (p \otimes e_{11}) \otimes K,$$

$K_1((p \otimes e_{11}) A_0 (p \otimes e_{11})) = 0$. Hence u is connected to the identity in $(pAp \otimes K)^{\sim}$.

Now we consider the case that B is not unital. Let u be a unitary in $B' = (\tilde{B} \otimes K)$. Again, it is enough to show that u is connected to the identity of $(\tilde{B} \otimes K)^{\sim}$. It is easy to see that u is close to a unitary of the form $(1 - \sum_{i=1}^{k} 1 \otimes e_{ii}) + w$, where w is a unitary in $(\sum_{i=1}^{k} 1 \times e_{ii}) B' (\sum_{i=1}^{k} 1 \otimes e_{ii})$. Since B has real rank zero, by [BP, 2.6], B has an approximate identity $\{d_\alpha\}$ consisting of projections. This implies that w is close to a unitary of the form $(\sum_{i=1}^{k} 1 \otimes e_{ii} - p) + w'$, where $p \leqq \sum_{i=1}^{k} 1 \otimes e_{ii}$ is a projection and w' is a unitary in $pB'p$. So u is connected to the unitary $(1 - p) + w'$. Since B is a hereditary C^*-subalgebra of A, we have that

$$pB'p \cong pM_k(A)p.$$

Since $K_1(M_k(A)) = 0$, from what we have shown, $K_1(pM_k(A)p) = 0$. We also have that $(pB'p \otimes K) \cong B'$. Therefore $(1 - p) + w'$ is connected to the identity of B'. This implies that u is connected to the identity of B'. This completes the proof.

DEFINITION 2.5. A C^*-algebra A of real rank zero is said to satisfy condition (a) if there is an integer k such that for every projection $p \in A$

$$\operatorname{cer}(pAp) \leqq k.$$

From [Li 3, 1.3], every C^*-algebra with (FU) satisfies the condition (a). It follows from 2.1 and [Zh 5, 3.3] that if A has stable rank one and satisfies condition (a), then $M_n(A)$ satisfies condition (a) (with different k though).

LEMMA 2.6. *Let A be a σ-unital C^*-algebra with real rank zero, stable rank one and $K_1(A) = 0$.* Suppose that A satisfies condition (a) *with the integer k. Then for any integer m, if B is a unital hereditary C^*-subalgebra of $M_m(M(A))$ $(= M(M_m(A)))$, then*

$$\operatorname{cer}(B) \leqq 2(\{\ln(m)/\ln 2\} + 1 + k) + \varepsilon$$

and the unitary group of B is connected.

PROOF. Fix an integer m. Let $B = pM(M_m(A))p$ for some projection p in $M(M_m(A))$. If $p \in M_m(A)$, by [Zh 5, 3.3], we may assume that $p = \sum_{i=1} p_i' \otimes e_{ii}$, where p_i' is a projection in A and $\{e_{ij}\}$ is a matrix unit for M_m. Moreover, we may assume that

$$p_1' \leqq p_2' \leqq \cdots \leqq p_m'.$$

Therefore the estimate of $\operatorname{cer}(B)$ follows from 2.1. Since $K_1(M_m(A)) = 0$, by 2.4, $K_1(B) = 0$. It follows from [Rff, 2.10] that the unitary group of B is connected.

Now we assume that $p \in M(M_m(A)) \setminus M_m(A)$. For any unitary u in B, by an Elliott's trick (see [Ell 1, 2.4], [Zh 6, 1.6] or [Li 3, 2.1]), for $\varepsilon > 0$ there are projections $\{e_n\}$ in $M_m(A)$ and unitaries u_1, u_2 in $pM_m(A)p$ such that

$$\|u - u_1u_2\| < \varepsilon/2$$

and

$$u_1 = \sum_{i=1}^{\infty} (e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2}),$$

$$u_2 = \sum_{i=1}^{\infty} (e_{2n+1} - e_{2n-1})u_2(e_{2n+1} - e_{2n-1}).$$

By 2.1 and our condtion (a), there are

$$b_n^{(i)} \in (e_{2n} - e_{2n-2})M_m(A)(e_{2n} - e_{2n-2})_{\text{s.a.}},$$

$i = Z, 2, \ldots l$ such that

$$\left\|(e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2}) - \prod_{k=1}^{l} \exp(ib_n^{(k)})\right\| < \varepsilon/2^{n+2},$$

where $l = \{\ln(n)/\ln 2\} + k + 1$. Furthermore, since A has real rank zero, we may assume that $\|b_n^{(i)}\| \leqq 2\pi$. Clearly,

$$h_1^{(k)} = \sum_{n=1}^{\infty} b_n^{(k)} \in pM(M_m(A))p_{\text{s.a.}}.$$

Hence

$$\left\|u_1 - \prod_{k=1}^{l} \exp(ih_1^{(k)})\right\| < \sum_{n=1}^{\infty} \varepsilon/2^{n+2} = \varepsilon/4.$$

Similarly, there are $h_2^{(h)} \in pM(M_m(A))p$, $k = 1, 2, \ldots, l$ such that

$$\left\|u_2 - \prod_{k=1}^{l} \exp(ih_2^{(k)})\right\| < \varepsilon/4.$$

Therefore

$$\left\|u - \sum_{k=1}^{l} \exp(ih_1^{(k)})\exp(ih_2^{(k)})\right\| < \varepsilon.$$

This completes the proof.

THEOREM 2.7. *If A is a σ-unital C^*-algebra with real rank zero, stable rank one and $K_1(A) = 0$, and satisfies condition* (a), *then for any hereditary C^*-subalgebra B of $M_m(M(A))$, $K_1(B) = 0$.*

PROOF. The proof is similar to that of [Zh 6, 2.17].

If B is unital, Lemma 2.6 applies. So we may assume that B is not unital. Moreover, by 2.4, we may assume that $B \not\subset A$. Set $B' = (\tilde{B} \otimes \lesssim)^{\sim}$. It is enough to show that a unitary $u \in B'$ can be connected to the identity of B' by a path of unitaries in B'. As in the second part of the proof of Lemma 2.4, without loss of generality, we may assume that $u = (1 - F_n) + w$, where $F_n = \sum_{i=1}^{n} 1 \otimes e_{ii}$ and w is a unitary in $F_n B' F_n$.

Now consider the map

$$\tau\colon F_n B' F_n \cong M_n(\tilde{B}) \to M_n(\tilde{B}/B) \cong M_n(\mathbf{C}).$$

Set $v = \tau(w)$. If we use the same notation v for the corresponding scalar matrix in $M_n(\tilde{B})$, then we may write $w = v + b$ for some $b \in M_n(B)$. Clearly w is connected to a unitary with form $F_n + b'$ in the unitary group of $M_n(\tilde{B})$ for some $b' \in M_n(B)$. Therefore we may assume that $u = 1 + b' = (1 - F_n) + (F_n + b')$.

Notice that for any integer $m \geqq 1$, $M_m(B)$ is a hereditary C^*-subalgebra of

$M_m(M(A)) \cong M(M_m(A))$. Since $M_m(A)$ has real rank zero (see [BP, 2.10]), it follows from [Zh 2, 1.1] that $M_m(B)$ has LP for any $m \geqq 1$ (*A C^*-algebra A is said to have* LP if A is the closed linear span of its projections). It is then routine to show that $B \otimes K$ has LP. By [Zh 6, 1.1], $B \otimes K$ has an approximate identity consisting of projections. Then, as in the second part of the proof of Lemma 2.4, it is easy to see that $u = 1 + b'$ is close to a unitary with the form $(1 - p) + v'$ where p is a projection in $B \otimes K$ and v' is a unitary in $p(B \otimes K)p$. So we may assume that $u = (1 - p) + v'$. It easy to see that p is close to a projection which is in $F_k(B \otimes K)F_k$ for some $k \geqq n$, without loss generality, we may further assume that $p \leqq F_k$.

We notice that $p(B \otimes K)p = pM_k(B)p = pM_k(M(A))p$. It follows from 2.6 that v' is connected to the identity of $F_k(B \otimes K)F_k$ by a path of unitaries in $F_k(B \otimes K)F_k$. This proves that u is connected to the identity of B' by a path of unitaries in B'.

COROLLARY 2.8. *Let A be a σ-unital C^*-algebra with* FU) *and stable rank one, then for any hereditary C^*-algebra B of $M_m(M(A))$ for any m, $K_1(B) = 0$.*

PROOF. It follows from [Li 3, 1.3], pAp has (FU) for all projections in A. Hence A satisfies condition (a). Moreover, by 2.1, $K_1(A) = 0$.

THEOREM 2.9. *Let A be a σ-unital C^*-algebra with real rank zero, stable rank one and $K_1(A) = 0$. If A satisfies condition* (a), *then $M(A)$ has real rank zero. Equivalently, for any $T \in M(A)_{s.a.}$ and $\varepsilon > 0$, there are an approximate identity $\{e_n\}$ of A consisting of projections and an element $a \in A_{s.a.}$ such that*

$$T = \sum_{i=1}^{\infty} \lambda_i(e_i - e_{i-1}) + a,$$

where $\|a\| < \varepsilon$ and $\{\lambda_i\}$ is a bounded sequence of real numbers.

PROOF. It is an immediate consequence of 2.7, Theorem A and [BP, 3.14].

COROLLARY 2.10. *Suppose that A is a σ-unital C^*-algebra with* (FU) *and stable rank one. Then $M(A)$ has real rank zero.*

COROLLARY 2.11 ([Li 3]). *If A is a σ-unital* AF-*algebra, then $M(A)$ has real rank zero.*

2. Corona algebras $M(A)/A$ with real rank zero.

If $M(A)$ has real rank zero, then the corona algebra $C(A) = M(A)/A$ has real rank zero. However, there are examples of C^*-algebras with $\mathrm{RR}(A) = \mathrm{RR}(C(A)) = 0$ but $\mathrm{RR}(M(A)) \neq 0$. It is shown in [Zh 3] that if $A = B \otimes K$, where B is the Bunce-Deddens algebra, then $\mathrm{RR}(A) = \mathrm{RR}(C(A)) = 0$ but $\mathrm{RR}(M(A)) \neq 0$. We se from Theorem A that if A is a σ-unital C^*-algebra such that for any n, $K_1(B) = 0$ for every hereditary C^*-subalgebra B of $M_n(M(A))$ which contains $M_n(A)$ but is not $M_n(A)$, then the corona algebras $C(A)$ has real rank zero. It was shown by Larry Brown that $K_1(M(B)) = 0$, where B is stably isomorphic to a Bunce-Deddens algebra. Notice that $K_1(B) \neq 0$. We show in this section that many simple C^*-algebras A with real rank zero have this phenominon. Hence corona algebras of these algebras have real rank zero.

LEMMA 3.1. *Let A be a non-elemetary simple C^*-algebra with real rank zero and p a non-zero projection in A, then for any positive integer k, there are k non-zero mutually equivalent and mutually orthogonal projections $q_i \leqq p$ $(i = 1, 2, \dots, k)$.*

PROOF. Since A is a non-elemetary, pAp is also non-elementary. Moreover pAp has real rank zero (See [BP, 2.8]). We may assume that $p = 1$. There is a nonzero projection q in A such that $1 - q \neq 0$. Suppose that a is a nonzero positive element in $(1 - q)A(1 - q)$. By [Cu 1, 1.8], there is a nonzero element y in A such that

$$y^*y \in qAq, \; yy^* \in (1 - q)A(1 - q).$$

Let $y = u\,|y|$ be the polar decomposition of y in A^{**}. Then by [Li, 1.2], the map

$$\phi(x) = uxu^*$$

is an isomorphism from $\mathrm{Her}(|y|)$ onto $\mathrm{Her}(|y^*|)$. Let q_1 be a nonzero projection in $\mathrm{Her}(|y|)$. Then $uq_1 \in A$ (See [Li 2, 1.2]). Moreover,

$$(uq_1)^*(uq_1) = q_1$$

and

$$(uq_1)(uq_1)^* = q_2$$

is a projection in

$$\mathrm{Her}(|y^*|) \subset (1 - p)A(1 - p).$$

The lemma then follows by induction.

LEMMA 3.2. *Let A be a simple C^*-algebra with stable rank one and p and q two nonzero, mutually orthogonal projections in A. Suppose that $u \in U(pAp)$, then there is a $v \in U(qAq)$ such that*

$$u + v \in U_0((p + q)A(p + q)).$$

PROOF. We may assume that $p + q = 1$. Working in $A \otimes K$, let

$$e = \mathrm{diag}(0, 1, 1, \dots).$$

By [Bn 1], there is a $W \in M(A \otimes K)$ such that

$$W^*W = p \otimes e, \; WW^* = q \otimes e.$$

By [Rff, 2.10], there is $v \in U(qAq)$ such that

$$[\mathrm{diag}(v, q, q, \ldots)] = [q + Wu^* \otimes e_{22} W^* + (q \otimes e - q_1)]$$

in $K_1(qAq)$, where $\{e_{ij}\}$ is a matrix unit for K and

$$q_1 = W(u \otimes e_{22})W^*W(u^* \otimes e_{22})W^* = W(p \otimes e_{22})W^*.$$

Set $W_0 = W(p \otimes e) + W^*(q \otimes e)$, and $W_1 = \mathrm{diag}(1, W_0)$. Then

$$[W_1 \,\mathrm{diag}(v + p, u + q, 1, 1, \ldots)W_1^*] = [v + W(u \otimes e_{22})W^* + (e - q_1)] = 0$$

in $K_1(A)$. So $[\mathrm{diag}(v + p,\ u + q,\ 1,\ 1 \ldots)] = 0$ in $K_1(A)$. Hence, by [Rff, 2.10], $u + v \in U_0(A)$. This completes the proof.

LEMMA 3.3. *Let A be a (non-unital) σ-unital simple C^*-algebra with real rank zero and stable rank one. Suppose that A satisfies condition* (a), *then the unitary group of $M(A)$ is connected.*

PROOF. We may assume that A is non-elementary. Suppose that v is a unitary in $M(A)$. It follows from [Ell 1, 2.4] (see 2.6 also) that there are unitaries u_1 and u_2 in $M(A)$ such that

$$\|u - u_1 u_2\| < 1$$

and

$$u_1 = \sum_{n=1}^{\infty} (e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2})$$

$$u_2 = \sum_{n=1}^{\infty} (e_{2n+1} - e_{2n-1})u_2(e_{2n+1} - e_{2n-1})$$

where $e_n (e_0 = 0)$ is an approximate identity for A. So u is connected with u_1u_2. To show that u is in $U_0(M(A))$, it is enough to show that both u_1 and u_2 are in $U_0(M(A))$. Therefore we may assume that

$$u = \sum_{n=1}^{\infty} (e_n - e_{n-1})u(e_n - e_{n-1}).$$

Set $u_n = (e_n - e_{n-1})u(e_n - e_{n-1})$.These u_n are unitaries in

$$(e_n - e_{n-1})A(e_n - e_{n-1}).$$

By factoring u further, we may assume that $u_{2n} = (e_{2n} - e_{2n-1})$, $n = 1, 2, \ldots$. By Lemma 3.1, we write

$$e_{2n} - e_{2n-1} = \sum_{i=1}^{2n+1} p_{2n}^{(i)},$$

where each $p_{2n}^{(i)}$ is a projection, $p_{2n}^{(i)} \neq 0$ if $i \neq 2n+1$, $p_{2n}^{(i)}$ are mutually orthogonal and $p_{2n}^{(i)} \sim p_{2n}^{(j)}$ if $i,j \neq 2n+1$. Suppose that $s_{2n}^{(ij)}$ are partial isometries in $(e_{2n} - e_{2n-1})A(e_{2n} - e_{2n-1})$ such that

$$(s_{2n}^{(ij)})(s_{2n}^{(ij)})^* = p_{2n}^{(i)}, \ (s_{2n}^{(ij)})^*(s_{2n}^{(ij)}) = p_{2n}^{(j)},$$

$i,j = 1,2,\ldots,2n$. By Lemma 3.2, there is a unitary $v_{2n}^{(2i-1)}$ in $p_{2n}^{(2i-1)}Ap_{2n}^{(2i-1)}$ such that $v_{2n}^{(2i-1)} + u_{2i-1}^* \in U_0(A)$, $i = 1,2,\ldots,n$. Set

$$s_{2n}^{(i)}(t) = p_{2n}^{(2i-1)}\cos t - s_{2n}^{(2i-1,2i)}\sin t + (s_{2n}^{(2i-1,2i)})^*\sin t + p_{2n}^{(2i)}\cos t,$$

$$y_{2n} = \sum_{i=1}^{n} v_{2n}^{(2i-1)}p_{2n}^{(2i-1)} + \sum_{i=1}^{n} p_{2n}^{(2i)} + p_{2n}^{(2n+1)},$$

$$z_{2n} = \sum_{i=1}^{n} p_{2n}^{(2i-1)} + \sum_{i=1}^{n} (v_{2n}^{(2i-1)})^* p_{2n}^{(2i)} + p_{2n}^{(2n+1)}$$

and

$$w_{2n}(t) = y_{2n}\left(\sum_{i=1}^{n} s_{2n}^{(i)}(t) + p_{2n}^{(2n+1)}\right) z_{2n}\left(\sum_{i=1}^{(i)} s_{2n}^{(i)}(t) + p_{2n}^{(2n+1)}\right)^*$$

$$w_{2n-1}(t) = u_{2n-1}.$$

So $\{w_n(t)\}$ is equi-continuous on $[0,\pi/2]$. Thus $w(t) = \sum_{n=1}^{\infty} w_n(t)$ is a norm continuous path in $U(M(A))$ with $w(0) = u$ and

$$w(\pi/2) = \sum_{n=1}^{\infty} u_{2n-1} + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} v_{2n}^{(2i-1)}p_{2n}^{(2i-1)} + \sum_{i=1}^{n} (v_{2n}^{(2i-1)})^* p_{2n}^{(2i)} + p_{2n}^{(2n+1)}\right)$$

By rearranging terms, we may write

$$w(\pi/2) = \sum_{n=1}^{\infty} w_n$$

where each w_n is a unitary in

$$U_0((e_n' - e_{n-1}')A(e_n' - e_{n-1}'))$$

and $\{e_n'\}$ is an approximate identity consisting of projections. Therefore, since A satisfies condition (a), there is an integer k such that for each n there are

$$h_n^{(i)} \in (e_n' - e_{n-1}')A(e_n' - e_{n-1}')_{\text{s.a.}}, \quad i = 1,2,\ldots,k$$

such that

$$\left\| w_n - \prod_{j=1}^{k} \exp(ih_n^{(j)}) \right\| < 1/2^n.$$

Since A has real rank zero, we may further assume that $0 \leqq h_n^{(j)} \leqq 2\pi$. Thus

$$\left\| w(\pi/2) - \sum_{n=1}^{\infty} \left(\prod_{j=1}^{k} \exp(ih_n^{(j)}) \right) \right\| < 1$$

Hence $w(\pi/2)$ and $\sum_{n=1}^{\infty} (\prod_{j=1}^{k} \exp(ih_n^{(j)}))$ are in the same connected component in $U(M(A))$. Notice that $\{\prod_{j=1}^{k} \exp(ih_n^{(j)}(1-t))\}$ is equi-continuous on $[0, 1]$. Set

$$v(t) = \sum_{n=1}^{\infty} \left(\prod_{j=1}^{k} \exp(ih_n^{(j)}(1-t)) \right).$$

Then $v(t)$ is a norm continuous path in $U(M(A))$ and

$$v(0) = \prod_{j=1}^{k} \exp\left(i \sum_{n=1}^{\infty} h_n^{(j)} \right), v(1) = 1$$

This completes the proof.

COROLLARY 3.4. *Let A be a σ-unital simple C^*-algebra with real rank zero and stable rank one. If A satisfies condition* (a), *then for any n and any unital hereditary C^*-subalgebra B of $M(M_n(A))$ which contains $M_n(A)$ but not $M_n(A)$, the unitary group of B is connected.*

PROOF. Suppose that $B = pM(M_n(A))p$ for some projection p in $M(M_n(A))$. If q is a projection in $pM_n(A)p \subset M_n(A)$, then by [Zh 5, 3.3], q has the form described in 2.1. By 2.1, $pM_n(A)p$ has the same properties A has. Since $B = M(pM_n(A)p)$, 3.4 follows from 3.3.

THEOREM 3.5. *Let A be a σ-unital simple C^*-algebra with real rank zero and stable rank one. If A satisfies condition* (a), *for any n and hereditary C^*-subalgebra B of $M(M_n(A))$ which contains $M_n(A)$ but is not $M_n(A)$, $K_1(B) = 0$.*

PROOF. It follows from 3.4 as in 2.7.

REMARK 3.6. A special case of 3.5 was proved by Larry Brown a few years ago. He showed that 3.5 is true for non-unital C^*-algebras which are stably isomorphic to Bunce-Deddens algebras.

THEOREM 3.7. *Let A be a σ-unital simple C^*-algebra with real rank zero and stable rank one. If A satisfies condition* (a), *then*

$$\mathrm{RR}(M(A)/A) = 0.$$

PROOF. This follows from 3.4 and Theorem A immediately.

3. The Weyl-von Neumann theorem for unitaries.

We have shown that the Weyl-von Neumann theorem for self-adjoint elements holds for AF-algebras and their multiplier algebras. In this section, we show that if u is a unitary in the multiplier algebra of a σ-unital C^*-algebra with stable rank one, (FU) and finitely many ideals in its corona algebra A, then for any $\varepsilon > 0$, there is an element $a \in A$ and an approximate identity $\{e_n\}$ consisting of projections such that

$$u = \sum_{n=1}^{\infty} \alpha_n(e_n - e_{n-1}) + a$$

with $|\alpha_n| = 1$ and $\|a\| < \varepsilon$. Consequently, in these cases, $\mathrm{cer}(C(A)) \leqq 1 + \varepsilon$.

Our 4.1 is inspired by a result of Mikael Rørdam that if both I and Q have stable (FU), then A has (FU), where I, Q and A are as in 4.1. However, we do not need the stable assumption and our proof is different.

THEOREM 4.1. *Let*

$$0 \to I \to A \to Q \to 0$$

be a short exact sequence of C^**-algebras.*

(i) *If* I *has* (FU), Q *has real rank zero and* $\mathrm{cer}(Q) \leqq 1 + \varepsilon$; *then* A *has real rank zero and* $\mathrm{cer}(A) \leqq 1 + \varepsilon$;

(ii) *In* ((i) *if* $\tilde{A}$ *has connected unitary group,* A *has* (FU);

iii) *If* A *has* (FU), *then* I *has* (FU) *and* Q *has real rank zero and* $\mathrm{cer}(Q) \leqq 1 + \varepsilon$;

(iv) *If* I *has* (FU), Q *has* (FU), *then* A *has* (FU).

PROOF. By [Ph 1, 1.4], we may assume that A is unital.

(i) Since I has (FU), it follows from [Zh 3, 3.3] and [Ch, 2] that A has real rank zero. Let

$$I^{\perp} = \{b \in A\colon bi = ib = 0, \forall i \in A\}$$

Then I is an ideal of A. Moreover $I^{\perp} + I$ is an essential ideal of A. By [P 2], we may assume that A is a C^*-subalgebra of $M(I^{\perp} + I) \cong M(I^{\perp}) \oplus M(I)$. Hence A may be written as $A_1 \oplus A_2$, where A_1 is a C^*-subalgebra of $M(I^{\perp})$ and A_2 is a C^*-subalgebra of $M(I)$. Since A_1 is isomorphic to a hereditary C^*-subalgebra of Q, by [Li 3, 1.4], A_1 has weak (FU). Therefore we may assume that I is an essential ideal of A. Next we assume that I is σ-unital. It follows from [M, Theorem 9 and Introduction] (see [Zh 1, 3.1] also) that for any selfadjoint element $h \in A_{\mathrm{s.a.}}$ and $\delta > 0$, there are an approximate identity $\{e_n\}$ consisting of projections and an element $c \in I_{\mathrm{a.a.}}$ such that

$$h = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}) + c$$

and $\|c\| < \delta$, where $\{\lambda_n\}$ is a bounded sequence of real numbers. Let $u \in U_0(A)$. Then $\pi(u) \in U_0(Q)$, where π is the map: $A \to Q$. For $1 > \varepsilon > 0$, there is $\bar{h} \in Q_{s.a.}$ such that

$$\|\pi(u) - \exp(i\bar{h})\| < \varepsilon/2^6.$$

Therefore, there is $h \in A_{s.a.}$ and $b \in I$ such that

$$\|u - \exp(ih) - b\| < \varepsilon/2^6.$$

Therefore, by choosing a small δ, one obtains

$$\left\|\exp(ih) - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1})\right\| < \varepsilon 2^6.$$

(Notice also that

$$\exp(ih) - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) \in I.)$$

There is an integer N such that

$$\|e_N b e_N - b\| < \varepsilon/2^7.$$

Therefore

$$\left\|u - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) - e_N b e_N\right\| < \sum_{k=1}^{3} \varepsilon/2^{k+4}.$$

Let $x = \sum_{n=1}^{N} e^{i\lambda_n}(e_n - e_{n-1}) + e_N b e_N$. Clearly

$$\|e_N u - u e_N\| < \sum_{k=1}^{3} \varepsilon/2^{k+2} + \sum_{k=1}^{3} \varepsilon/2^{k+4} = 4/16\varepsilon.$$

Set $v = x|x|^{-1}$ (the inverse is taken in $e_N A e_N$), then

$$\|v - x\| \leqq \|x\| \, \||x|^{-1} - e_N\| < (1 + 3/16\varepsilon)/(1 - 3/16\varepsilon)(3/16\varepsilon) < 5/16\varepsilon.$$

Hence

$$\left\|u - \sum_{n=N+1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) - v\right\| < \sum_{k=1}^{3} \varepsilon/2^{k+4} + 5/16\varepsilon = 6/16\varepsilon.$$

By [Li 3, 1.3], $e_N A e_N = e_N I e_N$ has (FU). So there is an $h' \in (e_N I e_N)_{s.a.}$ such that

$$\left\|v - e_N - \sum_{k=1}^{\infty} (ih')^n/n!\right\| < 10/16\varepsilon.$$

We conclude that there is an $h_0 \in A_{s.a.}$ such that

$$\|u - \exp(ih_0)\| < \varepsilon.$$

Hence $\operatorname{cer}(A) \leqq 1 + \varepsilon$.

Now we reduce the general case to the case that I is σ-unital. There are $\{h_n\}$ in $A_{s.a.}$ and $\{j_n\}$ in I such that

$$\|u - \exp(ih_n) - j_n\| \to 0.$$

Let A_0 be the C^*-subalgebra generated by $\{h_n, j_n\}$ and I_0 be the ideal $A^0 \cap I$. Since I_0 is separable and I has real ranks zero, there is an increasing sequence of projections $\{p(0,n)\}$ in I such that

$$\|a(1 - p(0,n))\| \to 0$$

for all $a \in I$. Let A_1 be the C^*-subalgebra generated by A_0 and $\{p(0,n)\}$ and I_1 be ideal $A_1 \cap I$. Suppose that $\{u_k\}$ is a dense sequence of normal partial isometries in I_1. Since I has (FU), there are projections $\{p(1,n)\}$ *in* I such that each u_k can be approximated by linear combinations of finitely many orthogonal projections in $\{p(1,n)\}$. Let A_2 be the C^*-subalgebra generated by A_1 and $\{p(1,n)\}$ and I_2 be the ideal $A_2 \cap I$. If A_m and I_m have been constructed, choose an dense sequence of normal partial isometries $\{v_k\}$ in I_m and à sequence of projections $\{p(m,n)\}$ in I such that each v_k can be approximated by linear combinations of finitely many orthogonal projections in $\{p(m,n)\}$. Then let A_{m+1} be the C^*-subalgebra generated by A_m and $\{p(m,n)\}$ and I be the ideal $A_{m+1} \cap I$. Set $A_\infty = (\cup A_m)^-$ and $I_\infty = (\cup I^m)^-$ $(= I \cap A_\infty)$. It is then easy to check A_∞ is separable and I_∞ is separable and has (FU). Now consider the elements $w_n = \exp(ih_n) + j_n$. As before, we may assume that I_∞ is essential in A_∞. Then we can apply the above argument to the elements w_n and (i) follows.

(ii) This is an immediate consequence of (i).

(iii) That I has (FU) follows from [Li 3, 1.3] and that Q has real rank zero and $\operatorname{cer}(Q) \leqq 1 + \varepsilon$ follows from [Ph 1, 1.6].

(iv) For every $u \in U(A)$, $\pi(u) \in U_0(A)$. So (iv) follows from the proof of (i).

THEOREM 4.2. *Let A be a σ-unital C^*-algebra with* (FU) *and with stable rank one. If there is a sequence of ideals*

$$A = I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = M(A)$$

such that I_k/I_{k-1} is simple for each $k = 1, 2, \ldots, n$, then $\operatorname{cer}(M(A)) \leqq 1 + \varepsilon$ *and $M(A)$ has real rank zero. Moreover, if $u \in M(A)$ and $\varepsilon > 0$, then there are $a \in A$ and an approximate identity $\{e_n\}$ consisting of projections such that*

$$u = \sum_{n=1}^{\infty} \alpha_n(e_n - e_{n-1}) + a$$

with $|\alpha_n| = 1$ *and* $\|a\| < \varepsilon$. *Consequently* $\mathrm{cer}(C(A)) = 1 + \varepsilon$.

PROOF. It follows [Zh 2, 1.3] that I_1/A is purely infinite. It then follows from [Ph 2] that $\mathrm{cer}(I_1/A) \leqq 1 + \varepsilon$. Moreover, I_1/A has real rank zero. It follows from 4.1 that $\mathrm{cer}(I_1) \leqq 1 + \varepsilon$ and I_1 has real rank zero. By induction and repeated application of 4.1, we conclude that $\mathrm{cer}(M(A) \leqq 1 + \varepsilon$ and $M(A)$ has real rank zero. Since $M(A)$ has connected unitary group, then by 4.1, $M(A)$ has (FU). To show that every unitary $u \in M(A)$ has the form

$$u = \sum_{n=1}^{\infty} \alpha_n(e_n - e_{n-1}) + a$$

as described in the theorem, we use the proof of 3.1 in [Zh 1]. Since $M(A)$ has (FU), for any $e > 0$, there is a selfadjoint element $h \in M(A)$ such that

$$\|u - \exp(ih)\| < \varepsilon.$$

Since $M(A)$ has real rank zero, this implies that

$$\left\| u - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) \right\| < \varepsilon,$$

where $\{\lambda_n\}$ is a bounded sequence of real numbers and $\{e_n\}$ is an approximate identity for A consisting of projections. It follows from [Zh 1, 3.9] (the equivalence of (b) and (e)) that we may assume that

$$u = \sum_{n=1}^{\infty} (e_n - e_{n-1})u(e_n - e_{n-1}) + a$$

with $a \in A$ and $\|a\| < \varepsilon/4$. Since

$$\|(1 - e_n)a\| \to 0, \quad \text{as} \quad n \to \infty,$$

by passing a subsequence if necessary, we may assume that

$$\|(e_n - e_{n-1})u - u(e_n - e_{n-1})\| \to 0$$

as $n \to \infty$. As in the proof of 4.1 (i), by a standard argument, we may write

$$(e_n - e_{n-1})u(e_n - e_{n-1}) = w_n + a_n,$$

where w_n is a unitary in $(e_n - e_{n-1})A(e_n - e_{n-1})$, $a_n \in (e_n - e_{n-1})A(e_n - e_{n-1})$ and $\|a_n\| \leqq \varepsilon/2^{n+4}$. For each n, there is a selfadjoint element $h_n \in (e_n - e_{n-1}) A(e_n - e_{n-1})$ with $\|h_n\| \leqq 2\pi$ such that

$$\|w_n - \exp(ih_n)\| < \varepsilon/2^{n+4}.$$

This implies that there is $x \in M(A)_{\text{s.a.}}$ such that

$$u = \exp(ix) + a + b,$$

where $b = \sum_{n=1}^{\infty} a_n \in A$ and $\|b_n\| < \varepsilon/4$. Again, since $M(A)$ has real rank zero, we obtain

$$u = \sum_{n=1}^{\infty} \alpha_n(e_n - e_{n-1}) + (a + b + c),$$

where $\|c\| < \varepsilon/2$. Finally, we notice that, since $M(A)$ has real zero, the last conclusion follows immediately from 4.1. This completes the proof.

REMARK 4.3. If we assume that A has stable rank one then it follows from section two that $M(A)$ has connected unitary group.

THEOREM 4.4. *The multiplier algebra $M(A)$ of any matroid C^*-algebra A has* (FU). *Moreover, for any $u \in U(M(A))$ and $\varepsilon > 0$, there is an approximate identity $\{e_n\}$ of A consisting of projections and an element $a \in A$ such that*

$$u = \sum_{i=1}^{\infty} \alpha_i(e_i - e_{i-1}) + a,$$

where $|\alpha_i| = 1$ and $\|a\| < \varepsilon$. Furthermore, $\mathrm{cer}(C(A) \leqq 1 + \varepsilon$.

PROOF. If A is a finite matroid C^*-algebra then, by [Ell 1, Theorem 3.1], $M(A)/A$ is simple. If A is infinite, then by [Ell 1, 3.2], $M(A)/A$ has only one nontrivial ideal. So 4.2 applies.

EXAMPLES 4.5. There are many examples of C^*-algebras satisfying the conditions in 4.2. It follows from [Li 1, Theorem 2] that every simple AF-algebra with trace space having finitely many extreme points satisfying the condition in 4.2. For simple AF-algebras with infinitely many points in their extreme sets of trace spaces, if they have continuous scales then, by [Li 1, Theorem 1], $M(A)/A$ are simple. In fact every σ-unital simple C^*-algebra with continuous scale has a simple corona algebra $M(A)/A$. We also notice that every σ-unital simple C^*-algebra with real rank zero has many hereditary C^*-subalgebras with continuous scales (see [Li 5]).

COROLLARY 4.6. *Let A be a σ-unital simple C^*-algebra with stable rank one,* (FU) *and continuous scale, then $M(A)$ has* (FU). *Moreover* $\mathrm{cer}(C(A)) \leqq 1 + \varepsilon$.

THEOREM 4.7. *Let A be a σ-unital purely infinite simple C^*-algebra. If $K_1(A) = 0$, then $M(A)$ has* (FU). *Moreover,* $\mathrm{cer}(C(A)) = 1$.

PROOF. It follows from [Zh 3, 1.2 and 2.6] A is stable, $M(A)$ has real rank zero and $M(A)$ has connected unitary group. By [Ph 2], $\mathrm{cer}(A) \leqq 1 + \varepsilon$ and $\mathrm{cer}(C(A)) \leqq 1 + \varepsilon$. Since A is stable and $K_1(A) = 0$, A has (FU). By 4.1,

$\mathrm{cer}(M(A)) \leqq 1 + \varepsilon$. Since $M(A)$ has real rank zero and connected unitary group, $M(A)$ has (FU).

4.8. We notice that C^*-algebras O_n are purely infinite, simple and have $K_1(O_n) = 0$. So, by 4.8, $M(O_n)$ has (FU).

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