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COPIES OF C_0 IN CERTAIN VECTOR-VALUED FUNCTION BANACH SPACES

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Abstract.

In this note we study some vector-valued function spaces which contain a copy of c_0, and have the property that the lack of a copy of c_0 in the range space guarantees the presence of a copy of ℓ_∞ in the whole space.

Assuming X and Y are two normed spaces over the field K of real or complex numbers, $\mathscr{L}(X, Y)$ will denote the linear space of all bounded linear operators from X to Y equipped with the operator norm topology. If X and Y are Banach spaces, then $\mathscr{K}(X, Y)$ will stand for the closed linear subspace of $\mathscr{L}(X, Y)$ of all those compact mappings. If X, Y and Z are normed spaces, $\mathscr{B}(X \times Y, Z)$ will denote the normed space of all continuous bilinear mappings from $X \times Y$ into Z with the supremum norm. As usual we will write $\mathscr{B}(X, Y)$ instead of $\mathscr{B}(X \times Y, \mathsf{K})$.

If Ω is a non-empty set, Σ a σ-algebra of subsets of Ω and X is some Banach space, $\mathrm{ba}(\Sigma, X)$ and $\mathrm{ca}(\Sigma, X)$ will respectively stand for the Banach space of all bounded vector measures and all countably additive vector measures $F: \Sigma \to X$, both equipped with the semivariation norm $\|F\| = \sup\{\sum_{A \in \Pi} |x^* F(A)| : x^* \in X^*, \|x^*\| \leqq 1, \Pi \in \mathscr{P}\}$, where $\mathscr{P}$ is the family of all finite partitions of Ω by elements of Σ. Finally, $\mathrm{bvca}(\Sigma, X)$ will represent the Banach space of all X-valued countably additive measures of bounded variation defined on Σ, provided with the variation norm $|F| = \sup\{\sum_{A \in \Pi} \|F(A)\| : \Pi \in \mathscr{P}\}$.

In this paper we are going to consider some classes of Banach spaces $E(X)$ of X-valued functions, containing a copy of c_0, with the property that the lack of a copy of c_0 in the range (Banach) space X guarantees the presence of a copy of ℓ_∞ in the whole space $E(X)$.

THEOREM 1. *Let X be a normed space and let Y be a Banach space. Then $\mathscr{L}(X, Y)$ contains a copy of c_0 if and only if one of the following two conditions holds:*

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(i) *Y contains a copy of c_0, or*
(ii) *$\mathscr{L}(X, Y)$ has a copy of ℓ_∞.*

PROOF. Assume that $\mathscr{L}(X, Y)$ has a copy of c_0 but Y does not contain a copy of c_0. Let J be an isomorphism from c_0 into $\mathscr{L}(X, Y)$ and set $T_n := Je_n$ for $n \in \mathsf{N}$. Then, the formal series $\sum_n T_n$ is weakly inconditionally Cauchy and hence there is some $C > 0$ so that $\sup_n \|\sum_{1 \leqq i \leqq n} \xi_i T_i\| \leqq C \|\xi\|_\infty$ for each $\xi \in \ell_\infty$. Since for each x in X the linear mapping $T \to T(x)$ from $\mathscr{L}(X, Y)$ to Y is bounded (and hence, weakly continuous), it follows that $\sum_n T_n(x)$ is weakly unconditionally Cauchy for each $x \in X$. But, as Y does not contain a copy of c_0, $\sum_n T_n(x)$ is unconditionally convergent in Y for each $x \in X$ as a consequence of a result of Pelczynski ([1, p. 45]).

Let us define $\phi : \ell_\infty \to \mathscr{L}(X, Y)$ by $\phi\xi(x) = \sum_n \xi_n T_n(x) \, \forall x \in X$. Given $x \in X$ and $\xi \in \ell_\infty$, let $\varepsilon > 0$ and let $n \in \mathsf{N}$ be so that $\|\sum_{j > n} \xi_j T_j(x)\| < \varepsilon$. Then, $\|\phi\xi(x)\| \leqq \|\sum_{1 \leqq k \leqq n} \xi_k T_k(x)\| + \|\sum_{k > n} \xi_k T_k(x)\| \leqq C \|\xi\|_\infty \|x\| + \varepsilon$. Hence, $\|\phi\xi(x)\| \leqq C \|\xi\|_\infty \|x\|$. This shows that $\phi\xi \in \mathscr{L}(X, Y) \, \forall \xi \in \ell_\infty$ and that ϕ is bounded. Since $\inf_n \|T_n\| > 0$, there exists an infinite subset M of N such that the restriction of ϕ to $\ell_\infty(\mathsf{M})$ is an isomorphism ([7]). Thus $\mathscr{L}(X, Y)$ contains a copy of ℓ_∞.

Conversely if Ycontains a copy of c_0, then $\mathscr{L}(X, Y)$ contains a copy of c_0 since, given some $x^* \in X^*$ with $\|x^*\| = 1$, the map $\varphi : Y \to \mathscr{L}(X, Y)$ defined by $\varphi(y)x = (x^*x)y$ for $x \in X$ and $y \in Y$ is an isometry into.

REMARK 1. If X is an infinite-dimensional normed space and Y is any Banach space containing a copy of c_0, then $\mathscr{L}(X, Y)$ contains a copy of ℓ_∞. Indeed, assuming without loss of generality that Y coincides with c_0 and choosing a Josefson-Nissenzweig sequence (x_n^*) in X^* [1, p. 219], we may define $\phi : \ell_\infty \to \mathscr{L}(X, c_0)$ by $\phi\xi(x) = (\xi_n x_n^* x)$. Clearly, $\|\phi\xi\| \leqq \|\xi\|_\infty$, and hence $\phi\xi \in \mathscr{L}(X, Y)$ and ϕ is bounded. Since $\inf_n \|\phi(e_n)\| = 1$, we conclude as in the theorem above that $\mathscr{L}(X, Y)$ has a copy of ℓ_∞.

COROLLARY 1. *Let X be an infinite-dimensional normed space and let Y be a Banach space. Then $\mathscr{L}(X, Y)$ contains a copy of c_0 if and only if it contains a copy of ℓ_∞.*

PROOF. Assume $\mathscr{L}(X, Y)$ contains a copy of c_0. If Y does not contain any copy of c_0 this is consequence of the previous theorem. If Y has a copy of c_0, then apply the preceding remark.

COROLLARY 2. *Assume that X is a Banach space and Σ is any infinite σ-algebra of subsets of a Ω. Then* $\mathrm{ba}(\Sigma, X)$ *has a copy of c_0 if and only if* $\mathrm{ba}(\Sigma, X)$ *has a copy of ℓ_∞.*

PROOF. With $\ell_0^\infty(\Sigma)$ standing for the space of all Σ-simple functions on Ω endowed with the supremum norm, the linear operator $\phi : \mathscr{L}(\ell_0^\infty(\Sigma), X) \to \mathrm{ba}(\Sigma, X)$

defined by $(\phi T)(E) = T(E)$ for $T \in \mathscr{L}(\ell_0^\infty(\Sigma), X)$ and $E \in \Sigma$ is an isometry between these two Banach spaces. So Corollary 1 applies.

COROLLARY 3. *If X and Y are normed spaces and Z is a Banach space, then $\mathscr{B}(X \times Y, Z)$ has a copy of c_0 if and only if either Z has a copy of c_0 or $\mathscr{B}(X \times Y, Z)$ has a copy of ℓ_∞.*

PROOF. This is a direct consequence of the previous theorem, since $\mathscr{B}(X \times Y, Z)$ is isometric to $\mathscr{L}(X \otimes_\pi Y, Z)$.

EXAMPLE 1. Assuming X is a Banach space without the compact range property (CRP) it has been proved in [4] that $\mathscr{K}(C[0,1], X)$ has a copy of c_0. On the other hand each copy of ℓ_1 in $C[0,1]$ is non-complemented (it is easy to construct a copy of ℓ_1 in real $C[0,1]$, [1, p. 203]; this copy cannot be complemented in $C[0,1]$ since the space rca $(\mathscr{B}_{[0,1]})$ of regular Borel measures defined on the σ-algebra of Borel sets of $[0,1]$ does not have a copy of c_0: otherwise there would be a $\lambda \in \text{rca}^+(\mathscr{B}_{[0,1]})$ so that $L_1(\lambda)$ would have a copy of c_0, a contradiction). As it has been proved in [6] that if X and Y are Banach spaces, $\mathscr{K}(X, Y)$ contains a copy of ℓ_∞ if and only if either X has a complemented copy of ℓ_1 or Y has a copy of ℓ_∞, it follows that if a Banach space X does not have the CRP and does not contain a copy of ℓ_∞, then $\mathscr{K}(C[0,1], X)$ has a copy of c_0 but not of ℓ_∞. According to Corollary 1, $\mathscr{L}(C[0,1], X)$ has a copy of ℓ_∞.

It is known that if Z is a Banach space having a non-complemented copy of ℓ_1, then Z^* does not have the CRP and does not have any copy of c_0. Thus $\mathscr{L}(C([0,1], C[0,1]^*)$ has a copy of ℓ_∞ but not $\mathscr{K}(C[0,1], C[0,1]^*)$.

On the other hand, $\mathscr{K}(\ell_p, \ell_p)$ has a copy of c_0 for $1 \leqq p < \infty$ (if (e_n) is the unit vector basis of ℓ_p, for each n define $T_n : \ell_p \to \ell_p$ by $T_n \xi = \xi_n e_n$; then (T_n) is a basic sequence in $\mathscr{K}(\ell_p, \ell_p)$ equivalent to the unit vector basis of c_0) and so $\mathscr{L}(\ell_p, \ell_p)$ contains a copy of ℓ_∞ for $1 \leqq p < \infty$. Moreover, because of the aforementioned result of [6], $\mathscr{K}(\ell_p, \ell_p)$ does not contain a copy of ℓ_∞ for $1 < p < \infty$.

Assuming that p, $q \geqq 1$ with $1/p + 1/q = 1$, if X contains a copy of ℓ_q while Y has a copy of ℓ_p, and moreover X^* or Y has the approximation property, one has $\mathscr{K}(\ell_p, \ell_p) \cong \ell_q \check{\otimes}_\varepsilon \ell_p \hookrightarrow X \check{\otimes}_\varepsilon Y \cong \mathscr{K}(X^*, Y)$. So $\mathscr{K}(X^*, Y)$ has a copy of c_0. Hence $\mathscr{L}(X^*, Y)$ has a copy of ℓ_∞. In particular, if (Ω, Σ, μ) is any finite measure space (Σ being infinite), then $\mathscr{L}(L_p(\mu), L_p(\mu))$ has a copy of ℓ_∞ for $1 < p < \infty$ while $\mathscr{K}(L_p(\mu), L_p(\mu))$ does not contain ℓ_∞ for $1 < p < \infty$.

THEOREM 2. *Assume that X is a Banach space. Then* ca(Σ, X) *contains a copy of c_0 if and only if one of the following two conditions holds:*

(i) *X contains a copy of c_0*, or

(ii) ca(Σ, X) *has a copy of ℓ_∞.*

PROOF. Let J be an isomorphism from c_0 into $\mathrm{ca}(\Sigma, X)$. Since the map $F \to F(E)$ from $\mathrm{ca}(\Sigma, X)$ into X is continuous for each $E \in \Sigma$, the series $\sum_n Je_n(E)$ is weakly unconditionally Cauchy in X for each $E \in \Sigma$. Assuming that X does not have any copy of c_0, then for every $E \in \Sigma$, the series $\sum_n Je_n(E)$ is unconditionally convergent in X. Then we define the linear operator $T: \ell_\infty \to \mathrm{ca}(\Sigma, X)$ by $T\xi(E) = \sum_n \xi_n Je_n(E)$ for each $\xi \in \ell_\infty$ and each $E \in \Sigma$. Setting $\xi^n := (\xi_1, \ldots, \xi_n, 0, \ldots, 0, \ldots)$ then we have $\|T\xi(E)\| = \|\sum_n \xi_n Je_n(E)\| = \lim_n \|J\xi^n(E)\| \leqq \sup_n \|J\xi^n\| \leqq \sup_n \|J\| \, \|\xi^n\|_\infty \leqq \|J\| \, \|\xi\|_\infty$. This shows that $T\xi$ is a bounded vector measure and $\|T\| \leqq 4\|J\|$. On the other hand, according to a theorem of Bartle-Dunford-Schwartz (see [2, p. 14]) there is a $\mu \in \mathrm{ca}^+(\Sigma)$ such that $J\xi^n \ll \mu$ for each $\xi \in \ell_\infty$ and each $n \in \mathbb{N}$. Since $\lim_n J\xi^n(E) = T\xi(E) \in X$ for $\xi \in \ell_\infty$ and $n \in \mathbb{N}$, the Vitali-Hahn-Saks theorem guarantees that $T\xi \ll \mu$ for each $\xi \in \ell_\infty$. Hence $T\xi$ is countably additive for each $\xi \in \ell_\infty$ and so $T(\ell_\infty) \subseteq \mathrm{ca}(\Sigma, X)$. Then as $\inf_n \|Te_n\| > 0$, $\mathrm{ca}(\mu, X)$ must contain a copy of ℓ_∞.

Conversely, assuming that X contains a copy of c_0 and $\mu \in \mathrm{ca}(\Sigma)$ has semivariation one, as the map $\psi: X \to \mathrm{ca}(\Sigma, X)$ defined by $\phi(x)(E) = \mu(E)x$ for $E \in \Sigma$ and $x \in X$ is an isometry into, $\mathrm{ca}(\Sigma, X)$ contains a copy of c_0.

EXAMPLE 2. If λ stands for the Lebesgue measure on $[0, 1]$ while R_1 denotes the closed subspace of $L_1[0, 1]$ (isomorphic to l_2) spanned by the Rademacher functions, then $R_1 \check{\otimes}_\varepsilon \ell_2$ is isometric to a subspace of $L_1[0, 1] \check{\otimes}_\varepsilon \ell_2$. This last space is isometric to a subspace of $\mathrm{ca}(\mathscr{A}, \ell_2)$, where here $\mathscr{A}$ denotes the σ-algebra of all λ-measurable sets of $[0, 1]$. Given that $\mathscr{K}(\ell_2, \ell_2)$ is isometric to $R_1 \check{\otimes}_\varepsilon \ell_2$, then $\mathrm{ca}(\mathscr{A}, \ell_2)$ has a copy of c_0. Thus, by Theorem 2, $\mathrm{ca}(\mathscr{A}, \ell_2)$ contains a copy of ℓ_∞.

THEOREM 3. *Assume that X is a Banach space. Then* $\mathrm{bvca}(\Sigma, X)$ *has a copy of* c_0 *if and only if one of the following two conditions holds:*

(i) *X contans a copy of c_0, or*

(ii) $\mathrm{bvca}(\Sigma, X)$ *has a copy of* ℓ_∞.

PROOF. Let J be an isomorphic from c_0 into $\mathrm{bvca}(\Sigma, X)$. Assuming X does not contain a copy of c_0, then as in the theorem above the series $\sum_n Je_n(E)$ is unconditionally convergent in X for each $E \in \Sigma$. Again the linear operator $T: \ell_\infty \to \mathrm{bvca}(\Sigma, X)$ defined as before is bounded. In fact, if $\{E_i, 1 \leqq i \leqq n\}$ is a partition of Ω by elements of Σ, then

$$\sum_{i=1}^{n} \|T\xi(E_i)\| = \sum_{i=1}^{n} \|\textstyle\sum_j \xi_j Je_j(E_i)\| = \sum_{i=1}^{n} \lim_k \|J\xi^k(E_i)\| \leqq \sup_k \sum_{i=1}^{n} \|J\xi^k(E_i)\| \leqq$$

$$\sup_k |J\xi^k| \leqq \sup_k \|J\| \, \|\xi^k\|_\infty \leqq \|J\| \, \|\xi\|_\infty$$

Setting $\mu := \sum_n 2^{-n} |Je_n|$, clearly $J\xi^n \ll \mu$ for $\xi \in \ell_\infty$ and $n \in \mathbb{N}$. Now, since

$\lim_n J\xi^n(E) = T\xi(E)$ for $\xi \in \ell_\infty$ and $E \in \Sigma$, then $T\xi \ll \mu$ for each $\xi \in \ell_\infty$ and so $T(\ell_\infty) \subseteq \mathrm{bvca}(\Sigma, X)$. As $\inf_n \|Te_n\| > 0$ the conclusion follows.

The converse is obvious, since X is isometric to a subspace of $\mathrm{bvca}(\Sigma, X)$.

REMARK 2. If X is a normed space, as $X^* = \mathscr{L}(X, \mathsf{K})$ Theorem 1 implies the well-known fact that X^* contains a copy of c_0 if and only if X^* has a copy of ℓ_∞. On the other hand, it has been shown in [3] that if each nonzero finite positive measure on Σ is purely atomic, then $\mathrm{ca}(\Sigma, X)$ has a copy of c_0 or ℓ_∞ if and only if X does, and it has been shown in [5] that under the same hypothesis on Σ, then $\mathrm{bvca}(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does.

Finally, it is worth mentioning that in [8] a Banach space X has been constructed which does not contain a copy of c_0 while $\mathrm{bvca}(\mathscr{B}_{[0,1]}, X)$ does contain a copy of ℓ_∞.

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