MATH. SCAND. 77 (1995), 161–174

G-DIMENSION

SIAMAK YASSEMI

0. Introduction.

Among the finitely generated modules over a local ring the ones of *finite projective dimension* are particulary nice. For example, if $M \neq 0$ is such a module, then the (classical) Auslander-Buchsbaum equality states that depth $A =$ depth $M +$ pd M, where pd M denotes the projective dimension of M.

In [1] Auslander and Bridger has generalized the notion of finite projective dimension to that of *finite G-dimension*, and they prove a variety of interesting results. For example, they extend the Auslander-Buchsbaum equality to this setup. Furthermore, they prove that a ring A is Gorenstein if and only if every finitely generated A-module M has finite G-dimension.

The notion of modules of finite projective dimension has also been generalized in another direction, namely to that of complexes of modules of finite projective dimension, cf. R. Hartshorne [5], and H.-B. Foxby [3] has proved that (most of) the formulas known for modules, including the Auslander-Buchsbaum equality, also hold for complexes of modules.

In this paper the notion of complexes of finite G-dimension will be introduced by defining the class of *reflexive complexes*. This is done by applying the derived functor of the Hom-functor of complexes. To describe this in classical terms let $I = 0 \to I^0 \to I^1 \to \cdots$ be an injective resolution of the A-module A. For any bounded complex

$$X = 0 \to X^i \to \cdots \to X^s \to 0.$$

set $X^* = \mathrm{Hom}(X, I)$ (which is a complex of A-modules). The complex X is said to be a reflexive complex if and only if

(1) $H^i(X)$ finitely generated for all i and $H^i(X) = 0$ for $|i| \gg 0$.

(2) $H^i(X^*)$ is vanishes for $i \gg 0$.

(3) The canonical map $X \to X^{**}$ is a homology isomorphism (that is, the induced map $H^i(X) \to H^i(X^{**})$ is an isomorphism for all i).

Received April 12, 1994.

It turns out that if the complex X is a module M (that is, $X^i = 0$ for $i \neq 0$ and $X^0 = M$) then X is a reflexive complex if and only if the G-dimension of the module M is finite.

This allows us to define the G-dimension of a complex X with finitely generated cohomology by

$$\text{G-dim}\, X = \sup\{i \in \mathbf{Z} \mid H^i(X^*) \neq 0\}$$

when X is a reflexive complex, and by G-dim $X = \infty$ otherwise. This is at the same time a generalization of the G-dimension of finitely generated modules and of the projective dimension of bounded complexes with finitely generated-cohomology modules. We prove that the Auslander-Buchsbaum equality

$$\operatorname{depth} A = \operatorname{depth} X + \text{G-dim}\, X$$

holds whenever G-dim X is finite as wll as many other formulas known previously only for finitely generated modules of finite G-dimension or for bounded complexes of finite projective dimension.

Throughout this paper all rings are commutative noetherian with a non-zero identity element. Rings will always denote by A. We write "f.g." for "finitely generated" and we use the notation "C-M" for Cohen-Macaulay. We shall also use the notation and terminology of [3] for complexes.

This paper will be included in the author's Ph.D. thesis. The author wishes to thank his supervisor, professor H.-B. Foxby for all his support, helpfulness, and in particular for suggesting many of the topics considered in this paper.

The author also thanks the university of Copenhagen for its hospitality and the facilities offered during the preparations of this paper.

1. G-dimension of modules.

In this section we bring the definition and some results of G-dimensions of A-modules M. Then we prove some new results. Assume that $(A, \mathfrak{m})$ is a local ring.

1.1. DEFINITION. A f.g. A-modules M is said to be of G-dimension zero, and we write G-dim $M = 0$, if and only if

(1) $\operatorname{Ext}^i_A(M, A) = 0$ for $i > 0$.

(2) $\operatorname{Ext}^i(\operatorname{Hom}(M, A), A) = 0$ for $i > 0$.

(3) The canonical map $M \to \operatorname{Hom}(\operatorname{Hom}(M, A), A)$ is an isomorphism.

For a non-negative integer n the module M is said to be of G-dimension at most n, and we write G-dim $M \leqq n$, if and only if there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

with G-dim $G_i = 0$ for all $0 \leqq i \leqq n$.

If there does not exist such an exact sequence then G-dim $M = \infty$.

1.2. Lemma ([1; 3.7, 3.14)]. *If* G-dim $M < \infty$ *then the following hold:*
(a) G-dim M + depth M = depth A.
(b) G-dim $M = \sup\{t \mid \operatorname{Ext}^t_A(M,A) \neq 0\}$.

1.3. Theorem ([1; 4.13]). *Let M be an A-module. Then the following hold:*
(a) *For any f.g. A-module N with* pd $N < \infty$ *we have*

$$\operatorname{Ext}^i_A(M,N) = 0 \quad \textit{for} \quad i > \text{G-dim}\, M.$$

(b) *For any f.g. A-module M we have* G-dim $M \leqq$ pd M. *If M has finite projective dimension then equality holds.*

In [12, 2.7] it is proved that if M and N are f.g. A-modules such that pd $M < \infty$ then

$$\dim \operatorname{Ext}^i_A(M,N) + i \leqq \operatorname{pd} M + \dim(M \otimes N) \quad \text{for all} \quad i.$$

There is a similar result to above inequality when pd $N < \infty$.

1.4. Theorem. *Let M, N be f.g. A-modules with* pd $N < \infty$. *Then*

$$\dim \operatorname{Ext}^i_A(M,N) + i \leqq \text{G-dim}\, M + \dim(M \otimes N) \quad \textit{for all} \quad i.$$

In particular $\dim \operatorname{Ext}^i_A(M,A) + i \leqq$ G-dim $M + \dim M$ *for all i.*

Proof. Since $\operatorname{Supp} \operatorname{Ext}^i_A(M,N) \subseteq \operatorname{Supp} M \cup \operatorname{Supp} N = \operatorname{Supp}(M \otimes N)$, for all i, we have $\dim \operatorname{Ext}^i_A(M,N) \leqq \dim(M \otimes N)$ for all i. Also by (1.3) when $\operatorname{Ext}^i_A(M,N) \neq 0$ we have $i \leqq$ G-dim M.

Now we show a result similar to the above when N has injective dimension.

1.5. Theorem. *Let M, N f.g. A-modules with* id $N < \infty$. *Then*

$$\dim \operatorname{Ext}^i_A(M,N) + i \leqq (\operatorname{depth} A - \operatorname{depth} M) + \dim(M \otimes N) \quad \textit{for all} \quad i.$$

Proof. We have $i \leqq \operatorname{depth} A - \operatorname{depth} M$ when $\operatorname{Ext}^i_A(M,N) \neq 0$ by [3, 6.46]. Also $\dim \operatorname{Ext}^i_A(M,N) \leqq \dim(M \otimes N)$.

1.6. Remark. If M, N are f.g. A-modules then $\operatorname{grade} \operatorname{Ext}^i_A(M,N) = \operatorname{depth} A_{\mathfrak{p}}$ for some $\mathfrak{p} \in \operatorname{Supp} \operatorname{Ext}^i_A(M,N)$. Assume that id $N < \infty$ (this implies that A is a C-M by [7; page 151]) then $i \leqq \operatorname{id}_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = \operatorname{depth} A_{\mathfrak{p}}$, so $\operatorname{grade} \operatorname{Ext}^i_A(M,N) \geqq i$, and hence $\dim \operatorname{Ext}^i(M,N) \leqq \dim A - i$. Therefore $\dim \operatorname{Ext}^i(M,N) + i \leqq \dim A$ for all i. In particular, if A is a Gorenstein then we have $\dim \operatorname{Ext}^i(M,A) + i \leqq \dim A$ for all i, and this is a beter result than (1.5) in this case.

2. G-dimension of complexes.

First we bring some definitions and results about complexes that we use in the rest of paper. The reader is referre to [3] for details of the following brief résumé of the homological theory of complexes of modules.

A complex X of modules over a ring A is a sequence of A-homomorphisms

$$X = \cdots \to X^{i-1} \xrightarrow{\partial^{i-1}} X^i \xrightarrow{\partial^i} X^{i+1} \to \cdots$$

such that $\partial^i \partial^{i-1} = 0$ for $i \in \mathbf{Z}$. (Note that we only use superscripts and that all differential have degree 1.) The notation $\mathscr{C}$ denotes the category of complexes and all morphisms between them; thus we write $X \in \mathscr{C}$.

The cohomology functors from complexes A-modules to graded A-modules is as usual denoted by $H(-)$. A complex of A-modules X is said to be homologically trivial if $H(X) = 0$.

We say a complex X is bounded above (respectively, bounded below, bounded) if there is $n \in \mathbf{Z}$ such that $X_i = 0$ for $i > n$ (respectively, $i < n$, $|i| > n$) and we write $X \in \mathscr{C}^-$ (respectively, $X \in \mathscr{C}^+$, $X \in \mathscr{C}^b$). Furthermore we set

$$\mathrm{s}(X) = \sup\{i \in \mathbf{Z} \mid H^i(X) \neq 0\} \quad \text{and}$$

$$\mathrm{i}(X) = \inf\{i \in \mathbf{Z} \mid H^i(X) \neq 0\}.$$

(Thus $\mathrm{s}(X) = -\infty$ and $\mathrm{i}(X) = \infty$ if X is homologically trivial.)

Once and for all we identify any module M with a complex A-modules, which has M in degree zero and is trivial elsewhere. We denote the class of all these modules by $\mathscr{M}$.

The full subcategory of complexes with finitely generated cohomology modules is denoted by $\mathscr{C}$, and we write $\mathscr{C}_{\mathrm{fg}}^+$ for $\mathscr{C}^+ \cap \mathscr{C}_{\mathrm{fg}}$, and likewise for $\mathscr{C}_{\mathrm{fg}}^-$ and $\mathscr{C}_{\mathrm{fg}}^b$.

If X and Y are complexes of A-modules, then $\mathrm{Hom}(X, Y)$ denotes the complex of A-modules with

$$\mathrm{Hom}(X, Y)^n = \prod_{i \in \mathbf{Z}} \mathrm{Hom}(X^i, Y^{i+n}) \quad \text{and}$$

$$\partial^n((\alpha^i)_{i \in \mathbf{Z}}) = (\partial^{i+n}\alpha^i - (-1)^{i+1}\alpha^{i+1}\partial^i)_{i \in \mathbf{Z}}$$

for $(\alpha^i)_{i \in \mathbf{Z}} \in \mathrm{Hom}(X, Y)^n$ and $n \in \mathbf{Z}$.

If X and Y are complexes of A-modules then $X \otimes Y$ denotes the complex of A-modules with

$$(X \otimes Y)^n = \prod_{i \in \mathbf{Z}} X^i \otimes Y^{n-i} \quad \text{and}$$

$$\partial^n((x^i \otimes y^{n-i})_{i \in \mathbf{Z}}) = (\partial^i(x^i) \otimes y^{n-i} + (-1)^i x^i \otimes \partial^{n-i}(y^{n-i}))_{i \in \mathbf{Z}}$$

for $(x^i \otimes y^{n-i})_{i \in \mathbf{Z}} \in (X \otimes Y)^n$ and $n \in \mathbf{Z}$.

A homology isomorphism is a morphism $\alpha: X \to Y$ such that $H(\alpha)$ is an isomorphism; homology isomorphisms are marked by placing the sign $\simeq$, while $\cong$ is used for isomorphisms. The equivalence relation generated by the homology isomorphisms is also denoted by $\simeq$.

When $X \in \mathscr{C}^-$, then the complex $F \in \mathscr{F}^-$ (respectively, $P \in \mathscr{P}^-$ or $L \in \mathscr{L}^-$) is said to be flat (respectively, projecive of f.g. free) resolution of X, if there exists a homology isomorphism $F \to X$ (respectively, $P \to X$ or $L \to X$). Here $\mathscr{F}^-$ denotes the set of bounded above complexes of flat modules, and $\mathscr{L}^-$ denotes the set of bounded above complexes of f.g. free modules.

When $X \in \mathscr{C}^+$, then the complex $I \in \mathscr{I}^+$ is said to be an injective resolution of X, if there exists a homology isomorphism $X \to I$.

For $(X, Y) \in \mathscr{C}^- \times \mathscr{C}$ the equivalence class of $\operatorname{Hom}(P, Y)$ for any P belonges to $\mathscr{P}^-$ (bounded above complexes of projective modules) with $P \simeq X$ is denoted by $\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)$. Similarly if $(X, Y) \in \mathscr{C} \times \mathscr{C}^+$ then $\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)$ denotes the equivalence class of $\operatorname{Hom}(X, I)$ when $Y \simeq I \in \mathscr{I}^+$ (bounded below complexes of injective modules). These two notations coincide when $(X, Y) \in \mathscr{C}^- \times \mathscr{C}^+$ and in this case $\operatorname{Hom}(P, I)$ represents $\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)$. Moreover, $\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)$ does not depend on the choice of P or I.

Let $(A, \mathfrak{m})$ be a local ring and $X \in \mathscr{C}^+$. Then we define

$$\operatorname{depth}_A X = \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X)).$$

For $X \in \mathscr{C}^-$ we define dimension of X by

$$\dim_A X = \sup_{\mathfrak{p}} (\dim A/\mathfrak{p} + s(X_{\mathfrak{p}}))$$

where the supremum is taken over all $\mathfrak{p} \in \operatorname{spec} A$. (Recall that $s(X_{\mathfrak{p}}) = -\infty$ if $\mathfrak{p} \notin \operatorname{Supp} X$.)

The *flat dimension* of $X \in \mathscr{C}^b$ is defined by

$$\operatorname{fd}_A X = \inf_F \sup \{ l \mid F^{-l} \neq 0 \},$$

where the supremum is taken over all flat resolutions of X.

The *projective dimension* of $X \in \mathscr{C}^b$ is defined by

$$\operatorname{pd}_A X = \inf_P \sup \{ l \mid P^{-l} \neq 0 \},$$

where the supremum is taken over all projecive resolutions of X.

The *injective dimension* of $X \in \mathscr{C}^b$ is defined by

$$\operatorname{id}_A X = \inf_I \sup \{ l \mid I^{l} \neq 0 \}$$

where the infimum is taken over all injective resolutions of X.

As in [3] for $X \in \mathscr{C}$, the complex $\Gamma_{\mathfrak{a}}(X)$ is introduced by

$\Gamma_{\mathfrak{a}}(X)^l = \{x \in X^l \mid \mathfrak{a}^n x = 0 \text{ for some } n > 0\}$ and $\partial_{\Gamma_{\mathfrak{a}}(X)} = \partial^l_{X \mid \Gamma_{\mathfrak{a}}(X)}$, the restriction, for $l \in \mathbb{Z}$.

Furthermore, if $X \in \mathscr{C}^+$ then $\underline{\Gamma}_{\mathfrak{a}}(X)$ denotes the equivalence class of $\Gamma_{\mathfrak{a}}(I)$ whenever I is an injective resolution of X.

We denote by $X \underline{\otimes} Y$ for the equivalence class of $F \otimes Y$ whenever $X \in \mathscr{C}^-$, $X \simeq F \in \mathscr{F}^-$ and $Y \in \mathscr{C}$.

Now we bring some result of [3] that we use in the rest of this paper.

2.1. Lemma ([3; 3.1.3]). *Let $X \in \mathscr{C}^-$ and $Y \in \mathscr{C}^+$ both be non-trivial and write $s = \mathrm{s}(X)$ and $i = \mathrm{i}(Y)$. Then $\mathrm{i}(\mathrm{Hom}(X, Y)) \geqq -s + i$ and*

$$\mathrm{Ext}^{-s+i}(X, Y) \cong \mathrm{Hom}(H^s(X), H^i(Y)).$$

2.2. Lemma ([3; 4.7]). *Let $X, Y \in \mathscr{C}^-$ both be non-trivial, and let $s = \mathrm{s}(X)$ and $t = \mathrm{s}(Y)$. Then $\mathrm{s}(X \otimes Y) \leqq s + t$ and*

$$\mathrm{Tor}_{-s-t}(X, Y) \cong H^s(X) \otimes H^t(Y).$$

There are three important equalities that we use many times.

2.3. Theorem ([3; 5.2, 5.4, 5.6]). (a) *For $X, Y \in \mathscr{C}^-$ and $Z \in \mathscr{C}^+$ we have*

$$\underline{\mathrm{H}}\mathrm{om}(X, \underline{\mathrm{H}}\mathrm{om}(Y, Z)) = \underline{\mathrm{H}}\mathrm{om}(X \underline{\otimes} Y, Z).$$

(b) *For $X \in \mathscr{C}^b_{\mathrm{fg}}$ and $Y, Z \in \mathscr{C}^b$ we have*

$$\underline{\mathrm{H}}\mathrm{om}(X, Y) \underline{\otimes} Z = \underline{\mathrm{H}}\mathrm{om}(X, Y \underline{\otimes} Z),$$

when $\mathrm{pd}\, X < \infty$ *or* $\mathrm{fd}\, Z < \infty$.

(c) *For $X \in \mathscr{C}^b_{\mathrm{fg}}$ and $Y, Z \in \mathscr{C}^b$ we have*

$$X \underline{\otimes} \underline{\mathrm{H}}\mathrm{om}(Y, Z) = \underline{\mathrm{H}}\mathrm{om}(\underline{\mathrm{H}}\mathrm{om}(X, Y), Z),$$

when $\mathrm{pd}\, X < \infty$ *or* $\mathrm{id}\, Z < \infty$.

Now we bring a definition that we need for the definition of G-dimension of a complex of modules.

2.4. Definition. A complex $X \in \mathscr{C}^b_{\mathrm{fg}}$ is said to be a *reflexive complex* if and only if $\mathrm{s}(\underline{\mathrm{H}}\mathrm{om}(X, A) < \infty$ and the canonical homomorphism $X \to \underline{\mathrm{H}}\mathrm{om}(\underline{\mathrm{H}}\mathrm{om}(X, A), A)$ is a homology isomorphism.

Note that A is Gorenstein if and only if all $X \in \mathscr{C}^b_{\mathrm{fg}}$ are reflexive.

2.5. Remark. Sometimes an A-module M is said to be a reflexive module if and only if the canonical homomorphism $M \to \mathrm{Hom}(\mathrm{Hom}(M, A), A)$ is isomorphism. Note that there is no relation between reflexive modules and reflexive complexes. In other words if M is a reflexive A-module we can not conclude that M is a reflexive complex or vice versa. See the next example.

2.6. EXAMPLE. (a) Let $(A, \mathfrak{m})$ be a local domain which is not Gorenstein, and let M be a 2nd syzygy of $k = A/\mathfrak{m}$, in other words there is exact sequence $0 \to M \to F_1 \to F_0 \to K \to 0$ where F_0 and F_1 are f.g. free A-modules. Then M is reflexive module by [1; (2.1), p. 48]. Since A is not Gorenstein we have $\text{G-dim}\, k = \infty$ and hence $\text{G-dim}\, M = \infty$. Therefore M is not a reflexive complex by (2.7).

(b) Let $(A, \mathfrak{m})$ be a local ring and depth $A > 0$. Let $M = A/(x)$ where $x \in \mathfrak{m} - z(A)$. We have $\text{G-dim}\, M = \text{pd}\, M = 1 < \infty$, and hence M is reflexive complex by (2.7). On the other hand $\text{Hom}(M, A) = 0$, and hence M is not a reflexive module.

The next theorem play important role in this paper and it is an unpublish result of H.-B. Foxby.

2.7. THEOREM. *Let M be a f.g. A-module. Then* $\text{G-dim}\, M < \infty$ *if and only if M is a reflexive complex.*

PROOF. (Due to H.-B. Foxby). Let $* = \underline{\underline{\mathrm{H}}}\mathrm{om}(-, A)$.

"only if": By induction on $g_M = \text{G-dim}\, M$.

$\underline{g_M = 0}$: Since $\text{Ext}^i_A(M, A) = 0$ for all $i > 0$ we have that $M^* = \text{Hom}(M, A) \in \mathscr{M}$. Also since $\text{Ext}^i(\text{Hom}(M, A), A) = 0$ for all $i > 0$ we have that $M^{**} = \text{Hom}(M, A)^* = \text{Hom}(\text{Hom}(M, A), A) \simeq M$.

$\underline{g_M > 0}$: Let $0 \to K \to G \to M \to 0$ be an exact sequence such that $\text{G-dim}\, G = 0$. Then $\text{G-dim}\, K = g_M - 1$ by [1; 3.15]. We know that $0 \to M^* \to G^* \to K^* \to 0$ is an exact sequence of complexes so we have a long exact sequence

$$\cdots H^{i-1}(K^*) \to H^i(M^*) \to H^i(G^*) \to \cdots.$$

Since $H^i(K^*)$ and $H^i(G^*)$ are bounded we have that $H^i(M^*)$ is bounded. Also since the canonical homomorphisms $K \to K^{**}$ and $G \to G^{**}$ are homology isomorphisms we have that the canonical homomorphism $M \to M^{**}$ is homology isomorphism, by the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & G^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

"If" Let $g_M = \text{s}(M^*)$. We have that

$$\begin{aligned} \underline{\underline{\mathrm{H}}}\mathrm{om}(k, M) &= \underline{\underline{\mathrm{H}}}\mathrm{om}(k, M^{**}) \\ &= \underline{\underline{\mathrm{H}}}\mathrm{om}(k \underline{\underline{\otimes}} M^*, A). \end{aligned} \tag{2.3a}$$

So $\text{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, M)) = -\text{s}(k \underline{\underline{\otimes}} M^*) + \text{depth}\, A$ by [3; 5.8].

Since $s(k \underline{\underline{\otimes}} M^*) = s(M^*)$ by (2.2), we have that $\operatorname{depth} M = -g_M + \operatorname{depth} A$. Now we prove this part by induction on g_M.

$\underline{g_M = 0}$: We have $\operatorname{depth} M = \operatorname{depth} A$ and $\operatorname{Ext}^i(M, A) = 0$ for $i \neq 0$ thus $M^* \simeq \operatorname{Hom}(M, A)$. Also the canonical map $M \to M^{**} = \operatorname{Hom}(M, A)^*$ is a homological isomorphism. Thus $H^i(M) \cong \operatorname{Ext}^i(\operatorname{Hom}(M, A), A)$ is zero for $i \neq 0$. In addition we have

$$\begin{array}{ccc} M^{**} & = & \operatorname{Hom}(\operatorname{Hom}(M, A), A) \\ \simeq \uparrow & \nearrow \simeq & \\ M & & \end{array}$$

Hence $\text{G-dim}\, M = 0$.

$\underline{g_M > 0}$: Let $0 \to K \to F \to M \to 0$ be an exact sequence such that F is a f.g. free module. Then $0 \to M^* \to F^* \to K^* \to 0$ is an exact sequence and hence we have long exact sequence

$$\cdots \to H^{i-1}(K^*) \to H^i(M^*) \to H^i(F^*) \to H^i(K^*) \to H^{i+1}(M^*) \to \cdots.$$

Therefore $g_K \leqq g_M - 1$. Since M and F are reflexive complexes we have that K is reflexive complex and hence by induction hypothesis we know that $\text{G-dim}\, K < \infty$ so $\text{G-dim}\, M < \infty$.

Theorem (2.7) makes – in view of (1.2b) – the next definition possible.

2.8. Definition. For a reflexive complex $X \in \mathscr{C}^b_{\mathrm{fg}}$ we define

$$\text{G-dim}\, X = s(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A)).$$

If X is not reflexive we write $\text{G-dim}\, X = \infty$.

If $(A, \mathfrak{m})$ be a local ring and M be an A-module with $\text{G-dim}\, M < \infty$. Then we have $\text{G-dim}\, M + \operatorname{depth} M = \operatorname{depth} A$ by (1.3). Now we bring the generalization of this theorem for complexes.

2.9. Theorem. *Let $(A, \mathfrak{m})$ be a local ring. For $X \in \mathscr{C}^b_{\mathrm{fg}}$ with finite G-dimension we have that*

$$\text{G-dim}\, X + \operatorname{depth} X = \operatorname{depth} A.$$

Proof. Let $X^* = \underline{\underline{\mathrm{H}}}\mathrm{om}(X, A)$. Since $\text{G-dim}\, X < \infty$ we have that X is a reflexive complex and hence the canonical map $X \to X^{**}$ is a homology isomorphism. We have

$$\begin{aligned}
\operatorname{depth} X &= \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X)) \\
&= \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X^{**})) \\
&= \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(k \underline{\underline{\otimes}} X^{*}, A)) && \text{by (2.3a)} \\
&= -\mathrm{s}(k \underline{\underline{\otimes}} X^{*}) + \operatorname{depth} A && \text{by [3; 6.5 and 6.6]} \\
&= -\mathrm{s}(X^{*}) + \operatorname{depth} A && \text{by (2.2)} \\
&= -(\text{G-dim } X) + \operatorname{depth} A.
\end{aligned}$$

The next result is generalization of (1.3a) and [1; 4.13II] in two ways complexes and $H^i(Y)$ not finitely generated. Also it can be viewed as a generalization of [3; 6.48d] with the extra condition, fd $Y < \infty$.

2.10. THEOREM. *For* $X \in \mathscr{C}^b_{\mathrm{fg}}$ *with* G-dim $X < \infty$ *and* $Y \in \mathscr{C}^b$ *with* fd $Y < \infty$ *the following hold:*

(a) $$\mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)) \leqq \mathrm{s}(Y) + \text{G-dim } X$$

(*b*) $$i(X \underline{\underline{\otimes}} Y) \geqq \mathrm{i}(Y) - \text{G-dim } X.$$

Here equality holds in both places if and only if $Y \in \mathscr{C}^b_{\mathrm{fg}}$.

In particular, for A-modules M and N if G-dim $M < \infty$ and N has finite flat dimension then the following hold:

(a′) $$\operatorname{Ext}^i(M, N) = 0 \quad \text{for} \quad i > \text{G-dim } M$$

(b′) $$\operatorname{Tor}_i(M, N) = 0 \quad \text{for} \quad i > \text{G-dim } M.$$

PROOF. (a):

$$\begin{aligned}
\mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y)) &= \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A) \underline{\underline{\otimes}} Y) && \text{(2.3b)} \\
&\leqq \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A)) + \mathrm{s}(Y) && \text{(2.2)} \\
&= \mathrm{s}(Y) + \text{G-dim } X.
\end{aligned}$$

(b):

$$\begin{aligned}
\mathrm{i}(X \underline{\underline{\otimes}} Y) &= \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X^{*}, A) \underline{\underline{\otimes}} Y) \\
&= \mathrm{i}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X^{*}, Y)) && \text{(2.3b)} \\
&\geqq \mathrm{i}(Y) - \mathrm{s}(X^{*}) \\
&= \mathrm{i}(Y) - \text{G-dim } X.
\end{aligned}$$

The next result is the dual of 2.10.

2.11. THEOREM. *For* $X \in \mathscr{C}^b_{\mathrm{fg}}$ *with* G-dim $X < \infty$ *and* $Y \in \mathscr{C}^b$ *with* id $Y < \infty$ *the following hold:*

(a) $$\mathrm{s}(\underline{\mathrm{H}}\mathrm{om}(X, Y)) \leqq \mathrm{s}(Y) + \text{G-dim}\, X$$

(b) $$\mathrm{i}(X \underline{\otimes} Y) \geqq \mathrm{i}(Y) - \text{G-dim}\, X.$$

Here equality holds in two places if and only if $Y \in \mathscr{C}^b_{\mathrm{fg}}$.

In particular, for f.g. A-module M with G-dim $M < \infty$ and A-module N with id $N < \infty$ the following hold:

(a′) $$\mathrm{Ext}^i(M, N) = 0 \quad \text{for} \quad i > \text{G-dim}\, M$$

(b′) $$\mathrm{Tor}_i(M, N) = 0 \quad \text{for} \quad i > \text{G-dim}\, M.$$

PROOF. (a):

$$\begin{aligned} \mathrm{s}(\underline{\mathrm{H}}\mathrm{om}(X, Y)) &= \mathrm{s}(\underline{\mathrm{H}}\mathrm{om}(\underline{\mathrm{H}}\mathrm{om}(X^*, A), Y)) \\ &= \mathrm{s}(X^* \underline{\otimes} Y) && (2.3\mathrm{c}) \\ &\geqq \mathrm{s}(Y) + \mathrm{s}(X^*) \\ &= \mathrm{s}(Y) + \text{G-dim}\, X \end{aligned}$$

(b):

$$\begin{aligned} \mathrm{i}(X \underline{\otimes} Y) &= \mathrm{i}(X \underline{\otimes} \underline{\mathrm{H}}\mathrm{om}(A, Y)) \\ &= \mathrm{i}(\underline{\mathrm{H}}\mathrm{om}(\underline{\mathrm{H}}\mathrm{om}(X, A), Y)) && (2.3\mathrm{c}) \\ &\geqq \mathrm{i}(Y) - \mathrm{s}(\underline{\mathrm{H}}\mathrm{om}(X, A)) && (2.1) \\ &= \mathrm{i}(Y) - \text{G-dim}\, X. \end{aligned}$$

Let $(A, \mathfrak{m})$ be a local ring and $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ with pd $X < \infty$. Then we have dim $\underline{\mathrm{H}}\mathrm{om}(X, Y) \leqq \mathrm{pd}\, X + \dim Y$ by [3; 8.29, 7.9 and 6.48d]. Now we prove similar result when pd $Y < \infty$.

2.12. THEOREM. *Let* $(A, \mathfrak{m})$ *be a local ring and* $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ *such that* pd $Y < \infty$. *Then*

$$\dim \underline{\mathrm{H}}\mathrm{om}(X, Y) \leqq \text{G-dim}\, X + \dim Y.$$

In particular, for f.g. A-modules M and N with pd $N < \infty$ such that there exists t with $\mathrm{Ext}^i(M, N) = 0$ for $i \neq t$ we have

$$\dim \mathrm{Ext}^t(M, N) + t \leqq \text{G-dim}\, M + \dim N.$$

PROOF. Suppose that G-dim $X < \infty$. Since pd $Y < \infty$ we have that fd $\underline{\Gamma}_{\mathfrak{m}}(Y) < \infty$ by [2; 6.5]. Now

$$\begin{aligned}
\dim \underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y) &= \mathrm{s}(\underline{\underline{\Gamma}}_{\mathfrak{m}}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y))) && [3;\, 8.29]\\
&= \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\Gamma}}_{\mathfrak{m}}(Y)) && [3;\, 7.9]\\
&= \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A) \underline{\underline{\otimes}} \underline{\underline{\Gamma}}_{\mathfrak{m}}(Y)) && (2.3\mathrm{b})\\
&\leq \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A)) + \mathrm{s}(\underline{\underline{\Gamma}}_{\mathfrak{m}}(Y)) && (2.2)\\
&= \text{G-dim}\, X + \dim Y && [3;\, 8.29]
\end{aligned}$$

Let $(A, \mathfrak{m})$ be a local ring and $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ with $\mathrm{pd}\, Y < \infty$. Then we have $\mathrm{depth}(X \otimes Y) = \mathrm{depth}\, X + \mathrm{depth}\, Y - \mathrm{depth}\, A$ by [3; 6.46]. Now we show the similar result when $\mathrm{id}\, Y < \infty$.

2.13. THEOREM. *Let $(A, \mathfrak{m})$ be a local ring and $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ with* $\text{G-dim}\, X < \infty$ *and* $\mathrm{id}\, Y < \infty$. *Then we have*

$$\mathrm{depth}(X \underline{\underline{\otimes}} Y) = \mathrm{depth}\, X + \mathrm{depth}\, Y - \mathrm{depth}\, A.$$

In particular, for f.g. A-modules M and N with $\text{G-dim}\, M$ and $\mathrm{id}\, N$ finite and $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$ we have

$$\mathrm{depth}(M \otimes N) = \mathrm{depth}\, M + \mathrm{depth}\, N - \mathrm{depth}\, A.$$

PROOF. We have

$$\begin{aligned}
\mathrm{depth}(X \underline{\underline{\otimes}} Y) &= \mathrm{depth}(X \underline{\underline{\otimes}} \underline{\underline{\mathrm{H}}}\mathrm{om}(A, Y))\\
&= \mathrm{depth}(\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A), Y)) && (2.3\mathrm{c})\\
&= \mathrm{depth}\, Y - \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, A)) && [3;\, 6.5]\\
&= \mathrm{depth}\, Y - \text{G-dim}\, X\\
&= \mathrm{depth}\, Y + \mathrm{depth}\, X - \mathrm{depth}\, A.
\end{aligned}$$

If $(A, \mathfrak{m})$ is local, $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ and $\mathrm{pd}\, X < \infty$ then $\mathrm{pd}\, \underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y) = \mathrm{pd}\, Y + \mathrm{s}(X)$ by [3; 6.48c]. Now we prove the similar result for G-dimension.

2.14. LEMMA. *Let $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$* and assume either $\mathrm{pd}\, X < \infty$ *and* $\text{G-dim}\, Y < \infty$ *or A is Gorenstein. Then*

$$\text{G-dim}\, \underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y) = \text{G-dim}\, Y + \mathrm{s}(X).$$

PROOF. We have

$$\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y), A) = X \underline{\underline{\otimes}} \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A) \quad \text{by (2.3c)}.$$

Thus

$$\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y), A), A) = \underline{\underline{\mathrm{H}}}\mathrm{om}(X \underline{\underline{\otimes}} \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A), A)$$
$$= \underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A), A)).$$

Since the canonical map $Y \to \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A), A)$ is a homology isomorphism we have the canonical map

$$\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y) \to \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y), A), A)$$

is homology isomorphism.

On the other hand

$$\begin{aligned} \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y), A)) &= \mathrm{s}(X \underline{\underline{\otimes}} \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A)) \\ &= \mathrm{s}(X) + \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A)) \qquad (2.2) \\ &= \mathrm{s}(X) + \text{G-dim}\, Y. \end{aligned}$$

It is easy to prove that for $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ with $\mathrm{pd}\, X < \infty$ and $\mathrm{pd}\, Y < \infty$ we have

$$\mathrm{pd}(X \underline{\underline{\otimes}} Y) = \mathrm{pd}\, X + \mathrm{pd}\, Y.$$

Now we extend this result to G-dim $Y < \infty$.

2.15. LEMMA. *For* $X, Y \in \mathscr{C}^b_{\mathrm{fg}}$ *with* $\mathrm{pd}\, X < \infty$ *and* $\text{G-dim}\, Y < \infty$ *we have*

$$\text{G-dim}(X \underline{\underline{\otimes}} Y) = \mathrm{pd}\, X + \text{G-dim}\, Y.$$

PROOF. We have

$$\underline{\underline{\mathrm{H}}}\mathrm{om}(X \underline{\underline{\otimes}} Y, A) = \underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A)) \qquad (2.3a)$$

Thus

$$\begin{aligned} \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X \underline{\underline{\otimes}} Y, A), A) &= \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A)), A) \\ &= X \underline{\underline{\otimes}} \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A), A) \qquad (2.3c) \end{aligned}$$

Since the canonical map $Y \to \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A), A)$ is homology isomorphism we have the canonical map $X \underline{\underline{\otimes}} Y \to \underline{\underline{\mathrm{H}}}\mathrm{om}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\otimes}}\, Y, A), A)$ is homology isomorphism. On the other hand

$$\begin{aligned} \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X \underline{\underline{\otimes}} Y, A)) &= \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A))) \qquad (2.3a) \\ &= \mathrm{pd}\, X + \mathrm{s}(\underline{\underline{\mathrm{H}}}\mathrm{om}(Y, A)) \qquad [3; 6.48d] \\ &= \mathrm{pd}\, X + \text{G-dim}\, Y. \end{aligned}$$

2.16. LEMMA. *Let* $(A, \mathfrak{m})$ *be a local ring and let* M, N *be f.g. A-modules. Then*

$$\mathrm{depth}\, N \leqq \dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M, N).$$

In particular $\mathrm{depth}\, A \leqq \dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M, A)$.

PROOF. We know that $\operatorname{depth}\underline{\underline{\mathrm{H}}}\mathrm{om}(M,N) = \operatorname{depth} N$ by [3; 6.5], and $\operatorname{depth}\underline{\underline{\mathrm{H}}}\mathrm{om}(M,N) \leqq \dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M,N)$ by [3; 6.15].

2.17. THEOREM. *Let $(A,\mathfrak{m})$ be a local ring and let M be f.g. A-module. Let N be a C-M A-module with* $\operatorname{pd} N < \infty$. *Then*

$$\dim N \leqq \dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M,N) \leqq \text{G-dim}\, M + \dim(M \otimes N).$$

In particular, if A is a C-M ring then

$$\dim A \leqq \dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) \leqq \text{G-dim}\, M + \dim M.$$

PROOF. We know that $\dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M,N) = \dim \operatorname{Ext}^i(M,N) + i$ for some i by [3; 6.12]. Now use (1.4) and (2.16).

We recall the standard measure of non-Cohen-Macaulayness, namely its Cohen-Macaulay defect $\operatorname{cmd}_A M = \dim M - \operatorname{depth} M$ (Grothendick calls $\operatorname{cmd}_A M$ the co-depth of M and denotes it by Coprof M, [5].)

In [12; 3.8] it is proved that for a f.g. A-module M with $\operatorname{pd} M < \infty$ that we have $\operatorname{cmd} \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) \leqq \operatorname{cmd} M$. Now we extend this result for A-module M with $\text{G-dim}\, M < \infty$.

2.18. THEOREM. *Let $(A,\mathfrak{m})$ be a local ring and let M be a f.g. A-module with finite G-dimension. Then* $\operatorname{cmd} \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) \leqq \operatorname{cmd} M$.

PROOF. We have $\dim \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) \leqq \dim M + \text{G-dim}\, M$ by (1.4). Also $\operatorname{depth} \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) = \operatorname{depth} M + \text{G-dim}\, M$ by (3; 6.5) and (1.2a). Thus $\operatorname{cmd} \underline{\underline{\mathrm{H}}}\mathrm{om}(M,A) \leqq \operatorname{cmd} M$.

2.19. REMARK. We know that by general intersection theorem if M, N are f.g. A-modules then $\dim N \leqq \operatorname{pd} M + \dim(M \otimes N)$.

In (2.17) we proved that for C-M A-module N with $\operatorname{pd} N < \infty$

$$\dim N \leqq \text{G-dim}\, M + \dim(M \otimes N).$$

Now it is natural to ask that, is it correct in general?

The next example shows that the answer is negative.

2.20. EXAMPLE. Let $(A,\mathfrak{m})$ be a local Gorenstein ring with $\dim A = 1$ and $\operatorname{spec} A = \{\mathfrak{m}, \mathfrak{p}, \mathfrak{q}\}$ (for example $A = k[[X,Y]]/(XY)$). Let $M = A/\mathfrak{p}$ and $N = A/\mathfrak{q}$. Then $M \otimes N = A/\mathfrak{p} + \mathfrak{q}$ and hence $\dim(M \otimes N) = 0$. Since A is a Gorenstein ring we have $\text{G-dim}\, M < \infty$ and hence $\text{G-dim}\, M = \operatorname{depth} A - \operatorname{depth} M = 1 - 1 = 0$. On the other hand $\dim N = 1$ so $\dim N > \text{G-dim}\, M + \dim(M \otimes N)$.

REFERENCES

1. M. Auslander and M. Bridger, *Stable module Theory*, Mem. Amer. Math. Soc. 94 (1964).
2. H.-B. Foxby, *Bounded complexes of flat modules*, J. Pure. Appl. Algebra 14 (1974), 149–172.
3. H.-B. Foxby, *A homological theory of complexes of modules*, Preprint series, Dept. of Math. Univ. Copenhagen 19a, 19b (1981).
4. A. Grothendick (notes by R. Hartshorne), *Local cohomology*, Lecture Notes in Math. 41, (1967).
5. R. Hartshorne, *Residues and duality*, Lecture Notes in Math. 20 (1966).
6. M. Hochster, *The equicharacteristic case of some homological conjectures on local rings*, Bull. Amer. Math. Soc. 80 (1974), 638–686.
7. H. Matsumura, *Commutative ring Theory*, Cambr. stud. Adv. Math., Cambridge Univ. Press, Cambridge (1986).
8. C. Peskine and L. Szpiro, *Annnneaux de Gorenstein, et torsion en algebere Commutative*, Secrétariat mathematique, rue Pierre Curie, Paris 5e (1967).
9. C. Peskine and L. Szpiro, *Dimension projective finite et cohomologie locale*, Publ. Math. I.H.E.S. 42 (1973), 47–119.
10. P. Roberts, *Cohen-Macaulay complexes and an analytic proof of the New Intersection Conjecture*, J. Algebra 66 (1980), 220–225.
11. P. Roberts, *Le Théoréme d'intersection*, C. R. Acad. sci. Paris, ser. I 304 (1987), 177–180.
12. S. Yassemi, *Generalized section functors*, J. Pure Appl. Algebra, to appear.

MATEMATISK INSTITUT
UNIVERSITETSPARKEN 5
DK-2100 København Ø
DENMARK
E-mail address: yassemi@math.ku.dk