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COHERENT RINGS AND HOMOLOGICALLY FINITE SUBCATEGORIES

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Introduction.

For an arbitrary ring R (always associative with 1) we denote by $\operatorname{Mod} R$ the category of all left R-modules and by $\operatorname{mod} R$ the full subcategory of $\operatorname{Mod} R$ with objects the finitely presented modules. Here we recall that a module M in $\operatorname{Mod} R$ is said to be finitely presented if there is an exact sequence $F \to F' \to M \to 0$ with F and F' finitely generated free modules. By a subcategory of $\operatorname{mod} R$ we always understand a full subcategory closed under isomorphisms. By a functor from a subcategory $\mathscr{C}$ of $\operatorname{mod} R$ to the category Ab of abelian groups we will understand an additive functor (covariant or contravariant).

For each module M in $\operatorname{mod} R$ we denote by $(M,)$ the functor $\operatorname{Hom}_R(M,)$ from $\operatorname{mod} R$ to the category of abelian groups and if $\mathscr{C}$ is a subcategory of $\operatorname{mod} R$ we denote by $(M,)|_{\mathscr{C}}$ the functor $(M,)$ restricted to $\mathscr{C}$.

A covariant functor $F\colon \mathscr{C} \to \mathrm{Ab}$, where $\mathscr{C}$ is a subcategory of $\operatorname{mod} R$, is said to be finitely generated if there is an exact sequence $(M,)|_{\mathscr{C}} \to F \to 0$ of functors with M in $\mathscr{C}$ and F is said to be finitely presented, or coherent, if there is an exact sequence $(N,)|_{\mathscr{C}} \to (M,)|_{\mathscr{C}} \to F \to 0$ with M and N in $\mathscr{C}$.

If $(M,)|_{\mathscr{C}}$ is a finitely generated functor on the subcategory $\mathscr{C}$ of $\operatorname{mod} R$ for all modules M in $\operatorname{mod} R$, we say that the subcategory $\mathscr{C}$ is covariantly finite in $\operatorname{mod} R$. In this paper we give a procedure for constructing new covariantly finite subcategories of $\operatorname{mod} R$ from two given covariantly finite subcategories in the case where R is a left coherent ring.

Preliminaries.

Recall that a ring R is said to be left coherent if every finitely generated left ideal in R is finitely presented. First we recall the fact that a ring R is left coherent if and only if $\operatorname{mod} R$ is an abelian category and for the convenience of the reader we include a proof of this.

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LEMMA 1. *If* $M \xrightarrow{f} N$ *is a morphism in* mod R, *then* Coker f *is also in* mod R.

PROOF. Choose free resolutions $F_1(M) \to F_0(M) \to M \to 0$ and $F_1(N) \xrightarrow{a} F_0(N) \xrightarrow{b} N \to 0$ of M and N with $F_i(M)$ and $F_i(N)$ finitely generated for $i = 0, 1$. This leads to the exact commutative diagram

$$\begin{array}{ccccccc}
F_1(M) & \longrightarrow & F_0(M) & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
F_1(N) & \xrightarrow{a} & F_0(N) & \xrightarrow{b} & N & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle p} & & \\
 & & & & \operatorname{Coker} f & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

The sequence $F_1(N) \coprod F_0(M) \xrightarrow{(a,g)} F_0(N) \xrightarrow{pb} \operatorname{Coker} f \to 0$ is now exact, showing thet Coker f is finitely presented.

LEMMA 2. *Assume R is a left coherent ring. Then any finitely generated R-submodule of a finitely generated free left R-module is finitely presented.*

PROOF. Let M be a finitely generated submodule of the free module nR. The proof goes by induction on n. The case $n = 1$ follows by the definition of a left coherent ring. Assume the result is proven for $n - 1$. We have an exact sequence $0 \to R \to nR \xrightarrow{f} (n-1)R \to 0$. This gives rise to the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}(f \mid M) & \longrightarrow & M & \longrightarrow & f(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & nR & \xrightarrow{f} & (n-1)R & \longrightarrow & 0
\end{array}$$

$f(M)$ is a finitely generated submodule of $(n-1)R$ since M is finitely generated and therefore $f(M)$ is finitely presented by the induction hypothesis. Since $f(M)$ is finitely presented and M is finitely generated it follows by Schanuel's lemma that $\operatorname{Ker}(f \mid M)$ is finitely generated and hence finitely presented since it is a submodule of R. But then we have that M is finitely presented.

Using this Lemma we may prove that any finitely generated submodule of a finitely presented module is finitely presented when the ring is left coherent.

LEMMA 3. *Assume R is a left coherent ring. Then any finitely generated R-submodule of a finitely presented left R-module is finitely presented.*

PROOF. Let $P_1 \to P_0 \xrightarrow{f} M \to 0$ be a free presentation of M with P_0 and P_1 finitely generated. Then $\operatorname{Ker} f$ is finitely presented by Lemma 2. Let M' be a submodule of M and $P' \to M'$ a surjection with P' finitely generated free. Consider the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & K' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \coprod P' & \longrightarrow & P' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & f^{-1}(M') & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $f^{-1}(M')$ is a finitely generated submodule of P_0 and therefore finitely presented. It follows by Schanuel's lemma that K'', and therefore also K' is finitely generated. This proves that M' is finitely presented.

Recall that an additive category $\mathscr{A}$ is said to be abelian if (*i*) every morphism in $\mathscr{A}$ has a kernel and a cokernel in $\mathscr{A}$; (ii) every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel; (iii) every morphism may be written as the composition of an epimorphism and a monomorphism.

PROPOSITION 4. *For a ring R the following are equivalent.*

(1) *R is left coherent.*

(2) $\operatorname{mod} R$ *is an abelian category.*

PROOF. Assume first that R is left coherent. Let $M \xrightarrow{f} N$ be a morphism in $\operatorname{mod} R$. Since $\operatorname{Coker} f$ is in $\operatorname{mod} R$ by Lemma 1 it is clear that $\operatorname{mod} R$ has cokernels. To prove that $\operatorname{Ker} f$ is in $\operatorname{mod} R$ it suffices by Lemma 3 to show that $\operatorname{Ker} f$ is finitely generated. Consider the exact sequence $0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$. Since M is finitely generated, $\operatorname{Im} f$ is finitely generated and since it is a submodule of the finitely presented module N, $\operatorname{Im} f$ is finitely presented. Using Schanuel's lemma it then follows that $\operatorname{Ker} f$ is finitely generated and hence finitely presented. This shows that $\operatorname{mod} R$ has kernels and it follows that $\operatorname{mod} R$ is an abelian category.

Assume conversely that mod R is an abelian category. Let $A \xrightarrow{f} B$ be a morphism between finitely presented left R-modules. Since mod R is abelian, f has a kernel L in mod R. Denoting by K the kernel of f in Mod R we want to show that K is finitely presented. For this it suffices to show that K is isomorphic to L. Consider the following diagram of morphisms in Mod R

$$\begin{array}{ccccccc} & & & & L & & \\ & & & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & A & \xrightarrow{f} & B \end{array}$$

Since $fj = 0$ and K is the kernel of f in Mod R there is a unique map $L \xrightarrow{g} K$ such that $j = ig$. We claim that g is an isomorphism. Let $x \in K$. Then there is a map $R \xrightarrow{h} K$ with $h(1) = x$. Now $f(ih) = 0$ and R is in mod R, so since L is the kernel of f in mod R, there is a unique map $R \xrightarrow{u} L$ with $ih = ju$. But then $x = ih(1) = ju(1) = igu(1)$. Hence x is in Im g, showing that g is an epimorphism. To prove that g is a monomorphism it suffices to prove that j is a monomorphism in Mod R. So assume $j(y) = 0$ with $y \in L$. Choose a map $R \xrightarrow{v} L$ with $v(1) = y$. Then $jv = 0$, so since j is a monomorphism in mod R, we have that $v = 0$, proving that j is mono in Mod R. Now let I be a finitely generated left ideal in R and let $nR \xrightarrow{f} R$ be a morphism with Im $f = I$. From the above we have that Ker f is finitely presented and hence I is finitely presented and therefore R is left coherent.

We now want to establish some facts concerning additive functors from subcategories of mod R to the category Ab of abelian groups. All these are well known results, so we only indicate some proofs for the convenience of the reader.

LEMMA 5. *Let* $0 \to F \to G \to H \to 0$ *be an exact sequence of functors from a subcategory* C *of* mod R *to* Ab (*either variance*).

(1) *If* G *is finitely generated then* H *is finitely generated.*

(2) *If* F *and* H *are finitely generated then* G *is finitely generated.*

LEMMA 6. *Let* R *be any ring and let* $0 \to F \to G \xrightarrow{g} H \to 0$ *be an exact sequence of covariant functors from* mod R *to* Ab. *If* G *is coherent and* F *is finitely generated, then* H *and* F *are coherent.*

PROOF. If G is coherent, there is an exact sequence

$$0 \to \operatorname{Ker} m \to (M,\ ) \xrightarrow{m} G \to 0$$

with M in mod R and Ker m finitely generated. This gives rise to the following exact commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \operatorname{Ker} m & \longrightarrow & \operatorname{Ker}(gm) & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & (M,\ ) & = & (M,\ ) & & & & \\
 & & & & {\scriptstyle m}\downarrow & & {\scriptstyle gm}\downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & G & \xrightarrow{g} & H & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

From the top row we see that $\operatorname{Ker}(gm)$ is finitely generated and hence H is coherent. Choose N in mod R with a surjection $(N,\ ) \xrightarrow{\eta} \operatorname{Ker}(gm)$. This induces an exact sequence

$$(+)\ \ 0 \to (\operatorname{Coker} \eta,\ ) \to (N,\ ) \xrightarrow{\eta} \operatorname{Ker}(gm) \to 0$$

where $M \xrightarrow{\eta} N$ is the map corresponding to $(N,\ ) \xrightarrow{\eta} \operatorname{Ker}(gm) \hookrightarrow (M,\ )$. Now $\operatorname{Coker} \eta$ is finitely presented by Lemma 1 so by the sequence $(+)$ it follows that $\operatorname{Ker}(gm)$ is coherent. It then follows the first part of the proof, applied to the exact sequence $0 \to \operatorname{Ker} m \to \operatorname{Ker}(gm) \to F \to 0$ that F is coherent.

Using Lemma 6 we immediately get the following result.

LEMMA 7. *Let R be any ring and let $0 \to F \to G \to H$ be an exact sequence of covariant functors from* mod R *to* Ab. *If G and H are coherent, then F is coherent.*

LEMMA 8. *Let R be any ring and $\mathscr{C}$ a subcategory of* mod R. *If $F \xrightarrow{\alpha} G$ is a morphism between two coherent functors from $\mathscr{C}$ to* Ab, *then there is a morphism $\tilde{F} \xrightarrow{\tilde{\alpha}} \tilde{G}$ between two coherent functors on* mod R *such that $\alpha = \tilde{\alpha}|_{\mathscr{C}}$.*

PROOF. Choose presentations $(A_0,\ )|_{\mathscr{C}} \xrightarrow{(a,\ )|_{\mathscr{C}}} (A_1,\ )|_{\mathscr{C}} \to F \to 0$ and $(B_0,\ )|_{\mathscr{C}} \xrightarrow{(b,\ )|_{\mathscr{C}}} (B_1,\ )_{\mathscr{C}} \to G \to 0$ of F and G respectively with A_0, A_1, B_0 and B_1 in $\mathscr{C}$. Let $\tilde{F}$ and $\tilde{G}$ be the cookernel of $(a,\ )$ and $(b,\ )$ respectively. Then $\tilde{F}$ and $\tilde{G}$ are coherent functors on mod R and there exists a morphism $\tilde{F} \xrightarrow{\tilde{\alpha}} \tilde{G}$ with $\alpha = \tilde{\alpha}|_{\mathscr{C}}$.

Let $\mathscr{C}$ be any category and $\mathscr{A}$ a full subcategory of $\mathscr{C}$. Recall that $\mathscr{A}$ is said to be covariantly finite in $\mathscr{C}$ if for each object C in $\mathscr{C}$ the functor $(C,\ )|_{\mathscr{A}}$ is a finitely generated functor on $\mathscr{A}$. Dually, $\mathscr{A}$ is said to be contravariantly finite in $\mathscr{C}$ if $(\ ,C)|_{\mathscr{A}}$ is finitely generated for each object C in $\mathscr{C}$.

LEMMA 9. *Let R be any ring and $\mathscr{C}$ a covariantly finite subcategory of* mod R. *If F is a coherent covariant functor from* mod R *to* Ab, *then $F|_{\mathscr{C}}$ is a coherent functor on $\mathscr{C}$.*

PROOF. If F is coherent, there is an exact sequence $(M',) \to (M,) \to F \to 0$ with M and M' in mod R. Using that restriction to a subcategory is an exact functor and that $\mathscr{C}$ is covariantly finite in mod R we get the following commutative diagram

$$\begin{array}{ccccccc}
(C'_1,)|_{\mathscr{C}} & \longrightarrow & (C'_0,)|_{\mathscr{C}} & \longrightarrow & (M',)|_{\mathscr{C}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(C_1,)|_{\mathscr{C}} & \longrightarrow & (C_0,)|_{\mathscr{C}} & \longrightarrow & (M,)|_{\mathscr{C}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & F|_{\mathscr{C}} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

with C'_1, C'_0, C_1 and C_0 in $\mathscr{C}$. Hence the exact sequence $(C_1,)|_{\mathscr{C}} \coprod (C'_0,)|_{\mathscr{C}} \to (C_0,)|_{\mathscr{C}} \to F|_{\mathscr{C}} \to 0$ shows that $F|_{\mathscr{C}}$ is coherent.

Using Lemma 8, Lemma 7 and Lemma 9 we have the following result.

LEMMA 10. *Let R be any ring and $\mathscr{C}$ a covariantly finite subcategory of* mod R. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of covariant functors from $\mathscr{C}$ to* Ab. *If G and H are coherent, then F is coherent.*

While Lemmas 5 to 10 are valid for any ring, we need for our next result that R is left coherent.

PROPOSITION 11. *Let R be a left coherent ring. Then* $\mathrm{Ext}^1_R(M,)$ *is a coherent functor from* mod R *to* Ab *for all modules M in* mod R.

PROOF. Let M be in mod R. Since R is left coherent there is an exact sequence $0 \to K \to P \to M \to 0$ with P finitely generated projective and K finitely presented. This induces an exact sequence of functors $(P,) \to (K,) \to \mathrm{Ext}^1_R(M,) \to 0$ showing that $\mathrm{Ext}^1_R(M,)$ is coherent.

From this proposition and Lemma 9 we get the following.

COROLLARY 12. *Let R be a left coherent ring and $\mathscr{C}$ a covariantly finite subcategory of* mod R. *Then* $\mathrm{Ext}^1_R(M,)|_{\mathscr{C}}$ *is a coherent functor from $\mathscr{C}$ to* Ab *for all modules M in* mod R.

REMARK. As we see the proof of Proposition 11 uses that the category mod R has enough projective modules. For the contravariant functors $\mathrm{Ext}^1_R(\ ,M)|_{\mathscr{C}}$ one would instead need that mod R has enough injective modules for the same proof to go through. For this reason Proposition 11, and therefore Corollary 12, are not necessarily valid in the contravariant case. Let for example R be a Dedekind ring. Then R is noetherian and hence left coherent, but mod R does not have enough injectives; viz. let $\underline{m}$ be a maximal ideal in R. Then the injective hull of the finitely presented module $R/\underline{m}$ is the Prüfer module $R(\underline{m}^\infty)$, which is not finitely generated.

The main result.

For subcategories $\mathscr{A}$ and $\mathscr{B}$ of mod R we denote by ${}_{\mathscr{A}}\varepsilon^{\mathscr{B}}$ the full subcategory of mod R with objects the modules E for which there is an exact sequence $0 \to A \to E \to B \to 0$ with A in $\mathscr{A}$ and B in $\mathscr{B}$. In [6] we considered the case when R is an artin algebra and proved that ${}_{\mathscr{A}}\varepsilon^{\mathscr{B}}$ is covariantly (contravariantly) finite in mod R whenever $\mathscr{A}$ and $\mathscr{B}$ are covariantly (contravariantly) finite in mod R. When R is a left coherent ring one can, as remarked above, not hope for the same proof to go through in the contravariant case. The proof in the covariant case is the same as in [6] but we include it here for the convenience of the reader.

THEOREM 13. *Let R be a left coherent ring and let $\mathscr{A}$ and $\mathscr{B}$ be subcategories of* mod R. *If $\mathscr{A}$ and $\mathscr{B}$ are covariantly finite in* mod R, *then ${}_{\mathscr{A}}\varepsilon^{\mathscr{B}}$ is covariantly finite in* mod R.

PROOF. Let M be any module in mod R. Since $\mathscr{B}$ is covariantly finite in mod R there is a surjection $(B,\ )|_{\mathscr{B}} \xrightarrow{(b,\ )|_{\mathscr{B}}} (M,\ )|_{\mathscr{B}}$ with B in $\mathscr{B}$. Consider the exact sequence

$$0 \to \mathrm{K} \to \mathrm{Ext}^1_R(B,\ )|_{\mathscr{A}} \xrightarrow{\mathrm{Ext}^1_R(b,\ )|_{\mathscr{A}}} \mathrm{Ext}^1_R(M,\ )|_{\mathscr{A}}$$

of functors on $\mathscr{A}$. Since $\mathscr{A}$ is covariantly finite in mod R it follows from Corollary 12 and Lemma 7 that K is finitely generated. Hence there exists a module A_K in $\mathscr{A}$ and a surjection $(A_K,\ )|_{\mathscr{A}} \xrightarrow{\phi} K$. Let $\phi_{A_K}(1_{A_K})$ be represented by the element $0 \to A_K \xrightarrow{i} E \xrightarrow{p} B \to 0$ in $\mathrm{Ext}^1_R(B, A_K)$. Now $\mathrm{Ext}^1_R(b, A_K)(\phi_{A_K})) = 0$, so in the following pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_K & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & A_K & \xrightarrow{i} & E & \xrightarrow{p} & B & \longrightarrow & 0
\end{array}$$

the upper sequence splits. This is equivalent to the existence of a morphism $M \xrightarrow{e} E$ with $pe = b$. Using that $\mathscr{A}$ is covariantly finite in mod R there is a surjection $(A,\ )|_{\mathscr{A}} \xrightarrow{(a,\ )|_{\mathscr{A}}} (M,\ )|_{\mathscr{A}}$ with A in $\mathscr{A}$. This gives us a morphism

$$\left(\begin{pmatrix} e \\ a \end{pmatrix}, \ \right)\Big|_{{}_{\mathscr{A}}\varepsilon^{\mathscr{B}}} : (E \coprod A, \ )|_{{}_{\mathscr{A}}\varepsilon^{\mathscr{B}}} \to (M, \ )|_{{}_{\mathscr{A}}\varepsilon^{\mathscr{B}}}$$

which we claim is an epimorphism.

So let $M \xrightarrow{f} E'$ be any morphism with E' in ${}_{\mathscr{A}}\varepsilon^{\mathscr{B}}$. We may assume that E' occurs as the middle term of an exact sequence $0 \to A' \xrightarrow{j} E' \xrightarrow{q} B' \to 0$ where A' is in $\mathscr{A}$ and B' is in $\mathscr{B}$. Since $(B, \ )|_{\mathscr{B}} \xrightarrow{(b, \ )|_{\mathscr{B}}} (M, \ )|_{\mathscr{B}}$ is epi there is a map $B \xrightarrow{\beta} B'$ such that $\beta b = qf$. Form the pullback

$$\begin{array}{ccccccccc} \delta: 0 & \longrightarrow & A' & \xrightarrow{u} & F & \xrightarrow{v} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \beta} & & \\ \varepsilon: 0 & \longrightarrow & A' & \xrightarrow{j} & E' & \xrightarrow{q} & B' & \longrightarrow & 0 \end{array}$$

Now $\beta(0, b) = (0, \beta b) = (0, qf) = (qj, qf) = q(j, f)$ so by the universal property of pullbacks there exists a unique map $A' \coprod M \xrightarrow{t} F$ such that $h't = (j, f)$ and $vt = (0, b)$. But we also have the following commutative diagram

$$\begin{array}{ccccccccc} \eta: 0 & \longrightarrow & A' & \xrightarrow{\binom{1}{0}} & A' \coprod M & \xrightarrow{(0,1)} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle (j,f)} & & \downarrow{\scriptstyle qf} & & \\ \varepsilon: 0 & \longrightarrow & A' & \xrightarrow{j} & E' & \xrightarrow{q} & B' & \longrightarrow & 0 \end{array}$$

and hence $\eta = \operatorname{Ext}^1_R(qf, A')(\varepsilon) = \operatorname{Ext}^1_R(b, A')(\delta)$ so since $\eta = 0$, we have that $\delta \in K(A')$. Therefore $\delta = \phi_{A'}(\alpha')$ for some map $A_K \xrightarrow{\alpha'} A'$. From this we get that $\delta = \phi_{A'}(\alpha') = \phi_{A'}(A_K, \alpha')(1_{A_K}) = K(\alpha')\phi_{A_K}(1_{A_K}) = \operatorname{Ext}^1_R(A_K, \alpha')(\phi_{A_K}(1_{A_K}))$ so δ is given in a pushout diagram

$$\begin{array}{ccccccccc} \phi(1): 0 & \longrightarrow & A_K & \xrightarrow{i} & E & \xrightarrow{p} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle h} & & \| & & \\ \delta: 0 & \longrightarrow & A' & \xrightarrow{u} & F & \xrightarrow{v} & B & \longrightarrow & 0 \end{array}$$

Now we have $vt\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0, b)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = b = pe = vhe$, so $v\left(he - t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0$. Hence $\operatorname{Im}\left(he - \begin{pmatrix} 0 \\ t \end{pmatrix}\right) \subset \operatorname{Ker} v = \operatorname{Im} u$, so there exists a unique map $M \xrightarrow{s} A'$ with $us = he - \begin{pmatrix} 0 \\ t \end{pmatrix}$. Further, since $(A, \ )|_{\mathscr{A}} \xrightarrow{(a, \ )|_{\mathscr{A}}} (M, \ )|_{\mathscr{A}}$ is epi, we have that $s = \alpha a$ for some map $A \xrightarrow{\alpha} A'$. But then we have $\begin{pmatrix} 0 \\ t \end{pmatrix} = he - us = he - u\alpha a$, and therefore

$$f = (j, f)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = h't\begin{pmatrix} 0 \\ 1 \end{pmatrix} = h'he - h'us = h'he - js = h'he - j\alpha a =$$
$\left(\begin{pmatrix} e \\ a \end{pmatrix}, E'\right)(h'h, -j\alpha)$, showing that $\left(\begin{pmatrix} e \\ a \end{pmatrix}, \right)\Big|_{\mathscr{A}\varepsilon^{\mathscr{B}}}$ is epi.

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