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THE IMAGE OF THE ENVELOPING ALGEBRA AND IRREDUCIBILITY OF INDUCED REPRESENTATIONS OF EXPONENTIAL LIE GROUPS

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Abstract.

It is shown that the image of the universal enveloping algebra of the Lie algebra of an exponential Lie group under a representation of the group induced from a character satisfying the Pukanszky condition is a dense algebra of differential operators. This result is used to prove irreducibility of certain families of nonunitary induced representations of exponential Lie groups.

I. Introduction.

Let G denote an exponential Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$. In the case where G is nilpotent a theorem due to A. A. Kirillov [Ki] (and to J. Dixmier [Di] in a generic case) states: Let π be a strongly continuous and unitary irreducible representation of G, then π may be realized on $L^2(\mathsf{R}^n)$ such that its derived representation $d\pi$ maps $\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ onto the complex algebra of all differential operators on R^n with polynomial coefficients. The present paper deals with the problem of describing this image of $\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ when G is a general exponential Lie group. We provide an extension of Kirillov's result which implies that the image is dense in the algebra of all differential operators on R^n with C^∞-coefficients.

We establish the result in a context of not necessarily unitary representations induced in a generalized sense from not necessarily unitary characters, with the unitary representations arizing as particular cases. The aim of our study is to extend irreducibility results known for the unitary representations to this wider class of representations. This was achieved for the nilpotent groups in [JS1], [Ja] and [JS2] by extensions of Kirillov's theorem adapted to the generalized setup. For the Heisenberg groups particular instances of these representations were treated in [Pe], [Li2], [Ho] and [LL1, 2]. Cf. also [Li1, 2] for related studies. In the present paper the results are inferred for general exponential groups via our density theorem.

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To be more specific, let $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ be a complex-valued and real-linear functional on $\mathfrak{g}$ and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ with $\alpha([\mathfrak{k}, \mathfrak{k}]) = \{0\}$. Let K denote the analytic subgroup of G corresponding to $\mathfrak{k}$ and let $\chi = \chi_{\alpha, \mathfrak{k}}$ be the continuous character on K determined by $\chi(\exp X) = e^{-\alpha(X)}$, $X \in \mathfrak{k}$. Let finally $\Lambda_{\alpha, \mathfrak{k}}$ denote the action of G by left translations on the distribution space

$$\mathscr{D}'_{\alpha, \mathfrak{k}}(G) := \{u \in \mathscr{D}'(G) \mid u(gk) = \chi(k)^{-1} u(g), \quad \forall g \in G, k \in K\}.$$

The representations we study, and call induced from χ, are the restrictions of $\Lambda_{\alpha, \mathfrak{k}}$ to any left invariant subspace of $\mathscr{D}'_{\alpha, \mathfrak{k}}(G)$ with a locally convex topology for which certain natural continuity and density conditions are satisfied.

The unitary representations fit into this setup as follows. If $\alpha = i\beta + \delta$, where $\beta, \delta \in \mathfrak{g}^*$ and $\delta|_{\mathfrak{k}} = \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{g}/\mathfrak{k}}$, then the unitary representation $\pi_{\beta, \mathfrak{k}}$ of G induced from the unitary character $\chi_{i\beta, \mathfrak{k}}$ is the restriction of $\Lambda_{\alpha, \mathfrak{k}}$ to the subspace of functions in $\mathscr{D}'_{\alpha, \mathfrak{k}}(G)$ satisfying an appropriate square-integrability condition.

It is well-known that $\pi_{\beta, \mathfrak{k}}$ is irreducible if and only if $\mathfrak{k}$ satisfies the so-called Pukanszky condition relative to β, and that every continuous unitary irreducible representation of G is unitarily equivalent to some $\pi_{\beta, \mathfrak{k}}$, cf. [Be], [Pu].

We prove that the Pukanszky condition also suffices for irreducibility when the induction from $\chi_{\alpha, \mathfrak{k}}$ is in the extended sense, and then in addition for more general complex-valued α than arizing in the unitary case.

Our results are obtained for α of the form $\alpha = c\beta + \delta$, where $c \in \mathbf{C} \backslash \{0\}$, $\beta \in g^*$ and $\delta \in (\mathfrak{g}^*)^{\mathbf{C}}$ with $\delta([\mathfrak{g}, \mathfrak{g}]) = \{0\}$, and with $\mathfrak{k}$ satisfying the Pukanszky condition relative to β. The induced representations are realized via coexponential bases Ξ of $\mathfrak{g}$ modulo $\mathfrak{k}$ as representations on subspaces of $\mathscr{D}'(\mathbf{R}^n)$, $n = \dim \mathfrak{g}/\mathfrak{k}$.

The main result of the paper is (Theorem 4.1): The basis Ξ may be chosen such that the image of $\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})$ in the algebra of differential operators on $\mathbf{R}^n$ in coordinates $x_1, \ldots, x_n$ contains the multiplication by either x_1 or e^{x_1} (if $n \geqq 1$). If $\mathfrak{g}$ is spanned by $\mathfrak{k}$ and the nilradical of $\mathfrak{g}$, then Ξ may be chosen such that the image of $\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})$ equals the complex algebra of all differential operators on $\mathbf{R}^n$ with polynomial coefficients.

From the main result follows by recursion on n that the image of $\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})$ is dense in a specific sense in the algebra of all differential operators on $\mathbf{R}^n$ with coefficients of class C^∞ (Theorem 4.2).

Combining this density theorem for the image of $\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})$ with an irreducibility criterion from [JS2] we extend irreducibility results proved for nilpotent groups in [JS2] to exponential groups. We show that:

(1) The considered induced representations are all scalar irreducible (the only continuous intertwining operators are the scalar multiples of the identity).

(2) The representations on the spaces which are invariant under the natural multiplication on $\mathscr{D}'_{\alpha, \mathfrak{k}}(G)$ by the functions in $C^\infty(G/K)$ are even ultra-irreducible

(the image of the group under the representation spans a dense subspace for the ultra-weak operator topology; cf. [JS2]).

The concept of ultra-irreducibility was introduced in [Li3], [Li4]. It strengthens the notion of topological complete irreducibility due to [Go], which on its side in particular implies topological as well as scalar irreducibility. General criteria for ultra-irreducibility and other density results are found in [LL1,2].

For unitary representations all the considered notions of irreducibility coincide. The irreducibility of the unitary induced representations $\pi_{\beta,\mathfrak{k}}$ for $\mathfrak{k}$ satisfying the Pukanszky condition relative to β is thus a particular instance of (1).

Part (2) applies to the representations on local distribution spaces, e.g. $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ and $C^\infty_{\alpha,\mathfrak{k}}(G) := C^\infty(G) \cap \mathscr{D}'_{\alpha,\mathfrak{k}}(G)$. These choices appear in particular natural when the covariance condition $u(gk) = \chi(k^{-1})u(g)$, $\forall g \in G$, $k \in K$, is given its equivalent differential formulation, $Xu = \alpha(X)u$, $\forall X \in \mathfrak{k}$. This relates our induced representations to the so-called eigenspace representations introduced by S. Helgason in [He1]; cf. Chap. II.4 of [He2]. The present work may thus be seen as solving for exponential groups modified cases of the program set up in [He1].

II. Notation, definitions and auxiliary results.

1. *General.* In the following G denotes a real exponential Lie group with Lie algebra $\mathfrak{g}$, i.e. the exponential map exp: $\mathfrak{g} \to G$ is a diffeomorphism. The universal enveloping algebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ is denoted $\mathfrak{U}(\mathfrak{g}^C)$. The dual space of $\mathfrak{g}$ is denoted $\mathfrak{g}^*$ and $(\mathfrak{g}^*)^C$ will be the set of the complex-valued and real-linear functionals on $\mathfrak{g}$. By $C\mathfrak{g}^*$ we denote the subset of $(\mathfrak{g}^*)^C$ consisting of the complex multiples of the elements of $\mathfrak{g}^*$.

For each $\alpha \in (\mathfrak{g}^*)^C$, $S(\alpha, \mathfrak{g})$ will denote the set of subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ subordinate to α, i.e. for which $\mathfrak{k} \subseteq \mathfrak{k}^\alpha$, where $\mathfrak{k}^\alpha := \{X \in \mathfrak{g} \mid \alpha([X, \mathfrak{k}]) = \{0\}\}$. For each $\alpha \in \mathfrak{g}^*$, $P(\alpha, \mathfrak{g})$ will denote the set of subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ satisfying the *Pukanszky condition* relative to α, i.e. for which $\mathfrak{k} = \mathfrak{k}^\beta$ for all $\beta \in \mathfrak{g}^*$ with $\beta = \alpha$ on $\mathfrak{k}$. If $\mathfrak{g}$ is nilpotent, then a subalgebra $\mathfrak{k} \in P(\alpha, \mathfrak{g})$, if only $\mathfrak{k} = \mathfrak{k}^\alpha$; [Be; Chap. IV.3].

2. *The induced representations.* Given $\alpha \in (\mathfrak{g}^*)^C$ and $\mathfrak{k} \in S(\alpha, \mathfrak{g})$, let K denote the analytic subgroup of G corresponding to $\mathfrak{k}$ and let $\chi = \chi_{\alpha,\mathfrak{k}}$ be the continuous character on K defined by $\chi(\exp(X)) = e^{-\alpha(X)}$ for all $X \in \mathfrak{k}$.

The left regular representation Λ of G on the space $\mathscr{D}'(G)$ of distributions on G then leaves invariant the subspace

$$\mathscr{D}'_{\alpha,\mathfrak{k}}(G) := \{u \in \mathscr{D}'(G) \mid R(k)u = \chi(k)^{-1}u, \quad \forall k \in K\},$$

where $R(k)$ denotes right translation by k. Left and right translations on $\mathscr{D}'(G)$ are defined so as to extend the actions on function spaces embedded in $\mathscr{D}'(G)$ by means of some chosen left Haar measure on G, i.e. $[\Lambda(g)u](\varphi) = u(\Lambda(g^{-1})\varphi)$ and

$[R(g)u](\varphi) = u(\Delta_G(g^{-1})R(g^{-1})\varphi)$ for all $g \in G$, $u \in \mathscr{D}'(G)$ and $\varphi \in \mathscr{D}(G)$, where Δ_G denotes the modular function on G.

Set $C^\infty_{\alpha,\mathfrak{k}}(G) = C^\infty(G) \cap \mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ and let $\mathscr{D}_{\alpha,\mathfrak{k}}(G)$ denote the subspace of $C^\infty_{\alpha,\mathfrak{k}}(G)$ consisting of the functions of compact support modulo K.

The spaces $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ and $C^\infty_{\alpha,\mathfrak{k}}(G)$ inherit their topologies as closed subspaces of the strong dual $\mathscr{D}'(G)$ and the Fréchet space $C^\infty(G)$, respectively, while the space $\mathscr{D}_{\alpha,\mathfrak{k}}(G)$ is equipped with the inductive limit topology from the family of subspaces $\{\varphi \in C^\infty_{\alpha,\mathfrak{k}}(G) \mid \operatorname{supp}\varphi \subseteq CK\}$, where C ranges over the compact subsets of G.

We adapt the notion of a normal space of distributions to the present setup: a *normal* subspace of $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ is a locally convex space E for which $\mathscr{D}_{\alpha,\mathfrak{k}}(G) \subseteq E \subseteq \mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ with weakly continuous inclusion maps and with $\mathscr{D}_{\alpha,\mathfrak{k}}(G)$ dense in E.

We then define a *representation induced from* χ to be the restriction Λ_E of Λ to a left invariant normal subspace E of $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$, provided that Λ_E is a strongly continuous representation of G by weakly continuous endomorphisms of E.

Examples are $E = \mathscr{D}'_{\alpha,\mathfrak{k}}(G)$, $C^\infty_{\alpha,\mathfrak{k}}(G)$ and $\mathscr{D}_{\alpha,\mathfrak{k}}(G)$. The representations on these spaces are even differentiable. The restrictions of Λ to $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ and $C^\infty_{\alpha,\mathfrak{k}}(G)$ are denoted by $\Lambda_{\alpha,\mathfrak{k}}$ and $\lambda_{\alpha,\mathfrak{k}}$, respectively.

The unitary representation $\pi_{\beta,\mathfrak{k}}$ induced from the unitary character $\chi_{i\beta,\mathfrak{k}}$, where $\beta \in \mathfrak{g}^*$, $\mathfrak{k} \in S(\beta, \mathfrak{g})$, is then in our setup induced from $\chi_{i\beta+\delta,\mathfrak{k}}$, where $\delta \in \mathfrak{g}^*$ fulfilling $\delta|_\mathfrak{k} = \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{k}}$ accounts for the square root of the quotient between the modular functions on K and G appearing in the unitary induction.

3. *Coexponential bases.* To establish isomorphisms $\mathscr{D}'_{\alpha,\mathfrak{k}}(G) \simeq \mathscr{D}'(\mathsf{R}^n)$, $C^\infty_{\alpha,\mathfrak{k}}(G) \simeq C^\infty(\mathsf{R}^n)$, $\mathscr{D}_{\alpha,\mathfrak{k}}(G) \simeq \mathscr{D}(\mathsf{R}^n)$ etc. we use coexponential bases:

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. An ordered basis $(X_1, \ldots, X_n)$, of $\mathfrak{g}$ modulo $\mathfrak{k}$, i.e. of a complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$, is called *coexponential*, cf. [Be], if the map

$$(x_1, \ldots, x_n, X) \in \mathsf{R}^n \times \mathfrak{k} \mapsto \exp(x_1X_1)\ldots\exp(x_nX_n)\exp(X) \in G$$

is a diffeomorphism of $\mathsf{R}^n \times \mathfrak{k}$ onto G.

An ordered basis $(X_1, \ldots, X_n)$ og $\mathfrak{g}$ modulo $\mathfrak{k}$ is called *normal*, if for every $i = 1, \ldots, n$, the subspace $\mathfrak{g}_i := \operatorname{span}\{X_{i+1}, \ldots, X_n\} + \mathfrak{k}$ is an ideal of $\mathfrak{g}_{i-1}$.

A normal basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ is automatically coexponential, cf. Theorem 3.18.11 of [Va]. If $\mathfrak{g}$ is nilpotent, then by Engel's theorem there is for every subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ a normal basis of $\mathfrak{g}$ modulo $\mathfrak{k}$. The existence of coexponential bases for general exponential $\mathfrak{g}$ is established by (c) of Lemma 2.1 below.

A coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ of the form in (c) of Lemma 2.1 below will be called *compatible* with the nilpotent ideal $\mathfrak{n}$ in question.

2.1. LEMMA. *Let* $\mathfrak{k}$ *denote a subalgebra of* $\mathfrak{g}$.

(a) *Let* $\mathfrak{i}$ *be an ideal of* $\mathfrak{g}$ *for which* $\mathfrak{g} = \mathfrak{i} + \mathfrak{k}$. *If* $(X_1, \ldots, X_n)$ *is a coexponential basis of* $\mathfrak{i}$ *modulo* $\mathfrak{i} \cap \mathfrak{k}$, *then it it also a coexponential basis of* $\mathfrak{g}$ *modulo* $\mathfrak{k}$.

(b) *Let* $\mathfrak{l}$ *be a subalgebra of* $\mathfrak{g}$ *with* $\mathfrak{l} \supseteq \mathfrak{k}$. *If* $(X_1, \ldots, X_m)$ *is a coexponential (resp.*

normal) basis of $\mathfrak{g}$ *modulo* $\mathfrak{l}$ *and* $(X_{m+1},\dots,X_n)$ *is a coexponential (resp. normal) basis of* $\mathfrak{l}$ *modulo* $\mathfrak{k}$, *then* $(X_1,\dots,X_n)$ *is a coexponential (resp. normal) basis of* $\mathfrak{g}$ *modulo* $\mathfrak{k}$.

(c) *Let* $\mathfrak{n}$ *denote a nilpotent ideal of* $\mathfrak{g}$ *with* $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}$. *Let* $(X_1,\dots,X_k)$ *denote a basis of* $\mathfrak{g}$ *modulo* $\mathfrak{n}+\mathfrak{k}$ *and let* $(X_{k+1},\dots,X_n)$ *denote a normal basis of* $\mathfrak{n}$ *modulo* $\mathfrak{n}\cap\mathfrak{k}$. *Then* $(X_1,\dots,X_n)$ *is a coexponential basis of* $\mathfrak{g}$ *modulo* $\mathfrak{k}$.

PROOF. (a): Since $\mathfrak{g}/\mathfrak{i}$ is solvable, there is a normal basis $(Y_1,\dots,Y_p)$ of $\mathfrak{g}$ modulo $\mathfrak{i}$, cf. Cor. 3.7.5 of [Va]. Since $\mathfrak{g}=\mathfrak{i}+\mathfrak{k}$, we can choose the $Y_1,\dots,Y_p$ in $\mathfrak{k}$, and the basis will be normal of $\mathfrak{k}$ modulo $\mathfrak{k}\cap\mathfrak{i}$. Let I, K and K_0 denote the analytic subgroups of G corresponding to $\mathfrak{i}$, $\mathfrak{k}$ and $\mathfrak{k}_0:=\mathfrak{k}\cap\mathfrak{i}$, respectively. Then we have the diffeomorphisms:

$$\begin{aligned}(x_1,\dots,x_n,a)\in\mathsf{R}^n\times K_0 &\mapsto \exp(x_1X_1)\dots\exp(x_nX_n)a\in I\\ (a,y_1,\dots,y_p)\in K_0\times\mathsf{R}^p &\mapsto a\exp(y_pY_p)\dots\exp(y_1Y_1)\in K\\ (b,y_1,\dots,y_p)\in I\times\mathsf{R}^p &\mapsto b\exp(y_pY_p)\dots\exp(y_1Y_1)\in G\end{aligned}$$

The map $(x_1,\dots,x_n,k)\in\mathsf{R}^n\times K\mapsto\exp(x_1X_1)\dots\exp(x_nX_n)k\in G$ is thus composed of diffeomorphisms $\mathsf{R}^n\times K\to\mathsf{R}^n\times K_0\times\mathsf{R}^p\to I\times\mathsf{R}^p\to G$ and is hence itself a diffeomorphism, so $(X_1,\dots,X_n)$ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$.

(b): Straightforward.

(c): Since $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}+\mathfrak{k}$, the basis $X_1,\dots,X_k$ of $\mathfrak{g}$ modulo $\mathfrak{n}+\mathfrak{k}$ is normal and hence coexponential. Since $\mathfrak{n}$ is an ideal of $\mathfrak{n}+\mathfrak{k}$, the normal and hence coexponential basis $X_{k+1},\dots,X_n$ of $\mathfrak{n}$ modulo $\mathfrak{n}\cap\mathfrak{k}$ is by (a) a coexponential basis of $\mathfrak{n}+\mathfrak{k}$ modulo $\mathfrak{k}$. The conclusion now follows by (b).

4. *Realizations.* Let $\alpha\in(\mathfrak{g}^*)^{\mathsf{C}}$ and $\mathfrak{k}\in S(\alpha,\mathfrak{g})$, and let $\Xi:=(X_1,\dots,X_n)$ be a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$. Then the map $S\colon C^\infty(G)\to C^\infty(\mathsf{R}^n)$ given by

$$(Sf)(x_1,\dots,x_n):=f(\exp(x_1X_1)\dots\exp(x_nX_n)),\qquad f\in C^\infty(G),$$

restricts to a topological vector space isomorphism $S_{\alpha,\mathfrak{k},\Xi}$ of $C^\infty_{\alpha,\mathfrak{k}}(G)$ onto $C^\infty(\mathsf{R}^n)$, realizing $\lambda_{\alpha,\mathfrak{k}}$ as a representation $\lambda=\lambda_{\alpha,\mathfrak{k},\Xi}$ of G on $C^\infty(\mathsf{R}^n)$ of the form

$$[\lambda(g)f](x)=e^{\alpha(\kappa(g^{-1},x))}f(g^{-1}\cdot x),\qquad g\in G,\ f\in C^\infty(\mathsf{R}^n),\ x\in\mathsf{R}^n.$$

Here $\kappa\colon G\times\mathsf{R}^n\to\mathfrak{k}$ and $(g,x)\in G\times\mathsf{R}^n\mapsto g\cdot x\in\mathsf{R}^n$ are the C^∞-maps determined by $g(x,0)=(g\cdot x,\kappa(g,x))$ for all $g\in G$, $x\in\mathsf{R}^n$, where G is identified with $\mathsf{R}^n\times\mathfrak{k}$ by means of the coexponential basis Ξ.

The map $S_{\alpha,\mathfrak{k},\Xi}$ extends and restricts to topological vector space isomorphisms of $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ and $\mathscr{D}_{\alpha,\mathfrak{k}}(G)$ onto $\mathscr{D}'(\mathsf{R}^n)$ and $\mathscr{D}(\mathsf{R}^n)$, respectively. The formula for $\lambda_{\alpha,\mathfrak{k},\Xi}$ extends by continuity to the realization $\Lambda_{\alpha,\mathfrak{k},\Xi}$ of $\Lambda_{\alpha,\mathfrak{k}}$ on $\mathscr{D}'(\mathsf{R}^n)$.

If Ξ is chosen compatible with a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$, then the Lebesgue measure on $\mathsf{R}^n\simeq G/K$ is relatively invariant under the action of G: Let $\delta_0\in\mathfrak{g}^*$ be defined by $\delta_0(\Xi)=\{0\}$ and $\delta_0|_{\mathfrak{k}}=\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{k}}$ and set $\chi_0(\exp X)=e^{\delta_0(X)}$,

$X \in \mathfrak{g}$. Then χ_0 is a character on G and it may be shown that $d(g \cdot x) = \chi_0(g)\,dx$, $g \in G$. The unitary representation $\pi_{\beta,\mathfrak{k}}$, $\beta \in \mathfrak{g}^*$, is realized for such Ξ as the restriction of $\Lambda_{\alpha,\mathfrak{k},\Xi}$ to $L^2(\mathsf{R}^n)$, where $\alpha = i\beta + \frac{1}{2}\delta_0$.

We denote by $\mathrm{DO}(\mathsf{R}^n)$ the complex algebra consisting of all the differential operators on R^n with C^∞-coefficients, and by $\mathrm{DP}(\mathsf{R}^n)$ the subalgebra consisting of the operators with polynomial coefficients.

The derived image of $\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ under the induced representations is determined by the representations on the spaces $C^\infty_{\alpha,\mathfrak{k}}(G) \simeq C^\infty(\mathsf{R}^n)$. If the basis Ξ is compatible with a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ with $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{n}$, then

$$d\lambda_{\alpha,\mathfrak{k},\Xi}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) \subseteq \mathrm{DO}(\mathsf{R}^k) \otimes \mathrm{DP}(\mathsf{R}^{n-k}) \subseteq \mathrm{DO}(\mathsf{R}^n), \tag{2.1}$$

where $k = \dim \mathfrak{g}/(\mathfrak{n} + \mathfrak{k})$ and $n = \dim \mathfrak{g}/\mathfrak{k}$.

The remaining two lemmas establish transformation properties of $\lambda_{\alpha,\mathfrak{k},\Xi}$ under a change in Ξ or the addition of a character on $\mathfrak{g}$ to α.

2.2. Lemma. *Let $\alpha \in (\mathfrak{g}^*)^{\mathsf{C}}$ and $\mathfrak{k} \in S(\alpha,\mathfrak{g})$. Let $\Xi = (X_1,\ldots,X_n)$ and $\Xi' = (X_1',\ldots,X_n')$ denote coexponential bases of $\mathfrak{g}$ modulo $\mathfrak{k}$ and set $\lambda = \lambda_{\alpha,\mathfrak{k},\Xi}$ and $\lambda' = \lambda_{\alpha,\mathfrak{k},\Xi'}$. Then the following hold:*

(1) *The isomorphism $\Phi := S_{\alpha,\mathfrak{k},\Xi} \circ S^{-1}_{\alpha,\mathfrak{k},\Xi'}\colon C^\infty(\mathsf{R}^n) \to C^\infty(\mathsf{R}^n)$ which intertwines λ' and λ is of the form*

$$(\Phi f)(x) = e^{\alpha(p(x))} f(\xi(x)). \qquad f \in C^\infty(\mathsf{R}^n),\ x \in \mathsf{R}^n,$$

where $\xi = (\xi_1,\ldots,\xi_n)\colon \mathsf{R}^n \to \mathsf{R}^n$ is the diffeomorphism and $p\colon \mathsf{R}^n \to \mathfrak{k}$ is the C^∞-map determined by

$$\exp(x_1X_1)\ldots\exp(x_nX_n) = \exp(\xi_1(x)X_1')\ldots\exp(\xi_n(x)X_n')\exp(p(x)) \tag{2.2}$$

for all $x = (x_1,\ldots,x_n) \in \mathsf{R}^n$.

(2) *If for some $D \in \mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$, $d\lambda'(D)$ equals the multiplication by a function $\psi \in C^\infty(\mathsf{R}^n)$, then $d\lambda(D)$ equals the multiplication by the function $\psi \circ \xi \in C^\infty(\mathsf{R}^n)$.*

(3) *Put $\mathfrak{g}_2 := \operatorname{span}\{X_2,\ldots,X_n\} + \mathfrak{k}$ and suppose that $\mathfrak{g}_2 = \operatorname{span}\{X_2',\ldots,X_n'\} + \mathfrak{k}$, that $\mathfrak{g}_2$ is an ideal of $\mathfrak{g}$ and that $X_1 - X_1' \in \mathfrak{g}_2$. If for some $D \in \mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$, $d\lambda'(D)$ equals the multiplication by a function $\psi(x_1)$, $\psi \in C^\infty(\mathsf{R})$, then $d\lambda(D)$ equals the multiplication by the same function $\psi(x_1)$.*

(4) *Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ with $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{n}$ and suppose that $\mathfrak{g} = \mathfrak{n} + \mathfrak{k}$ and that the coexponential bases Ξ and Ξ' of $\mathfrak{g}$ modulo $\mathfrak{k}$ are normal bases of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$. Then $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ equals $\mathrm{DP}(\mathsf{R}^n)$ if and only if $d\lambda'(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ equals $\mathrm{DP}(\mathsf{R}^n)$.*

Proof. (1): For every $f \in C^\infty(\mathsf{R}^n)$ and $x = (x_1,\ldots,x_n) \in \mathsf{R}^n$ we have

$$\begin{aligned}(\Phi f)(x) &= (S^{-1}_{\alpha,\mathfrak{k},\Xi'}f)(\exp(x_1X_1)\ldots\exp(x_nX_n))\\ &= (S^{-1}_{\alpha,\mathfrak{k},\Xi'}f)(\exp(\xi_1(x)X_1')\ldots\exp(\xi_n(x)X_n'))e^{\alpha(p(x))}\\ &= e^{\alpha(p(x))} f(\xi_1(x),\ldots,\xi_n(x)).\end{aligned}$$

(2): Let $f \in C^\infty(\mathsf{R}^n)$. Then $d\lambda(D)\Phi f = \Phi d\lambda'(D)f = \Phi(\psi f) = e^{\alpha \circ p}((\psi f) \circ \xi) = (\psi \circ \xi) e^{\alpha \circ p}(f \circ \xi) = (\psi \circ \xi)\Phi f$.

(3): Put $G_2 := \exp(\mathfrak{g}_2)$. Applying the quotient homomorphism $\pi\colon G \to G/G_2$ to (2.2) we get that $\exp(x_1 d\pi(X_1)) = \exp(\xi_1(x) d\pi(X_1'))$ for all $x = (x_1, \ldots, x_n) \in \mathsf{R}^n$. Hence $\xi_1(x) = x_1$, since $d\pi(X_1) = d\pi(X_1') \neq 0$ and since the quotient group G/G_2 is exponential. The conclusion now follows from (2).

(4): Since $N := \exp(\mathfrak{n})$ is a subgroup of G, p maps R^n into $\mathfrak{k} \cap \mathfrak{n}$. Since $\mathfrak{n}$ is nilpotent, the maps ξ, ξ^{-1} and p are all polynomial. Hence the isomorphism $E \mapsto \Phi \circ E \circ \Phi^{-1}$ of $\mathrm{DO}(\mathsf{R}^n)$ onto $\mathrm{DO}(\mathsf{R}^n)$ which maps $d\lambda'(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ onto $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ also maps $\mathrm{DP}(\mathsf{R}^n)$ onto $\mathrm{DP}(\mathsf{R}^n)$. This proves (4).

The representation $\lambda_{\alpha,\mathfrak{k}}$ is essentially unchanged when a character on $\mathfrak{g}$ is added to α:

2.3. LEMMA. *Let $\alpha \in (\mathfrak{g}^*)^{\mathsf{C}}$ and $\mathfrak{k} \in S(\alpha, \mathfrak{g})$. Let $\delta \in (\mathfrak{g}^*)^{\mathsf{C}}$ be a character on* $\mathfrak{g}$, i.e. $\delta([\mathfrak{g},\mathfrak{g}]) = \{0\}$, *and let χ_0 denote the corresponding character on G,* i.e. $\chi_0(\exp(X)) = e^{\delta(X)}$ *for all $X \in \mathfrak{g}$. Then $\mathfrak{k} \in S(\alpha + \delta, \mathfrak{g})$ and the following hold:*

(1) *There is by*

$$f \in C^\infty_{\alpha+\delta,\mathfrak{k}}(G) \mapsto \chi_0^{-1} f \in C^\infty_{\alpha,\mathfrak{k}}(G) \tag{2.3}$$

defined a topological vector space isomorphism of $C^\infty_{\alpha+\delta,\mathfrak{k}}(G)$ onto $C^\infty_{\alpha,\mathfrak{k}}(G)$ intertwining the representations $\lambda_{\alpha+\delta,\mathfrak{k}}$ and $\chi_0^{-1}\lambda_{\alpha,\mathfrak{k}}$ of G.

(2) *If Ξ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ with the property $\delta(\Xi) = \{0\}$, then*

$$d\lambda_{\alpha+\delta,\mathfrak{k},\Xi}(X) = d\lambda_{\alpha,\mathfrak{k},\Xi}(X) - \delta(X) \quad \textit{for all} \quad X \in \mathfrak{g}.$$

(3) *If Ξ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ compatible with $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$, and if $\delta' \in (\mathfrak{g}^*)^{\mathsf{C}}$ is the extension of $\delta|_{\mathfrak{k}}$ given by $\delta'(\Xi) = \{0\}$, then*

$$d\lambda_{\alpha+\delta,\mathfrak{k},\Xi}(X) = d\lambda_{\alpha,\mathfrak{k},\Xi}(X) - \delta'(X) \quad \textit{for all} \quad X \in \mathfrak{g}.$$

PROOF. (1): Set $\chi(\exp(X)) := e^{\alpha(X)}$ for $X \in \mathfrak{k}$. Let $f \in C^\infty_{\alpha+\delta,\mathfrak{k}}(G)$, $g, g' \in G$ and $k \in \exp(\mathfrak{k})$. Then $(\chi_0^{-1} f)(gk) = \chi_0^{-1}(gk) f(g)\chi(k)\chi_0(k) = (\chi_0^{-1} f)(g)\chi(k)$, which proves that $\chi_0^{-1} f \in C^\infty_{\alpha,\mathfrak{k}}(G)$. Also $\chi_0^{-1}(g') f(g^{-1}g') = \chi_0^{-1}(g)(\chi_0^{-1} f)(g^{-1}g')$, which proves that the map (2.3) intertwines $\lambda_{\alpha+\delta,\mathfrak{k}}$ and $\chi_0^{-1}\lambda_{\alpha,\mathfrak{k}}$.

(2): If $\delta(\Xi) = \{0\}$, then the equivalence between $\lambda_{\alpha+\delta,\mathfrak{k},\Xi}$ and $\chi_0^{-1}\lambda_{\alpha,\mathfrak{k},\Xi}$ given by (2.3) is the identity map.

(3): Since δ' is a character on $\mathfrak{g}$ and $d\lambda_{\alpha+\delta,\mathfrak{k},\Xi} = d\lambda_{\alpha+\delta',\mathfrak{k},\Xi}$, (3) follows from (2).

III. Lemmas on exponential Lie algebras.

Here we collect the results we need about the Lie algebra of a real exponential Lie group, i.e. a real solvable Lie algebra $\mathfrak{g}$ for which $\mathrm{ad}X$, $X \in \mathfrak{g}$, has no non-zero purely imaginary eigenvalues; cf. I.2 of [Be]. The center of $\mathfrak{g}$ is denoted $\mathfrak{z}$.

3.1. LEMMA. *Assume that* $\mathfrak{g} \neq \mathfrak{z}$ *and let* $\mathfrak{a}$ *be minimal among the non-central ideals of* $\mathfrak{g}$. *Put* $\mathfrak{i} := \{X \in \mathfrak{g} \mid [X, \mathfrak{a}] \subseteq \mathfrak{a} \cap \mathfrak{z}\}$. *Then*:

(i) $\mathfrak{a}$ *is abelian and* $1 \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \leqq 2$.

(ii) $\mathfrak{i}$ *is an ideal of* $\mathfrak{g}$, $\dim(\mathfrak{g}/\mathfrak{i}) \leqq 1$ *and for each* $X \in \mathfrak{g}\backslash\mathfrak{i}$, $\operatorname{ad}X$ *acts irreducibly on* $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. *In particular* $\mathfrak{i}$ *contains the nilradical of* $\mathfrak{g}$.

(iii) *If* $\mathfrak{i} = \mathfrak{g}$, *then* $\dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 1$.

PROOF. (i), first part of (ii): Follows by Lie's theorem, cf. Lemme 1.1 and 1.2 of Chap. VI of [Be]. To prove the last statement in (ii) let X be in the nilradical of $\mathfrak{g}$. Then $\operatorname{ad}X$ acts nilpotently on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. Hence $\operatorname{ad}X$ acts reducibly on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$ unless $\dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 1$ in which case it acts as zero. So $X \notin \mathfrak{g}\backslash\mathfrak{i}$ and therefore $X \in \mathfrak{i}$.

(iii) If $\mathfrak{i} = \mathfrak{g}$, then any subspace of $\mathfrak{a}$ containing $\mathfrak{a} \cap \mathfrak{z}$ properly is a non-central ideal of $\mathfrak{g}$. The minimality of $\mathfrak{a}$ therefore implies that $\dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 1$.

3.2. LEMMA. *Assume that* $\mathfrak{g} \neq \mathfrak{z}$ *and let* $\mathfrak{a}$ *and* $\mathfrak{i}$ *be as in Lemma* 3.1. *Let* $\alpha \in \mathfrak{g}^*$ *and assume that* $\ker\alpha$ *does not contain any non-zero ideal of* $\mathfrak{g}$. *Then*:

(i) $\dim\mathfrak{z} \leqq 1$ *and* $1 \leqq \dim(\mathfrak{g}/\mathfrak{a}^\alpha) \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \leqq 2$.

(ii) $\mathfrak{i} = \mathfrak{a}^\alpha \Leftrightarrow \mathfrak{a} \cap \mathfrak{z} = \{0\} \Rightarrow \dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1 \Leftrightarrow \dim\mathfrak{a} \leqq 2$.

(iii) *If* $\mathfrak{i} \neq \mathfrak{a}^\alpha$, *then the bilinear form* $\alpha([\cdot,\cdot])$ *on* $\mathfrak{i} \times \mathfrak{a}$ *factorizes to a non-degenerate form on* $\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha \times \mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. *In particular then* $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z})$.

(iv) $\mathfrak{a}^\alpha$ *is an ideal of* $\mathfrak{g}$ *if and only if* $[\mathfrak{a}^\alpha, \mathfrak{a}] = \{0\}$, *in which case* $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$.

(v) *If* $\mathfrak{g}_1$ *is a subalgebra of* $\mathfrak{g}$ *with* $\mathfrak{g}_1 \supseteq \mathfrak{a}^\alpha$, *then either* $\mathfrak{g}_1 = \mathfrak{g}$ *or* $\mathfrak{g}_1 = \mathfrak{a}^\alpha$.

PROOF. (i): By the assumption on $\ker\alpha$ we have $\mathfrak{z} \cap \ker\alpha = \{0\}$, so $\dim\mathfrak{z} \leqq 1$. Since $\mathfrak{a}$ is a non-central ideal of $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{a}]$ is a non-zero ideal of $\mathfrak{g}$, so $[\mathfrak{g}, \mathfrak{a}] \nsubseteq \ker\alpha$, i.e. $\mathfrak{g} \neq \mathfrak{a}^\alpha$. Hence $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) \geqq 1$. The bilinear form $\alpha([\cdot,\cdot])$ on $\mathfrak{g} \times \mathfrak{a}$ factorizes to a non-degerate form on $\mathfrak{g}/\mathfrak{a}^\alpha \times \mathfrak{a}/\mathfrak{a} \cap \mathfrak{g}^\alpha$. Hence $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{g}^\alpha) \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \leqq 2$, where the first inequality is due to $\mathfrak{z} \subseteq \mathfrak{g}^\alpha$.

(ii): Assume $\mathfrak{i} = \mathfrak{a}^\alpha$. Then $\mathfrak{g} \neq \mathfrak{i}$, so that $[\mathfrak{g}, \mathfrak{a}] \nsubseteq \mathfrak{z}$. Thus $[\mathfrak{g}, \mathfrak{a}]$ is a non-central ideal of $\mathfrak{g}$ contained in $\mathfrak{a}$, whence $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$ by the minimality property of $\mathfrak{a}$. Since $\mathfrak{i} = \mathfrak{a}^\alpha$, we also have $[\mathfrak{i}, \mathfrak{a}] \subseteq \mathfrak{z} \cap \ker\alpha = \{0\}$. So given $X \in \mathfrak{g}\backslash\mathfrak{i}$ we have that $[X, \mathfrak{a}] = [\mathsf{R}X + \mathfrak{i}, \mathfrak{a}] = [\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$, whence $\dim\mathfrak{a} = \dim[X, \mathfrak{a}] \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z})$. This implies that $\mathfrak{a} \cap \mathfrak{z} = \{0\}$ and $\dim\mathfrak{a} \leqq 2$.

Assume $\mathfrak{a} \cap \mathfrak{z} = \{0\}$. Then $[\mathfrak{i}, \mathfrak{a}] = \{0\}$ and thus $\mathfrak{i} \subseteq \mathfrak{a}^\alpha$. Since $\dim(\mathfrak{g}/\mathfrak{i}) \leqq 1 \leqq \dim(\mathfrak{g}/\mathfrak{a}^\alpha)$, this implies that $\mathfrak{i} = \mathfrak{a}^\alpha$ and $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$.

Assume $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) \neq 1$. Then, as just proved, $\mathfrak{a} \cap \mathfrak{z} \neq \{0\}$. Since $\dim\mathfrak{z} \leqq 1$, it follows that $\dim(\mathfrak{a} \cap \mathfrak{z}) = 1$. Also, by (i), $1 < \dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 2$. Hence $\dim\mathfrak{a} = 3 > 2$.

To prove the remaining implication of (ii), i.e. that $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$ implies $\dim\mathfrak{a} \leqq 2$, we use (iii), which is proved below. Assume $\dim\mathfrak{a} > 2$. Then by (i),

$\dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 2$ and $\dim(\mathfrak{a} \cap \mathfrak{z}) = 1$. Since $\mathfrak{i} = \mathfrak{a}^\alpha$ implies $\mathfrak{a} \cap \mathfrak{z} = \{0\}$, we also have $\mathfrak{i} \neq \mathfrak{a}^\alpha$. Hence by (iii), $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 2 \neq 1$.

(iii): The bilinear form $\alpha([\cdot,\cdot])$ on $\mathfrak{i} \times \mathfrak{a}$ factorizes to a non-degenerate form on $\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha \times \mathfrak{a}/\mathfrak{a}_0$, where $\mathfrak{a}_0 := \mathfrak{a} \cap \mathfrak{i}^\alpha$. Since $\mathfrak{z} \cap \ker\alpha = \{0\}$, we have $\mathfrak{a} \cap \mathfrak{i}^\alpha = \{Y \in \mathfrak{a} \mid [Y, \mathfrak{i}] = \{0\}\}$. Hence $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{i}^\alpha$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{a} \cap \mathfrak{z} \subseteq \mathfrak{a}_0 \subseteq \mathfrak{a}$, the minimality of $\mathfrak{a}$ therefore implies that either $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{z}$ or $\mathfrak{a}_0 = \mathfrak{a}$. If $\mathfrak{a}_0 = \mathfrak{a}$, then $\mathfrak{a} \subseteq \mathfrak{i}^\alpha$ and thus $\mathfrak{i} \subseteq \mathfrak{a}^\alpha$ so that $\mathfrak{i} = \mathfrak{a}^\alpha$ as seen above. Assuming $\mathfrak{i} \neq \mathfrak{a}^\alpha$, we therefore have $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{z}$. This proves the first part of (iii). Hence $\dim(\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \geqq \dim(\mathfrak{g}/\mathfrak{a}^\alpha) \geqq \dim(\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha)$, which finishes the proof of (iii).

(iv): If $\mathfrak{a}^\alpha$ is an ideal of $\mathfrak{g}$, then $[\mathfrak{a}^\alpha, \mathfrak{a}]$ is an ideal of $\mathfrak{g}$ contained in $\ker\alpha$, whence $[\mathfrak{a}^\alpha, \mathfrak{a}] = \{0\}$. Conversely, assume $[\mathfrak{a}^\alpha, \mathfrak{a}] = \{0\}$. Then $\mathfrak{a}^\alpha$ equals the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ which is an ideal of $\mathfrak{g}$. Moreover, $\mathfrak{a}^\alpha \subseteq \mathfrak{i}$. If $\mathfrak{a}^\alpha = \mathfrak{i}$, then $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$ by (ii). If $\mathfrak{a}^\alpha \neq \mathfrak{i}$, then we have by (iii) that $\mathfrak{g} = \mathfrak{i} + \mathfrak{a}^\alpha$ and so $\mathfrak{g} = \mathfrak{i}$. In this case $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) = 1$ by Lemma 3.1 (iii).

(v): The claim is trivial if $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$, so assume $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 2$. Then by (iv), $[\mathfrak{a}^\alpha, \mathfrak{a}] \neq \{0\}$ and thus $\mathfrak{a}^\alpha \nsubseteq \mathfrak{i}$. Hence there exists $V \in \mathfrak{a}^\alpha \backslash \mathfrak{i} \subseteq \mathfrak{g}_1 \backslash \mathfrak{i}$. Moreover $\mathfrak{a}^\alpha \neq \mathfrak{i}$, so (iii) applies. Since $[V, [\mathfrak{i}, \mathfrak{a}]] \subseteq [V, \mathfrak{a} \cap \mathfrak{z}] = \{0\} \subseteq \ker\alpha$, the action of $\operatorname{ad} V$ on $\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha$ equals minus the transpose w.r.t. $\alpha([\cdot,\cdot])$ of the action of $\operatorname{ad} V$ on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. Since the latter action is irreducible by Lemma 3.1 (ii), so is the former. Hence, since $(\mathfrak{g}_1 \cap \mathfrak{i})/(\mathfrak{i} \cap \mathfrak{a}^\alpha)$ is an $\operatorname{ad} V$-invariant subspace of $\mathfrak{i}/\mathfrak{i} \cap \mathfrak{a}^\alpha$, because $V \in \mathfrak{g}_1$ and $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$, we have that either $\mathfrak{g}_1 \cap \mathfrak{i} = \mathfrak{i} \cap \mathfrak{a}^\alpha$ or $\mathfrak{g}_1 \cap \mathfrak{i} = \mathfrak{i}$. If $\mathfrak{g}_1 \cap \mathfrak{i} = \mathfrak{i} \cap \mathfrak{a}^\alpha$, then $\mathfrak{g}_1 = \mathsf{R}V + (\mathfrak{g}_1 \cap \mathfrak{i}) = \mathsf{R}V + (\mathfrak{i} \cap \mathfrak{a}^\alpha) = \mathfrak{a}^\alpha$. If $\mathfrak{g}_1 \cap \mathfrak{i} = \mathfrak{i}$, then $\mathfrak{g}_1 \supseteq \mathfrak{i} + \mathfrak{a}^\alpha$, implying, since $\mathfrak{g} = \mathfrak{i} + \mathfrak{a}^\alpha$ by (iii), that $\mathfrak{g}_1 = \mathfrak{g}$.

3.3. LEMMA. *Let $\alpha \in \mathfrak{g}^*$ and $\mathfrak{k} \in S(\alpha, \mathfrak{g})$. Then $\mathfrak{k} \in P(\alpha, \mathfrak{g})$ if and only if for every $X \in \mathfrak{g} \backslash \mathfrak{k}$ there exists $V \in \mathfrak{k}$ for which $[X, V] \in \mathfrak{k} \backslash \ker\alpha$.*

PROOF. Let $X \in \mathfrak{g} \backslash \mathfrak{k}$ and suppose $[X, \mathfrak{k}] \cap \mathfrak{k} \subseteq \ker\alpha$. Then there exists $\beta \in \mathfrak{g}^*$ such that $\beta = \alpha$ on $\mathfrak{k}$ and $\beta = 0$ on $[X, \mathfrak{k}]$. Hence $X \in \mathfrak{k}^\beta \backslash \mathfrak{k}$ and thus $\mathfrak{k} \neq \mathfrak{k}^\beta$, proving that $\mathfrak{k} \notin P(\alpha, \mathfrak{g})$.

Conversely, assume that for all $X \in \mathfrak{g} \backslash \mathfrak{k}$ there exists $V \in \mathfrak{k}$ such that $[X, V] \in \mathfrak{k} \backslash \ker\alpha$, and let $\beta \in \mathfrak{g}^*$ with $\beta = \alpha$ on $\mathfrak{k}$. As $\mathfrak{k} \backslash \ker\beta = \mathfrak{k} \backslash \ker\alpha$, it follows that $(\mathfrak{g} \backslash \mathfrak{k}) \cap \mathfrak{k}^\beta = \emptyset$, i.e. $\mathfrak{k}^\beta \subseteq \mathfrak{k}$. Thus $\mathfrak{k}^\beta = \mathfrak{k}$, since $\mathfrak{k} \subseteq \mathfrak{k}^\beta$, as $\beta([\mathfrak{k}, \mathfrak{k}]) = \alpha([\mathfrak{k}, \mathfrak{k}]) = \{0\}$. This proves that $\mathfrak{k} \in P(\alpha, \mathfrak{g})$.

3.4. LEMMA. *Let $\alpha \in \mathfrak{g}^*$ and $\mathfrak{k} \in P(\alpha, \mathfrak{g})$, and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$.*

(i) *If $\mathfrak{a}$ is minimal among the non-zero ideals of $\mathfrak{g}$, then $\mathfrak{a} \subseteq \mathfrak{k}$.*

(ii) *If $\mathfrak{a}$ is abelian, then $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{a}^\alpha + \mathfrak{a} \in P(\alpha, \mathfrak{g})$.*

PROOF. (i): By minimality of $\mathfrak{a}$, either $\mathfrak{a} \cap \mathfrak{z} = \mathfrak{a}$ or $\mathfrak{a} \cap \mathfrak{z} = \{0\}$. If $[\mathfrak{k}, \mathfrak{a}] = \{0\}$, then $\mathfrak{a} \subseteq \mathfrak{k}^\alpha = \mathfrak{k}$. So assume $[\mathfrak{k}, \mathfrak{a}] \neq \{0\}$. Then the case $\mathfrak{a} \cap \mathfrak{z} = \{0\}$ holds. It follows by Lemma 3.1 that $\operatorname{ad} \mathfrak{k}$ acts irreducibly on $\mathfrak{a}$. Hence the $\operatorname{ad} \mathfrak{k}$-invariant subspace $\mathfrak{a} \cap \mathfrak{k}$ of $\mathfrak{a}$ equals either $\{0\}$ or $\mathfrak{a}$. The case $\mathfrak{a} \cap \mathfrak{k} = \{0\}$ is ruled out by Lemma 3.3. Hence $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a}$, i.e. $\mathfrak{a} \subseteq \mathfrak{k}$.

(ii): Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}^\alpha$ and $\mathfrak{k}'$ are subalgebras of $\mathfrak{g}$. Since $\mathfrak{a}$ is abelian, $\mathfrak{a} \subseteq \mathfrak{a}^\alpha$. Let $\gamma \in \mathfrak{g}^*$ with $\gamma = \alpha$ on $\mathfrak{k}'$. We shall then prove that $(\mathfrak{k}')^\gamma = \mathfrak{k}'$.

Since $\gamma = \alpha$ on $\mathfrak{k} \cap \mathfrak{a}^\alpha$, there exists $\beta \in \mathfrak{g}^*$ such that $\beta = \alpha$ on $\mathfrak{k}$ and $\beta = \gamma$ on $\mathfrak{a}^\alpha$. It follows that $\alpha = \beta = \gamma$ on $\mathfrak{a}$. Thus $\mathfrak{a}^\alpha = \mathfrak{a}^\beta = \mathfrak{a}^\gamma$, since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Also $\mathfrak{k} = \mathfrak{k}^\beta$, since $\mathfrak{k} \in P(\alpha, \mathfrak{g})$ and $\beta = \alpha$ on $\mathfrak{k}$. Hence

$$\begin{aligned}(\mathfrak{k}')^\gamma &= (\mathfrak{k} \cap \mathfrak{a}^\alpha + \mathfrak{a})^\gamma = (\mathfrak{k} \cap \mathfrak{a}^\alpha)^\gamma \cap \mathfrak{a}^\gamma = (\mathfrak{k} \cap \mathfrak{a}^\alpha)^\gamma \cap \mathfrak{a}^\alpha = (\mathfrak{k} \cap \mathfrak{a}^\alpha)^\beta \cap \mathfrak{a}^\alpha \\ &= (\mathfrak{k} \cap \mathfrak{a}^\beta)^\beta \cap \mathfrak{a}^\alpha = (\mathfrak{k}^\beta \cap \mathfrak{a}^\beta)^\beta \cap \mathfrak{a}^\alpha = ((\mathfrak{k} + \mathfrak{a})^\beta)^\beta \cap \mathfrak{a}^\alpha \\ &= (\mathfrak{k} + \mathfrak{a} + \mathfrak{g}^\beta) \cap \mathfrak{a}^\alpha = (\mathfrak{k} + \mathfrak{a}) \cap \mathfrak{a}^\alpha = (\mathfrak{k} \cap \mathfrak{a}^\alpha) + \mathfrak{a} = \mathfrak{k}'.\end{aligned}$$

3.5. Lemma. *Let* $\alpha \in \mathfrak{g}^*$ *and* $\mathfrak{k} \in P(\alpha, \mathfrak{g})$. *Assume* $\mathfrak{g} \neq \mathfrak{z}$ *and let* $\mathfrak{a}$ *be minimal among the non-central ideals of* $\mathfrak{g}$. *Set* $\mathfrak{k}_0 = \mathfrak{k} \cap \alpha^\alpha$ *and* $\mathfrak{i} = \{X \in \mathfrak{g} \mid [X, \mathfrak{a}] \subseteq \mathfrak{a} \cap \mathfrak{z}\}$. *Then:*

(i) $\dim(\mathfrak{k}/\mathfrak{k}_0) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k})$.

(ii) $[\mathfrak{k} \cap \mathfrak{i}, \mathfrak{k} \cap \mathfrak{i}] \subseteq \mathfrak{k}_0 \cap \mathfrak{i}$.

(iii) *If* $\mathfrak{k} \nsubseteq \mathfrak{i}$, *then* $[\mathfrak{k}, \mathfrak{k}] + \mathfrak{k}_0 = \mathfrak{k}$.

(iv) *If* $\mathfrak{k} \nsubseteq \mathfrak{i}$ *and* $\mathfrak{k} \neq \mathfrak{k}_0$, *then* $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{z}$.

(v) *The bilinear form* $\alpha([\cdot, \cdot])$ *on* $\mathfrak{k} \cap \mathfrak{i} \times \mathfrak{a}$ *factorizes to a non-degenerate form on* $\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i} \times \mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}$. *In particular* $\dim(\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i}) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}) = \dim(\mathfrak{k}/\mathfrak{k}_0)$.

Proof. (i): Since $\mathfrak{k} = \mathfrak{k}^\alpha$, the bilinear form $\alpha([\cdot, \cdot])$ on $\mathfrak{k} \times \mathfrak{a}$ factorizes to a non-degenerate form on $\mathfrak{k}/\mathfrak{k}_0 \times \mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}$.

(ii): Since $[[\mathfrak{i}, \mathfrak{i}], \mathfrak{a}] = \{0\}$, as $[\mathfrak{i}, \mathfrak{a}] \subseteq \mathfrak{z}$, we have $[\mathfrak{i}, \mathfrak{i}] \subseteq \mathfrak{a}^\alpha$. Hence $[\mathfrak{k} \cap \mathfrak{i}, \mathfrak{k} \cap \mathfrak{i}] \subseteq \mathfrak{k} \cap \mathfrak{a}^\alpha \cap \mathfrak{i} = \mathfrak{k}_0 \cap \mathfrak{i}$.

(iv): We have $\mathfrak{z} \subseteq \mathfrak{k}^\alpha = \mathfrak{k}$. Assume that $\mathfrak{k} \neq \mathfrak{k}_0$ and $\mathfrak{a} \cap \mathfrak{k} \neq \mathfrak{a} \cap \mathfrak{z}$. The ad $\mathfrak{k}$-invariant subspace $\mathfrak{a} \cap \mathfrak{k}/\mathfrak{a} \cap \mathfrak{z}$ of $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$ is then by (i) non-trivial. So by Lemma 3.1 ad $\mathfrak{k}$ acts trivially on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$, whence $\mathfrak{k} \subseteq \mathfrak{i}$.

(v): The proof of (i) covers the case in which $\mathfrak{k} \subseteq \mathfrak{i}$ and makes the claim trivial if $\mathfrak{k} = \mathfrak{k}_0$, so we may assume $\mathfrak{k} \neq \mathfrak{k}_0$ and $\mathfrak{k} \nsubseteq \mathfrak{i}$. Then by (iv), $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{z}$. Clearly $(\mathfrak{k} \cap \mathfrak{i}) \cap \mathfrak{a}^\alpha = \mathfrak{k}_0 \cap \mathfrak{i}$, so it remains to be proven that $\mathfrak{a} \cap (\mathfrak{k} \cap \mathfrak{i})^\alpha = \mathfrak{a} \cap \mathfrak{k}$. Let $A \in \mathfrak{a} \backslash \mathfrak{k}$. Then by Lemma 3.3 there exists $V \in \mathfrak{k}$ such that $[A, V] \in \mathfrak{k} \backslash \ker \alpha$. In particular $[A, V] \in \mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{z}$, so ad V does not act irreducibly on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. Hence $V \in \mathfrak{i} \cap \mathfrak{k}$ and thus $A \notin (\mathfrak{k} \cap \mathfrak{i})^\alpha$. This proves that $\mathfrak{a} \cap (\mathfrak{k} \cap \mathfrak{i})^\alpha \subseteq \mathfrak{a} \cap \mathfrak{k}$, whence $\mathfrak{a} \cap (\mathfrak{k} \cap \mathfrak{i})^\alpha = \mathfrak{a} \cap \mathfrak{k}$.

(iii): Assume $\mathfrak{k} \nsubseteq \mathfrak{i}$ and $\mathfrak{k} \neq \mathfrak{k}_0$. Then by (iv), $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{z}$. By (v), $\dim(\mathfrak{k}/\mathfrak{k}_0) = \dim(\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i})$, so $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{i} + \mathfrak{k}_0$. Hence also $\mathfrak{k}_0 \nsubseteq \mathfrak{i}$. Let $V \in \mathfrak{k}_0 \backslash \mathfrak{i}$. Then ad V acts irreducibly on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}$. Since $[V, [\mathfrak{k} \cap \mathfrak{i}, \mathfrak{a}]] \subseteq [V, \mathfrak{z}] = \{0\} \subseteq \ker \alpha$, the action of ad V on $\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i}$ (defined since $V \in \mathfrak{k}_0$) equals minus the transpose w.r.t. $\alpha([\cdot, \cdot])$ of the action of ad V on $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}$. Hence ad V also acts irreducibly on $\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i}$. In particular $[V, \mathfrak{k} \cap \mathfrak{i}] + \mathfrak{k}_0 \cap \mathfrak{i} = \mathfrak{k} \cap \mathfrak{i}$, so as $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{i} + \mathfrak{k}_0$, we have $[V, \mathfrak{k} \cap \mathfrak{i}] + \mathfrak{k}_0 = \mathfrak{k}$. This proves (iii).

IV. The image of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ and irreducibility of the group representations.

This section contains the results of the paper. The first theorem is the main one. It gives a partial description of the derived image of the enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ under the induced representations $\Lambda_{\alpha,\mathfrak{k}}$ of G realized as representations on $\mathscr{D}'(\mathsf{R}^n)$ via coexponential bases. It is assumed that $\mathfrak{k}$ satisfies a Pukanszky condition relative to α. The result implies irreducibility in various senses of the associated induced representations of G realized as representations on subspaces of $\mathscr{D}'(\mathsf{R}^n)$.

To determine the image of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ it suffices to consider the restriction $\lambda_{\alpha,\mathfrak{k}}$ of $\Lambda_{\alpha,\mathfrak{k}}$ to the space $C^\infty_{\alpha,\mathfrak{k}}(G)$ of C^∞-functions in $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$.

4.1. THEOREM. *Let G denote a real exponential Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha \in (\mathfrak{g}^*)^{\mathbb{C}}$ be of the form $\alpha = c\beta + \delta$, where $c \in \mathbb{C}\backslash\{0\}$, $\beta \in \mathfrak{g}^*$ and $\delta([\mathfrak{g}, \mathfrak{g}]) = \{0\}$, and let $\mathfrak{k} \in P(\beta, \mathfrak{g})$. Let $\mathfrak{n}$ denote a nilpotent ideal of $\mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$ and let $(X_{k+1}, \ldots, X_n)$ be a normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$, where $k = \dim \mathfrak{g}/(\mathfrak{n} + \mathfrak{k})$ and $n = \dim \mathfrak{g}/\mathfrak{k}$.*

(1) *If $k = 0$, set $\Xi := (X_1, \ldots, X_n)$.* Then the image $d\lambda_{\alpha,\mathfrak{k},\Xi}(\mathfrak{U}(\mathfrak{g}^{\mathbb{C}}))$ *equals the algebra* $\mathrm{DP}(\mathsf{R}^n)$ *of all differential operators on* R^n *with polynomial coefficients.*

(2) *If $k > 0$, there exists a basis $(X_1, \ldots, X_k)$ of $\mathfrak{g}$ modulo $\mathfrak{n} + \mathfrak{k}$ such that if $\Xi := (X_1, \ldots, X_k, X_{k+1}, \ldots, X_n)$, then the image $d\lambda_{\alpha,\mathfrak{k},\Xi}(\mathfrak{U}(\mathfrak{g}^{\mathbb{C}}))$ contains the multiplication by either the function x_1 or the function e^{x_1}.*

REMARKS. (i) If G is nilpotent, we may choose $\mathfrak{n} = \mathfrak{g}$, and then case (1) applies. Moreover, with $c = i$ and $\delta = 0$ the restriction of $\Lambda_{\alpha,\mathfrak{k},\Xi}$ to $L^2(\mathsf{R}^n)$ is a realization of the unitary representation $\pi_{\beta,\mathfrak{k}}$. Hence Theorem 4.1 is an extension of Kirillov's result mentioned in the introduction, [Ki; Theorem 7.1] (modulo [Ki; Theorem 5.2]). Actually it extends the more explicit version [CGP; Theorem 3.1], which says that $d\pi_{\beta,\mathfrak{k},\Xi}(\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})) = \mathrm{DP}(\mathsf{R}^n)$ for each $\mathfrak{k} \in P(\beta, \mathfrak{g})$ and each normal basis Ξ. (In [Ki] this is concluded, for given β, merely for some $\mathfrak{k}$ and some Ξ).

(ii) Theorem 4.1 is for nilpotent G not as general as [Ja; Theorem 3.2], which is a version of Kirillov's theorem for arbitrary complex-valued α, but it contains [JS1; Theorem 4.1] where α is proportional to a real-valued functional.

The proof of Theorem 4.1 is given in section V below. Here we derive some consequences of it. First we show that the image of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ is dense in the algebra $\mathrm{DO}(\mathsf{R}^n)$ of all differential operators on R^n with C^∞-coefficients. The density will in particular be with respect to pointwise convergence of operators on $C^\infty(\mathsf{R}^n)$. For our application, however, we need to consider convergence in a more controlled manner: For each subspace $\mathscr{A}$ of $\mathrm{DO}(\mathsf{R}^n)$, $\tilde{\mathscr{A}}$ will denote the subspace of $\mathrm{DO}(\mathsf{R}^n)$ spanned by the operators $M_\psi D$, where $D \in \mathscr{A}$ and M_ψ is multiplication by a function $\psi \in C^\infty(\mathsf{R}^n)$ which is the limit in $C^\infty(\mathsf{R}^n)$ of a sequence $\{\psi_i\}_{i=1}^\infty$ such that $M_{\psi_i} \in \mathscr{A}$ and $M_{\psi_i} D \in \mathscr{A}$ for all $i \in \mathsf{N}$. For each $m \in \mathsf{N}$, $\overline{\mathscr{A}}^{(m)}$ will denote the result of applying this "closure" operation m times to $\mathscr{A}$.

4.2. THEOREM. *Let G, $\mathfrak{g}$, α, $\mathfrak{k}$, $\mathfrak{n}$ and k, n be as in Theorem 4.1. Let Ξ denote a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$. Then*

$$\overline{d\lambda_{\alpha,\mathfrak{k},\Xi}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))}^{(k+1)} = \mathrm{DO}(\mathsf{R}^n).$$

PROOF. Put $\lambda = \lambda_{\alpha,\mathfrak{k},\Xi}$. The conclusion of the theorem is by Lemma 2.2 easily seen to be independent of the choice of coexponential basis Ξ. If $k = 0$, we have by (1) of Theorem 4.1 that $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ equals $\mathrm{DP}(\mathsf{R}^n)$ for a suitable choice of Ξ. In this case the conclusion holds because the polynomials are dense in $C^\infty(\mathsf{R}^n)$.

Suppose $k > 0$ and let the coexponential basis $\Xi = (X_1, \ldots, X_n)$ of $\mathfrak{g}$ modulo $\mathfrak{k}$ be chosen in accordance with (2) of Theorem 4.1. Since both x and e^x generate a dense subalgebra of $C^\infty(\mathsf{R})$, it follows that the "closure" $\overline{d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))}$ contains all the operators of the form $M_\psi D$, where $\psi \in C^\infty(\mathsf{R}^n)$ is a function of x_1 alone and where $D \in d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$. Note also that $\partial/\partial x_1 = -d\lambda(X_1) \in d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$.

Set $\mathfrak{g}_0 = \operatorname{span}\{X_2, \ldots, X_n\} + \mathfrak{k}$, $\alpha_0 = \alpha|_{\mathfrak{g}_0}$, $\Xi_0 = (X_2, \ldots, X_n)$ and $\lambda_0 = \lambda_{\alpha_0,\mathfrak{k},\Xi_0}$. Then λ_0 is a representation of the subgroup $G_0 = \exp(\mathfrak{g}_0)$ of G. Since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ we have for each $D \in \mathfrak{U}(\mathfrak{g}_0^{\mathsf{C}})$ that

$$[d\lambda(D)\varphi](x_1, \tilde{x}) = [d\lambda_0(e^{-x_1 \operatorname{ad} X_1} D)\varphi(x_1, .)](\tilde{x})$$

for all $\varphi \in C^\infty(\mathsf{R}^n)$, $x_1 \in \mathsf{R}$ and $\tilde{x} = (x_2, \ldots, x_n) \in \mathsf{R}^{n-1}$. For every $D \in \mathfrak{U}(\mathfrak{g}_0^{\mathsf{C}})$ there exist finitely many $\psi_\gamma \in C^\infty(\mathsf{R})$ and $D_\gamma \in \mathfrak{U}(\mathfrak{g}_0^{\mathsf{C}})$ such that

$$e^{x_1 \operatorname{ad} X_1} D = \sum_\gamma \psi_\gamma(x_1) D_\gamma, \qquad x_1 \in \mathsf{R},$$

and so

$$1_{\mathsf{R}} \otimes d\lambda_0(D) = \sum_\gamma \psi_\gamma(x_1)\, d\lambda(D_\gamma) \in \overline{d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))}.$$

From this it readily follows that

$$\mathrm{DO}(\mathsf{R}_{x_1}) \otimes d\lambda_0(\mathfrak{U}(\mathfrak{g}_0^{\mathsf{C}})) \subseteq \overline{d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))}.$$

If $k - 1 > 0$ we may repeat the argument with $\mathfrak{g}$ replaced by $\mathfrak{g}_0$. Carrying out the argument k times in all we conclude that

$$\mathrm{DO}(\mathsf{R}_{x_1}) \otimes \cdots \otimes \mathrm{DO}(\mathsf{R}_{x_k}) \otimes \mathrm{DP}(\mathsf{R}^{n-k}) \subseteq \overline{d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))}^{(k)} \subseteq \mathrm{DO}(\mathsf{R}^n).$$

From this the conclusion of the theorem follows.

As an application of this density theorem we extend to exponential Lie groups the irreducibility result proved for nilpotent Lie groups in [JS2; Theorem VI.1].

The class of functionals α covered here in the exponential case, however, is somewhat smaller than in the nilpotent case, where arbitrary complex-valued α is handled. Depending on the choice of representation space we establish irreducibility in the weak sense of scalar irreducibility or in the strong sense of ultra-irreducibility, cf. the definitions in the introduction.

Ultra-irreducibility implies topological complete irreducibility, and hence again topological irreducibility and operator irreducibility (the latter meaning that the only densely defined and closed intertwining operators are the constant multiples of the identity). For the unitary representations all the considered notions of irreducibility coincide.

For the exponential groups of dimension $\leqq 3$ irreducibility results for the representations on the spaces $C^{\infty}_{\alpha,\mathfrak{k}}(G)$ and $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ are found in [St; Sec. III].

4.3. Corollary. *Let G denote a real exponential Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ be of the form $\alpha = c\beta + \delta$, where $c \in \mathbf{C}\backslash\{0\}$, $\beta \in \mathfrak{g}^*$ and $\delta([\mathfrak{g},\mathfrak{g}]) = \{0\}$, and let $\mathfrak{k} \in P(\beta, \mathfrak{g})$. Then the following hold:*

(1) *Any representation Λ_E induced from $\chi_{\alpha,\mathfrak{k}}$ on a normal subspce E of $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ is scalar irreducible.*

(2) *Let furthermore the topology of E be semi-complete and let E be stable under the natural multiplications by the functions from $C^{\infty}(G/K) \simeq C^{\infty}_{0,\mathfrak{k}}(G)$ on $\mathscr{D}'_{\alpha,\mathfrak{k}}(G)$ with the corresponding bilinear map $(\psi, u) \mapsto \psi u$ of $C^{\infty}(G/K) \times E$ into E being separately continuous. Then the representation Λ_E is ultra-irreducible.*

Examples. Part (1) in particular entails the well-known irreducibility of the unitary representation $\pi_{\beta,\mathfrak{k}}$ induced from $\chi_{i\beta,\mathfrak{k}}$ with $\mathfrak{k} \in P(\beta, \mathfrak{g})$. Indeed, if $c = i$ and $\delta|_{\mathfrak{k}} = \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{k}}$, then in a suitable realization $\mathscr{D}'_{\alpha,\mathfrak{k}}(G) \simeq \mathscr{D}'(\mathbf{R}^n)$, $\pi_{\beta,\mathfrak{k}}$ is the restriction of $\Lambda_{\alpha,\mathfrak{k}}$ to $L^2(\mathbf{R}^n)$, cf. sec. II.4.

Part (2) applies to the local spaces of distributions in $\mathscr{D}'_{\alpha,\mathfrak{k}}(G) \simeq \mathscr{D}'(\mathbf{R}^n)$. The representation $\Lambda_{\alpha,\mathfrak{k}}$ realized on $\mathscr{D}'(\mathbf{R}^n)$ restricts to ultra-irreducible representations on e.g. the spaces $C^r_0(\mathbf{R}^n)$ and $C^r(\mathbf{R}^n)$ for $0 \leqq r \leqq \infty$, $\mathscr{D}'(\mathbf{R}^n)$ and $\mathscr{E}'(\mathbf{R}^n)$, and $L^p_{\text{loc}}(\mathbf{R}^n)$ and $L^p_c(\mathbf{R}^n)$ for $1 \leqq p < \infty$.

Proof of Corollary. Choose a realization $\tilde{\Lambda} := \Lambda_{\alpha,\mathfrak{k},\Xi}$ of $\Lambda_{\alpha,\mathfrak{k}}$ corresponding to a coexponential basis Ξ of $\mathfrak{g}$ modulo $\mathfrak{k}$ and identify E with its associated image in $\mathscr{D}'(\mathbf{R}^n)$. Then $\mathscr{D}(\mathbf{R}^n) \subseteq E \subseteq \mathscr{D}'(\mathbf{R}^n)$ with weakly continuous inclusions.

(1): Let $A\colon E \to E$ be a continuous linear operator commuting with $\tilde{\Lambda}$. It must be proved that A is a scalar multiple of the identity on E. The restriction A_0 of A to $\mathscr{D}(\mathbf{R}^n)$ is a continuous operator into $\mathscr{D}'(\mathbf{R}^n)$. Since $\mathscr{D}(\mathbf{R}^n)$ is dense in E, it suffices to prove that A_0 is a scalar multiple of the inclusion map $i\colon \mathscr{D}(\mathbf{R}^n) \to \mathscr{D}'(\mathbf{R}^n)$. If A_0 commutes with every operator in a subspace $\mathscr{A}$ of $\mathrm{DO}(\mathbf{R}^n)$, then A_0 commutes with every operator in $\bar{\mathscr{A}}$. Since A_0 intertwines the actions of $\tilde{\Lambda}$, Theorem 4.2 therefore implies that A_0 commutes with every operator in $\mathrm{DO}(\mathsf{R}^n)$. Hence A_0 is a scalar multiple of the inclusion map.

(2): Let now the further assumptions in (2) be in force. Then we may form the integrated representation $\tilde{\Lambda}_E\colon \mathscr{D}(G) \to B(E)$ corresponding to $\tilde{\Lambda}_E\colon G \to S(E)$. Here for any given topological vector spaces E and F we let $L(E, F)$, $S(E, F)$ and $B(E, F)$ denote the spaces consisting of those linear operators from E into F which are continuous, weakly continuous and maps bounded subsets of E into bounded subsets of F, respectively. For each $\varphi \in \mathscr{D}(G)$ we then have $\tilde{\Lambda}(\varphi) \in L(\mathscr{D}'(\mathsf{R}^n), C^\infty(\mathsf{R}^n))$ with $\tilde{\Lambda}(\varphi)|_E = \tilde{\Lambda}_E(\varphi) \in B(E)$.

The representation $\tilde{\Lambda}_E$ is ultra-irreducible, according to Theorem IV.2 of [JS2], if the following two conditions hold:

(i) The closure of $L(\mathscr{D}'(\mathsf{R}^n), \mathscr{D}(\mathsf{R}^n))|_E$ in $S(E)$ w.r.t. the ultraweak topology contains the identity operator I_E on E.

(ii) The closure $\mathscr{F}$ of $\tilde{\Lambda}(\mathscr{D}(G))|_E = \tilde{\Lambda}_E(\mathscr{D}(G))$ in $B(E)$ w.r.t. the ultraweak topology contains the operators $M_\psi \tilde{\Lambda}(\varphi)|_E \in L(\mathscr{D}'(\mathsf{R}^n), \mathscr{D}(\mathsf{R}^n))|_E$ for all $\psi \in \mathscr{D}(\mathsf{R}^n)$ and $\varphi \in \mathscr{D}(G)$.

Condition (i) is satisfied as an immediate consequence of the assumptions on E, cf. the proof of Corollary IV.3 of [JS2]. Condition (ii) is verified by means of Theorem 4.2 above: Fix $\varphi \in \mathscr{D}(G)$. Then for each $D \in \mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ we have

$$d\tilde{\Lambda}(D)\tilde{\Lambda}_E(\varphi) = d\tilde{\Lambda}(D)\tilde{\Lambda}(\varphi)|_E = \tilde{\Lambda}(d\Lambda(D)\varphi)|_E \in \tilde{\Lambda}(\mathscr{D}(G))|_E \subseteq \mathscr{F}.$$

By Theorem 4.2 it thus suffices to prove the following: If $\mathscr{A}$ is a subspce of $\mathrm{DO}(\mathsf{R}^n)$ such that $A\tilde{\Lambda}_E(\varphi) \in \mathscr{F}$ for all $A \in \mathscr{A}$, then $B\tilde{\Lambda}_E(\varphi) \in \mathscr{F}$ for all $B \in \bar{\mathscr{A}}$.

Now, such B is a sum of operators of the form $M_\psi A$, where $A \in \mathscr{A}$ and where $\psi \in C^\infty(\mathsf{R}^n)$ is a limit of $C^\infty(\mathsf{R}^n)$ of a sequence $\{\psi_i\}_{i=1}^\infty$ such that $M_{\psi_i} A \in \mathscr{A}$ for all $i \in \mathsf{N}$. We shall prove that $M_\psi A \tilde{\Lambda}_E(\varphi) \in \mathscr{F}$.

Since $M_{\psi_i} A \in \mathscr{A}$, we have $M_{\psi_i} A\tilde{\Lambda}_E(\varphi) \in \mathscr{F}$ for all $i \in \mathsf{N}$, so it suffices to prove that $M_{\psi_i} A\tilde{\Lambda}_E(\varphi) \to M_\psi A\tilde{\Lambda}_E(\varphi)$ ultraweakly in $B(E)$ as $i \to \infty$. But $M_{\psi_i} \to M_\psi$ ultraweakly in $L(E)$ as $i \to \infty$ (cf. Lemma II.2 in [JS2]), $A\tilde{\Lambda}_E(\varphi)$ belongs to $\mathscr{F} \subseteq B(E)$ and the map $C \mapsto CD$ is for each $D \in B(E)$ continuous from $L(E)$ into $B(E)$ w.r.t. the ultraweak topologies. Hence this is indeed the case.

V. Proof of Theorem 4.1.

For a given G it follows by (3) and (4) of Lemma 2.2 that if the conclusion of the theorem holds for one choice of normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$, then it holds for any other. In particular, since such a basis may be chosen compatible with $[\mathfrak{g}, \mathfrak{g}]$, we have by Lemma 2.3 (3) that the conclusion of the theorem holds for any character δ, if it holds for δ,= 0. Hence we may assume $\delta = 0$.

The proof is by induction on dim G. If dim $G = 1$, then $\mathfrak{g} = \mathfrak{k}$. In this case $\lambda_{\alpha, \mathfrak{k}}$ is one-dimensional and the conclusion of (1) thus trivially satisfied. Therefore assume that $\dim G \geqq 2$ and that the theorem holds for all the groups of dimension $< \dim G$.

If $\ker \alpha$ contains a non-zero ideal $\mathfrak{i}$ of $\mathfrak{g}$, then the result follows by an application of the induction hypothesis to the quotient group $G/\exp(\mathfrak{i})$.

Therefore, from now on we assume that $\ker \alpha$ does not contain any non-zero ideal of $\mathfrak{g}$. Since α is of the form $\alpha = c\beta$, where $c \in \mathbf{C}\backslash\{0\}$ and $\beta \in \mathfrak{g}^*$, this implies that the center $\mathfrak{z}$ of $\mathfrak{g}$ is of $\dim \mathfrak{z} \leqq 1$.

Let $\mathfrak{a}$ denote a minimal non-central ideal of $\mathfrak{g}$ and set $\mathfrak{i} = \{X \in \mathfrak{g} \mid [X, \mathfrak{a}] \subseteq \mathfrak{a} \cap \mathfrak{z}\}$.

Assume first that $\mathfrak{a} \cap \mathfrak{z} = \{0\}$. Then $\mathfrak{a}$ is minimal among the non-zero ideals of $\mathfrak{g}$, so by Lemma 3.4 (i), $\mathfrak{a} \subseteq \mathfrak{k} = \mathfrak{k}^\alpha \subseteq \mathfrak{a}^\alpha$. By (ii) and (iv) of Lemma 3.2 we get $\mathfrak{i} = \mathfrak{a}^\alpha$, $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$ and $[\mathfrak{a}, \mathfrak{a}^\alpha] = \{0\}$. By Lemma 3.1, $\mathfrak{n} \subseteq \mathfrak{i}$, so also $\mathfrak{n} \subseteq \mathfrak{a}^\alpha$. Hence $\mathfrak{n} + \mathfrak{k} \subseteq \mathfrak{a}^\alpha$, in particular $k > 0$.

Moreover, $\dim \mathfrak{a} = 1$ or 2. In any case there exist $Y_1, Y_2 \in \mathfrak{a}$ and $X_1 \in \mathfrak{g}\backslash\mathfrak{a}^\alpha$ such that $\mathfrak{a} = \operatorname{span}\{Y_1, Y_2\}$ and $[X_1, Y_1 \pm iY_2] = (-\frac{1}{2} \pm ib)(Y_1 \pm iY_2)$ for some $b \in \mathbf{R}$ (where $b = 0$ if $\dim \mathfrak{a} = 1$ and $b \neq 0$ if $\dim \mathfrak{a} = 2$).

Let $(X_2, \ldots, X_k)$ denote a basis of $\mathfrak{a}^\alpha$ modulo $\mathfrak{n} + \mathfrak{k}$. Then $(X_1, X_2, \ldots, X_k)$ is a basis of $\mathfrak{g}$ modulo $\mathfrak{n} + \mathfrak{k}$. Let $(X_{k+1}, \ldots, X_n)$ be a normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$ and set $\Xi = (X_1, \ldots, X_n)$. Then, since $Y_1, Y_2 \in \mathfrak{k}$ are central in $\mathfrak{a}^\alpha$, we find that

$$d\lambda_{\alpha, \mathfrak{k}, \Xi}(Y_1 \pm iY_2) = -\alpha(e^{-x_1 \operatorname{ad} X_1}(Y_1 \pm iY_2)) = -\alpha(Y_1 \pm iY_2)e^{-(-\frac{1}{2} \pm ib)x_1}.$$

Here $\alpha(Y_1 \pm iY_2) \neq 0$, because $\operatorname{span}\{Y_1, Y_2\} = \mathfrak{a} \nsubseteq \ker \alpha$ and $\alpha = c\beta \in \mathbf{C}\mathfrak{g}^*$. Since $e^{x_1} = e^{(\frac{1}{2}+ib)x_1}e^{(\frac{1}{2}-ib)x_1}$, this finishes the proof in the case $\mathfrak{a} \cap \mathfrak{z} = \{0\}$.

Assume from now on that $\mathfrak{a} \cap \mathfrak{z} \neq \{0\}$. Then $\dim \mathfrak{z} = 1$, so that $\mathfrak{a} \cap \mathfrak{z} = \mathfrak{z} = \mathbf{R}Z$ with $\alpha(Z) \neq 0$.

The proof is now divided according to whether (I): $\mathfrak{k} \subseteq \mathfrak{a}^\alpha$ or (II): $\mathfrak{k} \nsubseteq \mathfrak{a}^\alpha$.

(I) Assume $\mathfrak{k} \subseteq \mathfrak{a}^\alpha$. Then $\mathfrak{a} \subseteq \mathfrak{k}^\alpha = \mathfrak{k}$. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$, $\mathfrak{n} + \mathfrak{a}^\alpha$ is an ideal of $\mathfrak{g}$, so by Lemma 3.2 (v) either (a): $\mathfrak{n} \subseteq \mathfrak{a}^\alpha$ or (b): $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}^\alpha$.

(a) Assume $\mathfrak{n} \subseteq \mathfrak{a}^\alpha$. Then in particular $k > 0$. Moreover, then $\mathfrak{a}^\alpha$ is an ideal of $\mathfrak{g}$, so by Lemma 3.2 (iv), $\dim(\mathfrak{g}/\mathfrak{a}^\alpha) = 1$. Hence $\dim \mathfrak{a} = 2$, say $\mathfrak{a} = \operatorname{span}\{Y, Z\}$, where $\alpha(Y) = 0$. By Lemma 3.2 (ii), $\mathfrak{i} \neq \mathfrak{a}^\alpha$, so by Lemma 3.2 (iii), $\mathfrak{g} = \mathfrak{i} + \mathfrak{a}^\alpha$.

Let $X_1 \in \mathfrak{i}\backslash\mathfrak{a}^\alpha$ be normalized so that $[X_1, Y] = Z$. Let $(X_2, \ldots, X_k)$ denote a basis of $\mathfrak{a}^\alpha$ modulo $\mathfrak{n} + \mathfrak{k}$. Then $(X_1, X_2, \ldots, X_k)$ is a basis of $\mathfrak{g}$ modulo $\mathfrak{n} + \mathfrak{k}$. Let $(X_{k+1}, \ldots, X_n)$ denote a normal basis of $\mathfrak{g}$ modulo $\mathfrak{n} \cap \mathfrak{k}$ and set $\Xi = (X_1, \ldots, X_n)$. Since $[X_1, Y] = Z \in \mathfrak{z} \subseteq \mathfrak{k}$ and $Y \in \mathfrak{a} \cap \ker \alpha \subseteq \mathfrak{k}$, where $\mathfrak{a} \cap \ker \alpha$ is an ideal of $\mathfrak{a}^\alpha$ (in the present case in fact $[\mathfrak{a}, \mathfrak{a}^\alpha] = \{0\}$), we find that

$$d\lambda_{\alpha, \mathfrak{k}, \Xi}(Y) = \alpha(e^{-x_n \operatorname{ad} X_n} \ldots e^{-x_1 \operatorname{ad} X_1}(-Y)) = \alpha(Z)x_1, \tag{5.1}$$

which concludes the proof in case (I) (a).

(b) Next, assume $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}^\alpha$. We shall here apply the induction hypothesis to the subgroup $\exp(\mathfrak{a}^\alpha)$ of G. Set $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{a}^\alpha$. Then $\mathfrak{n}_0$ is a nilpotent ideal of $\mathfrak{a}^\alpha$ containing $[\mathfrak{a}^\alpha, \mathfrak{a}^\alpha]$. Also, $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{a}^\alpha \cap \mathfrak{n} = \mathfrak{n}_0$ since $[\mathfrak{n}, \mathfrak{a}] \subseteq [\mathfrak{i}, \mathfrak{a}] \subseteq \mathfrak{z}$.

Set $p = \dim(\mathfrak{g}/\mathfrak{a}^\alpha) = \dim(\mathfrak{n}/\mathfrak{n}_0)$. By Lemma 3.2, $p = 1$ or $p = 2$.

Since $\mathfrak{g} = \mathfrak{a}^\alpha + \mathfrak{n}$, a basis of $\mathfrak{a}^\alpha$ modulo $\mathfrak{n}_0 + \mathfrak{k}$ is also a basis of $\mathfrak{g}$ modulo $\mathfrak{n} + \mathfrak{k}$. Let $(X_1, \dots, X_k)$ be such a basis.

We have $\dim \mathfrak{a} = \mathrm{p} + 1$, so $\mathfrak{a} = \operatorname{span}\{Z, Y_1, \dots, Y_p\}$, where $Y_1, \dots, Y_p$ is a basis of $\mathfrak{a} \cap \ker \alpha$. Since $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}^\alpha$ and $\mathfrak{n} \subseteq \mathfrak{i}$, it follows by (iii) of Lemma 3.2 that the real bilinear form $\langle \cdot , \cdot \rangle = \alpha(Z)^{-1}\alpha([\cdot , \cdot])$ on $\mathfrak{n} \times \mathfrak{a}$ factorizes to a non-degenerate form on $\mathfrak{n}/\mathfrak{n}_0 \times \mathfrak{a}/\mathbb{R}Z$. Also, $[X, Y] = \langle X, Y \rangle Z$ for all $X \in \mathfrak{n}$ and $Y \in \mathfrak{a}$.

Hence there exists a basis $(X_{k+1}, \dots, X_{k+p})$ of $\mathfrak{n}$ modulo $\mathfrak{n}_0$, normal since $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n}_0$, such that $[X_{k+i}, Y_j] = \delta_{ij} Z$ for all $i, j = 1, \dots, p$. Since $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}^\alpha$ this is by Lemma 2.1 (c) also a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{a}^\alpha$.

Finally, let $(X_{k+p+1}, \dots, X_n)$ denote a normal basis of $\mathfrak{n}_0$ modulo $\mathfrak{n}_0 \cap \mathfrak{k} = \mathfrak{n} \cap \mathfrak{k}$, so that $(X_{k+1}, \dots, X_n)$ is a normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$.

Then $\Xi := (X_1, \dots, X_n)$ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ compatible with $\mathfrak{n}$ and $\Xi_0 := (X_1, \dots, X_k, X_{k+p+1}, \dots, X_n)$ is a coexponential basis of $\mathfrak{a}^\alpha$ modulo $\mathfrak{k}$ compatible with $\mathfrak{n}_0$. Also $\Xi' := (X_{k+1}, \dots, X_{k+p}) \oplus \Xi_0$ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{k}$ since $(X_{k+1}, \dots, X_{k+p})$ is a coexponential basis of $\mathfrak{g}$ modulo $\mathfrak{a}^\alpha$ and Ξ_0 is a coexponential basis of $\mathfrak{a}^\alpha$ modulo $\mathfrak{k}$.

For notational convenience put $(W_1, \dots, W_n) := \Xi'$. Set $\lambda = \lambda_{\alpha, \mathfrak{k}, \Xi}$, $\lambda' = \lambda_{\alpha, \mathfrak{k}, \Xi'}$ and $\lambda_0 = \lambda_{\alpha_0, \mathfrak{k}, \Xi_0}$, where $\alpha_0 = \alpha|_{\mathfrak{a}^\alpha}$. Since $\operatorname{span}\{Y_1, \dots, Y_p\} = \mathfrak{a} \cap \ker \alpha$ is an ideal of $\mathfrak{a}^\alpha$, the relations $[W_i, Y_j] = [X_{k+i}, Y_j] = \delta_{ij} Z$, $i, j = 1, \dots, p$, then imply

$$d\lambda'(Y_i) = \alpha(Z) w_i, \qquad i = 1, \dots, p. \tag{5.2}$$

Moreover, we claim that

$$\left\{ \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_p} \right\} \cup (1_{\mathbb{R}^p} \otimes d\lambda_0(\mathfrak{U}((\mathfrak{a}^\alpha)^{\mathbb{C}})) \subseteq d\lambda'(\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})). \tag{5.3}$$

Indeed, let $\varphi \in C^\infty_{\alpha, \mathfrak{k}}(G)$ and put $\tilde{\varphi}(w_1, \dots, w_n) = \varphi(\exp(w_1 W_1) \dots \exp(w_n W_n))$. Then for every $V \in \mathfrak{g}$ and $i \in \{1, \dots, p + 1\}$ we have

$$[d\lambda(V)\tilde{\varphi}](w_1, \dots, w_n)$$

$$= \frac{d}{dt}\bigg|_{t=0} \varphi(\exp(-tV)\exp(w_1 W_1) \dots \exp(w_n W_n))$$

$$= \frac{d}{dt}\bigg|_{t=0} \varphi(\exp(w_1 W_1) \dots \exp(w_{i-1} W_{i-1}) \exp(e^{-w_{i-1}\operatorname{ad} W_{i-1}} \dots e^{-w_1 \operatorname{ad} W_1}(-tV))$$

$$\cdot \exp(w_i W_i) \dots \exp(w_n W_n)),$$

so that

$$[d\lambda(e^{w_1 \operatorname{ad} W_1} \dots e^{w_{i-1} \operatorname{ad} W_{i-1}} V)\tilde{\varphi}](w_1, \dots, w_n)$$

$$= \frac{d}{dt}\bigg|_{t=0} \varphi(\exp(w_1 W_1) \dots \exp(w_{i-1} W_{i-1}) \exp(-tV) \exp(w_i W_i) \dots \exp(w_n W_n)).$$

In particular for $i \in \{1, \dots, p\}$ and $V = W_i$ we get

$$\sum_{n_1=0}^{\infty} \dots \sum_{n_{i-1}=0}^{\infty} \frac{w_1^{n_1}}{n_1!} \dots \frac{w_{i-1}^{n_{i-1}}}{n_{i-1}!} d\lambda((\operatorname{ad} W_1)^{n_1} \dots (\operatorname{ad} W_{i-1})^{n_{i-1}} W_i) = -\frac{\partial}{\partial w_i},$$

while for $i = p + 1$ and $V \in \mathfrak{a}^\alpha$,

$$\sum_{n_1=0}^{\infty} \dots \sum_{n_p=0}^{\infty} \frac{w_1^{n_1}}{n_1!} \dots \frac{w_p^{n_p}}{n_p!} d\lambda((\operatorname{ad} W_1)^{n_1} \dots (\operatorname{ad} W_p)^{n_p} V) = 1_{\mathsf{R}^p} \otimes d\lambda_0(V).$$

Since $W_1, \dots, W_p \in \mathfrak{n}$, these series are finite, and since $w_1, \dots, w_p \in d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$, their sums belong to $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$. This proves (5.3).

(1) Assume $k = 0$. Then $\mathfrak{a}^\alpha = \mathfrak{n}_0 + \mathfrak{k}$. Hence by the induction hypothesis applied to $\exp(\mathfrak{a}^\alpha)$

$$d\lambda_0(\mathfrak{U}((\mathfrak{a}^\alpha)^{\mathsf{C}})) = \mathrm{DP}(\mathsf{R}^{n-p}). \tag{5.4}$$

Also, $\Xi = \Xi'$ and $\lambda = \lambda'$, so by (5.2)–(5.4) it follows that $\mathrm{DP}(\mathsf{R}^n) \subseteq d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$, whence (cf. (2.1))

$$d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = \mathrm{DP}(\mathsf{R}^n).$$

(2) Assume $k > 0$. Then also $\dim(\mathfrak{a}^\alpha/(\mathfrak{n}_0 + \mathfrak{k})) = k > 0$. Hence by the induction hypothesis applied to $\exp(\mathfrak{a}^\alpha)$, the basis $(X_1, \dots, X_k) = (W_{p+1}, \dots, W_{p+k})$ of $\mathfrak{a}^\alpha$ modulo $\mathfrak{n}_0 + \mathfrak{k}$ may be chosen such that $d\lambda_0(\mathfrak{U}((\mathfrak{a}^\alpha)^{\mathsf{C}}))$ contains the multiplication by $(w_{p+1}, \dots, w_n) \mapsto \psi_0(w_{p+1})$, where either $\psi_0(x) = x$ or $\psi_0(x) = e^x$ for all $x \in \mathsf{R}$.

Thus by (5.3) there exists $D \in \mathfrak{U}(\mathfrak{g}^{\mathsf{C}})$ such that

$$d\lambda'(D) = \psi',$$

where $\psi'(w_1, \dots, w_n) = \psi_0(w_{p+1})$ for all $(w_1, \dots, w_n) \in \mathsf{R}^n$.

By (1) and (2) of Lemma 2.2 it then follows that

$$d\lambda(D) = \psi' \circ \xi,$$

where ξ is the diffeomorphism of R^n onto R^n defined by

$$\exp(x_1 X_1) \dots \exp(x_n X_n) \in \exp(\xi_1(x) W_1) \dots \exp(\xi_n(x) W_n) \exp(\mathfrak{k}) \tag{5.5}$$

for all $x = (x_1, \dots, x_n) \in \mathsf{R}^n$. Since $\operatorname{span}\{X_2, \dots, X_n\} + \mathfrak{k}$ is an ideal of $\mathfrak{g}$, as it contains $\mathfrak{n} \supseteq [\mathfrak{g}, \mathfrak{g}]$, (5.5) implies that $\xi_{p+1}(x) = x_1$ for all $x = (x_1, \dots, x_n) \in \mathsf{R}^n$.

Hence $(\psi' \circ \xi)(x) = \psi_0(\xi_{p+1}(x)) = \psi_0(x_1)$ for all $x = (x_1, \dots, x_n) \in \mathsf{R}^n$, which proves that $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))$ contains the multiplication by either x_1 or e^{x_1}.

This concludes the proof in case (I).

If the conclusion of the theorem holds for one nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ with $\mathfrak{n} \supseteq [\mathfrak{g},\mathfrak{g}]$, then it holds for every smaller such. Indeed, if $\mathfrak{n}_0$ is a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supseteq \mathfrak{n}_0 \supseteq [\mathfrak{g},\mathfrak{g}]$ and $r = \dim((\mathfrak{n} + \mathfrak{k})/(\mathfrak{n}_0 + \mathfrak{k}))$, then a normal basis $(X_{k+1},\ldots,X_n)$ of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$ can be chosen such that $(X_{k+r+1},\ldots,X_n)$ is a normal basis of $\mathfrak{n}_0$ modulo $\mathfrak{n}_0 \cap \mathfrak{k}$. Since the conclusion of the theorem holds independently of the choice of normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$, the claim easily follows.

Hence we may and will assume in the rest of the proof that $\mathfrak{n}$ equals the nilradical of $\mathfrak{g}$. Then in particular $\mathfrak{a} \subseteq \mathfrak{n}$.

(II) Assume $\mathfrak{k} \nsubseteq \mathfrak{a}^\alpha$. Set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{a}^\alpha$ and $\mathfrak{k}' = \mathfrak{a} + \mathfrak{k}_0$. Then by Lemma 3.4 (ii), $\mathfrak{k}' \in P(\alpha,\mathfrak{g})$. This case will be reduced to (I) essentially by relating suitable realizations of $\lambda_{\alpha,\mathfrak{k}}$ and $\lambda_{\alpha,\mathfrak{k}'}$ to each other. We begin by choosing appropriate coexponential bases, which, however, will be adjusted along the way.

Set $p = \dim(\mathfrak{k}/\mathfrak{k}_0)$ and $q = \dim(\mathfrak{k} \cap \mathfrak{n}/\mathfrak{k}_0 \cap \mathfrak{n})$. Then $0 < p$ and $0 \leqq q \leqq p$, and by Lemma 3.5, $p = \dim(\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i}) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}) \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \leqq 2$.

Set $\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{k}$. Then $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{k}'$ with $\mathfrak{k} \cap \mathfrak{k}' = \mathfrak{k}_0$. By Lemma 3.5 (ii) we have $[\mathfrak{k} \cap \mathfrak{i}, \mathfrak{k} \cap \mathfrak{i}] \subseteq \mathfrak{k}_0 \cap \mathfrak{i}$. Since $\mathfrak{g}_0 \cap \mathfrak{i} = \mathfrak{a} + (\mathfrak{k} \cap \mathfrak{i})$ and $[\mathfrak{a},\mathfrak{i}] \subseteq \mathfrak{z} \subseteq \mathfrak{k}_0 \cap \mathfrak{i}$, therefore also $[\mathfrak{g}_0 \cap \mathfrak{i}, \mathfrak{g}_0 \cap \mathfrak{i}] \subseteq \mathfrak{k}_0 \cap \mathfrak{i}$. As $\mathfrak{n} \subseteq \mathfrak{i}$, it follows that $[\mathfrak{g}_0 \cap \mathfrak{n}, \mathfrak{g}_0 \cap \mathfrak{n}] \subseteq \mathfrak{k}_0 \cap \mathfrak{n}$. Also, $\dim(\mathfrak{g}/\mathfrak{g}_0) = \dim(\mathfrak{g}/\mathfrak{k}) - \dim(\mathfrak{g}_0/\mathfrak{k}) = n - p$.

Let $(W_1,\ldots,W_q)$ be a basis of $\mathfrak{k} \cap \mathfrak{n}$ modulo $\mathfrak{k}_0 \cap \mathfrak{n}$. Since $\mathfrak{n} \subseteq \mathfrak{i}$, we can extend it to a basis $(W_1,\ldots,W_p)$ of $\mathfrak{k} \cap \mathfrak{i}$ modulo $\mathfrak{k}_0 \cap \mathfrak{i}$. Adding suitable multiplies of Z to each W_i we may assume that $\alpha(W_i) = 0$ for all $i = 1,\ldots,p$.

The set $(W_1,\ldots,W_q)$ is a basis of $\mathfrak{g}_0 \cap \mathfrak{n}$ modulo $\mathfrak{k}' \cap \mathfrak{n}$. It is a normal such, since $[\mathfrak{g}_0 \cap \mathfrak{n}, \mathfrak{g}_0 \cap \mathfrak{n}] \subseteq \mathfrak{k}_0 \cap \mathfrak{n}$. The set $(W_1,\ldots,W_p)$ is a basis of $\mathfrak{g}_0 \cap \mathfrak{i}$ modulo $\mathfrak{k}' \cap \mathfrak{i}$. It is a normal such, since $[\mathfrak{g}_0 \cap \mathfrak{i}, \mathfrak{g}_0 \cap \mathfrak{i}] \subseteq \mathfrak{k}_0 \cap \mathfrak{i}$. Since $\mathfrak{g}_0 \cap \mathfrak{i}$ is an ideal of $\mathfrak{g}_0$ with $\mathfrak{g}_0 \cap \mathfrak{i} + \mathfrak{k}' = \mathfrak{g}_0$ (as $\mathfrak{k} \cap \mathfrak{i} + \mathfrak{k}_0 = \mathfrak{k}$) and $\mathfrak{g}_0 \cap \mathfrak{i} \cap \mathfrak{k}' = \mathfrak{k}' \cap \mathfrak{i}$, it follows by Lemma 2.1 (a) that $(W_1,\ldots,W_p)$ is a coexponential basis of $\mathfrak{g}_0$ modulo $\mathfrak{k}'$.

We have $[X,Y] = \langle X,Y\rangle Z$, where $\langle X,Y\rangle = \alpha(Z)^{-1}\alpha([X,Y])$, for all $X \in \mathfrak{k} \cap \mathfrak{i}$, $Y \in \mathfrak{a}$. By Lemma 3.5 (v) the real bilinear form $\langle\cdot,\cdot\rangle$ factorizes to a non-degenerate form on $(\mathfrak{k} \cap \mathfrak{i}/\mathfrak{k}_0 \cap \mathfrak{i}) \times (\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k})$. Hence there exists a basis $(Y_1,\ldots,Y_p)$ of $\mathfrak{a}$ modulo $\mathfrak{a} \cap \mathfrak{k}$, with $\alpha(Y_1) = 0$, such that $[W_i, Y_j] = \delta_{ij}Z$ for $i,j = 1,\ldots,p$.

The set $(Y_1,\ldots,Y_p)$ is a basis of $\mathfrak{g}_0 \cap \mathfrak{n}$ modulo $\mathfrak{k} \cap \mathfrak{n}$, and it is a normal such since $[\mathfrak{g}_0 \cap \mathfrak{n}, \mathfrak{g}_0 \cap \mathfrak{n}] \subseteq \mathfrak{k}_0 \cap \mathfrak{n} \subseteq \mathfrak{k} \cap \mathfrak{n}$. Lemma 2.1 (a) it is thus a coexponential basis of $\mathfrak{g}_0$ modulo $\mathfrak{k}$.

Let $(X_1,\ldots,X_k)$ denote a basis of $\mathfrak{g}$ modulo $\mathfrak{n} + \mathfrak{k} = \mathfrak{n} + \mathfrak{g}_0$, automatically normal since $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{n}$. Let $(X_{k+1},\ldots,X_{n-p})$ denote a normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{g}_0$, by Lemma 2.1 (a) then a coexponential basis of $\mathfrak{n} + \mathfrak{g}_0$ modulo $\mathfrak{g}_0$. Then by Lemma 2.1 (b),

$$\Xi := (X_1, \ldots, X_{n-p}, Y_1, \ldots, Y_p) \quad \text{and} \quad \Xi' := (X_1 \ldots, X_{n-p}, W_1, \ldots, W_p)$$

are coexponential bases of $\mathfrak{g}$ modulo $\mathfrak{k}$ and $\mathfrak{k}'$, respectively. Here Ξ is compatible with $\mathfrak{n}$, since $(X_{k+1}, \ldots, X_{n-p}, Y_1, \ldots, Y_p)$ is a normal basis of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{k}$.

Claim (*): Let Φ denote the algebra isomorphism of $\mathrm{DO}(\mathrm{R}^{n-p}) \otimes \mathrm{DP}(\mathrm{R}^p)$ onto $\mathrm{DO}(\mathrm{R}^{n-p}) \otimes \mathrm{DP}(\mathrm{R}^p)$ given by $\Phi = I \otimes \Phi_0$, where I is the identity on $\mathrm{DO}(\mathrm{R}^{n-p})$ and Φ_0 is the algebra isomorphism of $\mathrm{DP}(\mathrm{R}^p_y)$ onto $\mathrm{DP}(\mathrm{R}^p_w)$ determined by

$$\Phi_0(y_i) = \alpha(Z)^{-1} \frac{\partial}{\partial w_i}, \qquad \Phi_0\left(\frac{\partial}{\partial y_i}\right) = -\alpha(Z) w_i, \qquad i = 1, \ldots, p.$$

Then $d\lambda_{\alpha, \mathfrak{k}, \Xi}(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}})) \subseteq \mathrm{DO}(\mathrm{R}^{n-p}) \otimes \mathrm{DP}(\mathrm{R}^p)$ and

$$\Phi(d\lambda_{\alpha, \mathfrak{k}, \Xi}(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}}))) = d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}})).$$

PROOF. Abbreviate $\lambda = \lambda_{\alpha, \mathfrak{k}, \Xi}$ and $\lambda' = \lambda_{\alpha, \mathfrak{k}', \Xi'}$. We will show that Φ intertwines $d\lambda$ and $d\lambda'$ after certain characters are added to α in the two representations.

Let $\delta \in \mathfrak{g}^*$ denote the linear extension of $\frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{g}_0/\mathfrak{k}} \in \mathfrak{k}^*$ given by $\delta(\Xi) = \{0\}$ and let $\delta' \in \mathfrak{g}^*$ denote the one of $\frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{g}_0/\mathfrak{k}'} \in (\mathfrak{k}')^*$ given by $\delta'(\Xi') = \{0\}$. Then δ and δ' are characters on $\mathfrak{g}$. Indeed, since nilpotent endomorphisms have zero trace, we have $\delta = 0$ on $\mathfrak{n} \cap \mathfrak{k}$ and $\delta' = 0$ on $\mathfrak{n} \cap \mathfrak{k}'$. Moreover, $\mathfrak{n} \cap \mathfrak{g}_0 = \sum_{i=1}^p \mathrm{R} Y_i + \mathfrak{n} \cap \mathfrak{k}$ and $\mathfrak{n} \cap \mathfrak{g}_0 = \mathfrak{n} \cap \mathfrak{k}' + \sum_{i=1}^q \mathrm{R} W_i$, so $\mathfrak{n} = \mathfrak{n} \cap \operatorname{span}(\Xi) + \mathfrak{n} \cap \mathfrak{k}$ and $\mathfrak{n} = \mathfrak{n} \cap \operatorname{span}(\Xi') + \mathfrak{n} \cap \mathfrak{k}'$. Hence δ and δ' vanish on $\mathfrak{n} \supseteq [\mathfrak{g}, \mathfrak{g}]$.

For later reference, note that since $[W_i, \mathfrak{g}_0] = [W_i, \sum_{i=1}^p \mathrm{R} Y_i + \mathfrak{k}] \subseteq \mathrm{R} Z + \mathfrak{k} = \mathfrak{k}$, we have $\delta(W_i) = \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{g}_0/\mathfrak{k}}(W_i) = 0$ for all $i = 1, \ldots, p$, so that both δ and δ' vanish on $\operatorname{span}\{Y_i, Z, W_i \mid 1 \leqq i \leqq p\}$.

Set $\tilde{\lambda} := \lambda_{\alpha+\delta, \mathfrak{k}, \Xi}$ and $\tilde{\lambda}' := \lambda_{\alpha+\delta', \mathfrak{k}', \Xi'}$. Then by (2) of Lemma 2.3, $d\tilde{\lambda} = d\lambda - \delta$ and $d\tilde{\lambda}' = d\lambda' - \delta'$, so that $d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}})) = d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}}))$ and $d\lambda'(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}})) = d\tilde{\lambda}'(\mathfrak{U}(\mathfrak{g}^{\mathrm{C}}))$.

Hence the claim (*) will follow if $d\tilde{\lambda}(\mathfrak{g}) \subseteq \mathrm{DO}(\mathrm{R}^{n-p}) \otimes \mathrm{DP}(\mathrm{R}^p)$ and

$$\Phi(d\tilde{\lambda}(V)) = d\tilde{\lambda}'(V) \quad \text{for all} \quad V \in \mathfrak{g}. \tag{5.6}$$

We shall reduce (5.6) to an assertion involving only $\mathfrak{g}_0$. Set $\tilde{\lambda}_0 := \lambda_{\alpha_0+\delta_0, \mathfrak{k}, \Xi_0}$ and $\tilde{\lambda}'_0 := \lambda_{\alpha_0+\delta'_0, \mathfrak{k}', \Xi'_0}$, where α_0, δ_0 and δ'_0 denote the restrictions of α, δ and δ' to $\mathfrak{g}_0$, and where $\Xi_0 := (Y_1, \ldots, Y_p)$ and $\Xi'_0 := (W_1, \ldots, W_p)$.

Given $V \in \mathfrak{g}$, there exist $\xi_i \in C^\infty(\mathrm{R} \times \mathrm{R}^{n-p})$ and $U \in C^\infty(\mathrm{R} \times \mathrm{R}^{n-p}, \mathfrak{g}_0)$ such that for all $(t, x) \in \mathrm{R} \times \mathrm{R}^{n-p}$,

$$\exp(-tV)\exp(x_1 X_1)\ldots\exp(x_{n-p}X_{n-p})$$
$$= \exp(\xi_1(t,x)X_1)\ldots\exp(\xi_{n-p}(t,x)X_{n-p})\exp(-U(t,x)),$$

implying that

$$d\tilde{\lambda}(V) = \sum_{i=1}^{n-p} \frac{\partial \xi_i}{\partial t}(0,x)\frac{\partial}{\partial x_i} + d\tilde{\lambda}_0\left(\frac{\partial U}{\partial t}(0,x)\right)$$

and

$$d\tilde{\lambda}'(V) = \sum_{i=1}^{n-p} \frac{\partial \xi_i}{\partial t}(0,x)\frac{\partial}{\partial x_i} + d\lambda'_0\left(\frac{\partial U}{\partial t}(0,x)\right).$$

Thus to prove (5.6) it suffices to prove that $d\tilde{\lambda}_0(\mathfrak{g}_0) \subseteq \mathrm{DP}(\mathrm{R}^p)$ and

$$\Phi_0(d\tilde{\lambda}_0(V)) = d\tilde{\lambda}'_0(V) \quad \text{for all} \quad V \in \mathfrak{g}_0. \tag{5.7}$$

Put $\lambda_0 := \lambda_{\alpha_0, \mathfrak{k}, \Xi_0}$ and $\lambda'_0 := \lambda_{\alpha_0, \mathfrak{k}', \Xi'_0}$. Then $d\tilde{\lambda}_0 = d\lambda_0 - \delta_0$ and $d\tilde{\lambda}'_0 = d\lambda'_0 - \delta'_0$ by Lemma 2.3 (2). Put $\mathfrak{l} = \mathfrak{k}_0 \cap \ker \alpha$. Then $\mathfrak{g}_0 = \operatorname{span}\{Y_i, Z, W_i \mid 1 \leqq i \leqq p\} + \mathfrak{l}$.

First we observe that (5.7) holds for $V \in \operatorname{span}\{Y_i, Z, W_i \mid 1 \leqq i \leqq p\}$. Indeed, using the relations $[W_i, Y_j] = \delta_{ij} Z$ and $[W_i, W_j + \mathfrak{l} \cap \mathfrak{i}] \subseteq \mathfrak{l} \cap \mathfrak{i}$ and that $W_i, Y_i \in \ker \alpha$, we easily find for $i = 1, \ldots, p$, that

$$d\lambda_0(W_i) = -\alpha(Z) y_i, \qquad d\lambda_0(Y_i) = -\frac{\partial}{\partial y_i}, \qquad d\lambda_0(Z) = -\alpha(Z),$$

$$d\lambda'_0(W_i) = -\frac{\partial}{\partial w_i}, \qquad d\lambda'_0(Y_i) = \alpha(Z) w_i, \qquad d\lambda'_0(Z) = -\alpha(Z).$$

Since $\delta = \delta' = 0$ on $\operatorname{span}\{W_i, Y_i, Z \mid 1 \leqq i \leqq p\}$, as noted above, it follows that (5.7) holds for $V \in \operatorname{span}\{W_i, Y_i, Z \mid 1 \leqq i \leqq p\}$.

It remains to prove (5.7) for $V \in \mathfrak{l}$. Note that $[\mathfrak{l}, \mathfrak{a}] \subseteq \operatorname{span}\{Y_1, \ldots, Y_p\}$. Indeed, $[\mathfrak{l}, \mathfrak{a}] \subseteq [\mathfrak{a}^\alpha, \mathfrak{a}] \subseteq \mathfrak{a} \cap \ker \mathfrak{a}$, and if $\mathfrak{k} \subseteq \mathfrak{i}$, then $[\mathfrak{l}, \mathfrak{a}] \subseteq \mathrm{R}Z$, so that actually $[\mathfrak{l}, \mathfrak{a}] = \{0\}$, while if $\mathfrak{k} \nsubseteq \mathfrak{i}$, then by Lemma 3.5 (iv), $\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{z} = \mathrm{R}Z$, so that $\mathfrak{a} \cap \ker \alpha = \operatorname{span}\{Y_1, \ldots, Y_p\}$.

Let A denote the matrix of $\operatorname{ad}_{\mathfrak{g}_0/\mathfrak{k}}(V)$ wrt. the basis $Y_1, \ldots, Y_p$ of $\mathfrak{g}_0$ modulo $\mathfrak{k}$. Then, since $\operatorname{ad} V$ leaves $\operatorname{span}\{Y_1, \ldots, Y_p\}$ invariant,

$$\operatorname{ad} V\left(\sum_{i=1}^{p} y_i Y_i\right) = \sum_{i=1}^{p} y'_i Y_i, \quad \text{with} \quad y' = Ay, \; y \in \mathrm{R}^p.$$

Let B denote the matrix of $\operatorname{ad}_{\mathfrak{g}_0/\mathfrak{k}'}(V)$ wrt. the basis $W_1, \ldots, W_p$ of $\mathfrak{g}_0$ modulo $\mathfrak{k}'$. Then, since $[V, W_i] \subseteq [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \cap \mathfrak{i} \cap \ker \alpha = \operatorname{span}\{W_1, \ldots, W_p\} + \mathfrak{l} \cap \mathfrak{i}$, there are $V_i \in \mathfrak{l} \cap \mathfrak{i}$, $i = 1, \ldots, p$, such that

$$\operatorname{ad} V\left(\sum_{i=1}^{p} w_i W_i\right) = \sum_{i=1}^{p} w'_i W_i + \sum_{i=1}^{p} w_i V_i, \quad \text{with} \quad w' = Bw, \; w \in \mathrm{R}^p.$$

Applying $\operatorname{ad} V$ to

$$\left[\sum_{i=1}^{p} w_i W_i, \sum_{j=1}^{p} y_j Y_j\right] = (w \cdot y) Z, \qquad w, y \in \mathrm{R}^p,$$

and using $[\mathfrak{l} \cap \mathfrak{i}, Y_i] = \{0\}$, we then get that $(Bw)\cdot y + w\cdot(Ay) = 0$ for all $w, y \in \mathsf{R}^p$. Hence $B = -A^t$, so that $\delta(V) = \frac{1}{2}\operatorname{tr} A = -\frac{1}{2}\operatorname{tr} B = -\delta'(V)$.

Put $G_0 = \exp(\mathfrak{g}_0)$. Let $\varphi \in C^\infty_{\alpha_0,\mathfrak{k}'}(G_0)$ and set $\tilde{\varphi}(w_1,\ldots,w_p) = \varphi(\exp(w_1 W_1)\ldots\exp(w_p W_p))$ for all $w = (w_1,\ldots,w_p) \in \mathsf{R}^p$. Since $\mathfrak{g}_1 := \operatorname{span}\{W_1,\ldots,W_p\} + \mathfrak{l} \cap \mathfrak{i}$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{l} \cap \mathfrak{i}$, there exists $U \in C^\infty(\mathsf{R}^p \times \mathsf{R}^p, \mathfrak{l} \cap \mathfrak{i})$ such that

$$\exp\left(\sum_{i=1}^p w_i W_i + \sum_{i=1}^p v_i V_i\right) = \exp(w_1 W_1)\ldots\exp(w_p W_p)\exp(U(w,v))$$

for all $(w, v) \in \mathsf{R}^p \times \mathsf{R}^p$. Since $\varphi \in C^\infty_{\alpha_0,\mathfrak{k}'}(G_0)$ is invariant under right translations by elements from $\exp(\mathfrak{k}' \cap \ker\alpha) \supseteq \exp(\mathfrak{l})$ we thus find

$$\begin{aligned}
[d\lambda_0'(V)\tilde{\varphi}](w) &= \frac{d}{dt}\Big|_{t=0} \varphi(\exp(-tV)\exp(w_1 W_1)\ldots\exp(w_p W_p)) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi\left(\exp(-tV)\exp\left(\sum_{i=1}^p w_i W_i\right)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi\left(\exp(-tV)\exp\left(\sum_{i=1}^p w_i W_i\right)\exp(tV)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi\left(\exp\left(\sum_{i=1}^p w_i W_i - t\left[V, \sum_{i=1}^p w_i W_i\right]\right)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi\left(\exp\left(\sum_{i=1}^p (w_i - tw_i')W_i - \sum_{i=1}^p tw_i V_i\right)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi(\exp((w_1 - tw_1')W_1)\ldots\exp((w_p - tw_p')W_p)) \\
&= -\sum_{i=1}^p w_i' \frac{\partial\tilde{\varphi}}{\partial w_i}, \quad \text{where} \quad w' = Bw.
\end{aligned}$$

Since $d\tilde{\lambda}_0' = d\lambda_0' - \delta_0'$, this gives

$$d\tilde{\lambda}_0'(V) = -\sum_{i=1}^p w_i' \frac{\partial}{\partial w_i} - \tfrac{1}{2}\operatorname{tr} B = -(Bw)\cdot\nabla_w - \tfrac{1}{2}\operatorname{tr} B,$$

where $\nabla_w := (\partial/\partial w_1,\ldots,\partial/\partial w_p)$.

By similar, but simpler, computations

$$d\tilde{\lambda}_0(V) = -\sum_{i=1}^p y_i' \frac{\partial}{\partial y_i} - \tfrac{1}{2}\operatorname{tr} A = -(Ay)\cdot\nabla_y - \tfrac{1}{2}\operatorname{tr} A.$$

Hence $d\tilde{\lambda}_0(V) \in DP(\mathsf{R}^p)$ and

$$\begin{aligned}\Phi_0(d\tilde{\lambda}_0(V)) &= -(A(\alpha(Z)^{-1}\nabla_w))\cdot(-\alpha(Z)w) - \tfrac{1}{2}\operatorname{tr} A\\ &= \nabla_w \cdot A^t w - \tfrac{1}{2}\operatorname{tr} A\\ &= (A^t w)\cdot\nabla_w + \operatorname{tr} A^t - \tfrac{1}{2}\operatorname{tr} A\\ &= d\tilde{\lambda}_0'(V).\end{aligned}$$

This proves the claim.

Now, either (a): $q = p$ or (b): $q < p$.

(a) Assume $q = p$. Then $(W_1, \ldots, W_p)$ is a normal basis of $\mathfrak{g}_0 \cap \mathfrak{n}$ modulo $\mathfrak{k}' \cap \mathfrak{n}$. Hence Ξ' is compatible with $\mathfrak{n}$, with $(X_{k+1}, \ldots, X_{n-p}, W_1, \ldots, W_p)$ being a normal basis of $\mathfrak{n}$ modulo $\mathfrak{k}' \cap \mathfrak{n}$. We have $\dim(\mathfrak{g}/(\mathfrak{n} + \mathfrak{k}')) = \dim(\mathfrak{g}/(\mathfrak{n} + \mathfrak{k})) = k$.

Since $\mathfrak{a} \subseteq \mathfrak{k}' \subseteq \mathfrak{a}^\alpha$, we have by (I) that the conclusion of the theorem holds for the representation $\lambda_{\alpha, \mathfrak{k}'}$:

(1) In the case $k = 0$ this means that

$$d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})) = \mathrm{DP}(\mathbf{R}^n).$$

An application of the claim (*) then gives

$$d\lambda_{\alpha, \mathfrak{k}, \Xi}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})) = \Phi^{-1}(d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}}))) = \Phi^{-1}(\mathrm{DP}(\mathbf{R}^n)) = \mathrm{DP}(\mathbf{R}^n).$$

(2) In the case $k > 0$ it means that $(X_1, \ldots, X_k)$ may be chosen such that

$$\psi(x_1) \in d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}}))$$

for either $\psi(x_1) = x_1$ or $\psi(x_1) = e^{x_1}$. Here an application of the claim (*) gives

$$\psi(x_1) = \Phi^{-1}(\psi(x_1)) \in \Phi^{-1}(d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}}))) = d\lambda_{\alpha, \mathfrak{k}, \Xi}(\mathfrak{U}(\mathfrak{g}^{\mathbf{C}})).$$

This finishes the proof in case II (a).

(b) Assume $q < p$. Since $W_{q+1}, \ldots, W_p$ do not belong to $\mathfrak{n}$, we shall replace them by their nilpotent parts in an extension of $\mathfrak{g}$.

The nilpotent part in the Jordan decomposition of a derivation is again a derivation, cf. Cor. 3.1.14 of [Va], so we may form the successive semi-direct product

$$\bar{\mathfrak{g}} := \mathbf{R} \times_{(\operatorname{ad} W_{q+1})_n} \cdots \mathbf{R} \times_{(\operatorname{ad} W_p)_n} \mathfrak{g},$$

where $(\operatorname{ad} W_i)_n$ denotes the nilpotent part in the Jordan decomposition of $\operatorname{ad} W_i \in \operatorname{End}(\mathbf{R} \times_{(\operatorname{ad} W_{i+1})_n} \cdots \mathbf{R} \times_{(\operatorname{ad} W_p)_n} \mathfrak{g})$ for $i = p, p-1, \ldots, q+1$. Since nilpotent endomorphisms have zero as only eigenvalue, the Lie algebra $\bar{\mathfrak{g}}$ is exponential.

We may consider $\mathfrak{g}$ as a subalgebra (in fact ideal, cf. (5.8) below) of $\bar{\mathfrak{g}}$ and write as a direct sum of subspaces

$$\bar{\mathfrak{g}} = \mathbf{R} W_{q+1}^{(n)} + \ldots + \mathbf{R} W_p^{(n)} + \mathfrak{g},$$

where $[W_i^{(n)}, X] = (\text{ad } W_i)_n(X)$ for $X \in \text{span}\{W_{i+1}^{(n)}, \ldots, W_p^{(n)}\} + \mathfrak{g}, i = q+1, \ldots, p$.

Since for each W the nilpotent part $(\text{ad } W)_n$ of ad W is a polynomial in ad W without constant term, we have

$$(5.8) \qquad [\mathfrak{g}, \mathfrak{g}] = [\mathbb{R}W_p^{(n)} + \mathfrak{g}, \mathbb{R}W_p^{(n)} + \mathfrak{g}] = \ldots = [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}].$$

In the present case $\mathfrak{k} \subseteq \mathfrak{i}$. Indeed, $\dim(\mathfrak{k} \cap \mathfrak{n}/\mathfrak{k}_0 \cap \mathfrak{n}) = q < p = \dim(\mathfrak{k}/\mathfrak{k}_0)$ implies that $[\mathfrak{k}, \mathfrak{k}] + \mathfrak{k}_0 \subseteq \mathfrak{k} \cap \mathfrak{n} + \mathfrak{k}_0 \neq \mathrm{k}$, so by Lemma 3.5 (iii), $\mathfrak{k} \subseteq \mathfrak{i}$.

Set $W_i^{(s)} := W_i - W_i^{(n)}, i = q+1, \ldots, p$. Since $W_i \in \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{a}] \subseteq [\mathfrak{i}, \mathfrak{a}] \subseteq \mathfrak{z}$, each ad W_i acts nilpotently on $\mathfrak{a}$, so by the uniqueness of the Jordan decomposition, $[W_i^{(s)}, \mathfrak{a}] = \{0\}$ for $i = q+1, \ldots, p$.

Set $\bar{\mathfrak{k}} := \mathfrak{k} + \text{span}\{W_{q+1}^{(s)}, \ldots, W_p^{(s)}\}$ and $\bar{\mathfrak{k}}' := \mathfrak{k}' + \text{span}\{W_{q+1}^{(s)}, \ldots, W_p^{(s)}\}$. Since $\mathfrak{k}' = \mathfrak{k}_0 + \mathfrak{a}$, where $\mathfrak{a}$ commutes with $\mathfrak{a}$, $\mathfrak{k}_0$ and each $W_i^{(s)}$, we have $[\bar{\mathfrak{k}}', \bar{\mathfrak{k}}'] \subseteq [\bar{\mathfrak{k}}, \bar{\mathfrak{k}}]$. Since $W_i \in \mathfrak{k}$, we conclude as in (5.8) that $[\bar{\mathfrak{k}}, \bar{\mathfrak{k}}] = [\mathfrak{k}, \mathfrak{k}]$. From $\mathfrak{k} \subseteq \mathfrak{i}$ we get by (ii) of Lemma 3.5 that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}_0 \cap \ker \alpha$. Hence $[\bar{\mathfrak{k}}', \bar{\mathfrak{k}}'] \subseteq [\bar{\mathfrak{k}}, \bar{\mathfrak{k}}] \subseteq \mathfrak{k}_0 \cap \ker \alpha$.

We form a modification $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by exchanging W_i for $W_i^{(n)}$, $i = q+1, \ldots, p$: Setting $\mathfrak{g}_1 := \text{span}\{X_1, \ldots, X_k\} + (\mathfrak{n} + \mathfrak{k}') = \text{span}\{X_1, \ldots, X_{n-p}, W_1, \ldots, W_q\} + \mathfrak{k}'$, which satisfies $\mathfrak{g} = \mathfrak{g}_1 + \text{span}\{W_{q+1}, \ldots, W_p\}$, we define

$$\tilde{\mathfrak{g}} := \mathfrak{g}_1 + \text{span}\{W_{q+1}^{(n)}, \ldots, W_p^{(n)}\} \quad \text{and} \quad \tilde{\mathfrak{n}} := \mathfrak{n} + \text{span}\{W_{q+1}^{(n)}, \ldots, W_p^{(n)}\}.$$

Then $\tilde{\mathfrak{g}} = \text{span}\{X_1, \ldots, X_k\} + (\tilde{\mathfrak{n}} + \mathfrak{k}')$ is an ideal of $\bar{\mathfrak{g}}$ with $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n} \subseteq \tilde{\mathfrak{n}} \subseteq \tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{n}}$ is a nilpotent ideal of $\tilde{\mathfrak{g}}$ containing $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Also, $\dim \tilde{\mathfrak{g}} = \dim \mathfrak{g}$.

Set $\tilde{\mathfrak{g}}_0 := \text{span}\{W_1, \ldots, W_q, W_{q+1}^{(n)}, \ldots, W_p^{(n)}\} + \mathfrak{k}'$. Then $[\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_0] \subseteq \mathfrak{k}_0 \subseteq \mathfrak{k}'$, since $\tilde{\mathfrak{g}}_0 \subseteq \bar{\mathfrak{k}} + \mathfrak{a}$ with $[\bar{\mathfrak{k}}, \mathfrak{a}] \subseteq \mathfrak{z} \subseteq \mathfrak{k}_0$ and $[\bar{\mathfrak{k}}, \bar{\mathfrak{k}}] \subseteq \mathfrak{k}_0$. Hence $(W_1, \ldots, W_q, W_{q+1}^{(n)}, \ldots, W_p^{(n)})$ is a normal basis of $\tilde{\mathfrak{g}}_0 \cap \tilde{\mathfrak{n}}$ modulo $\mathfrak{k}' \cap \tilde{\mathfrak{n}}$.

The normal basis $(X_{k+1}, \ldots, X_{n-p})$ of $\mathfrak{n}$ modulo $\mathfrak{n} \cap \mathfrak{g}_0$ may be chosen such that it is also a normal basis of $\tilde{\mathfrak{n}}$ modulo $\tilde{\mathfrak{n}} \cap \tilde{\mathfrak{g}}_0 = \mathfrak{n} \cap \mathfrak{g}_0 + \text{span}\{W_{q+1}^{(n)}, \ldots, W_p^{(n)}\}$. Then

$$\tilde{\Xi}_0 := (X_{k+1}, \ldots, X_{n-p}, W_1, \ldots, W_q, W_{q+1}^{(n)}, \ldots, W_p^{(n)})$$

is a normal basis of $\tilde{\mathfrak{n}}$ modulo $\mathfrak{k}' \cap \tilde{\mathfrak{n}}$, and thus

$$\tilde{\Xi} := (X_1, \ldots, X_k, X_{k+1}, \ldots, X_{n-p}, W_1, \ldots, W_q, W_{q+1}^{(n)}, \ldots, W_p^{(n)})$$

is a coexponential basis of $\tilde{\mathfrak{g}}$ modulo $\mathfrak{k}'$ compatible with $\tilde{\mathfrak{n}}$.

By Lemma 2.1 (a) $\tilde{\Xi}$ is also a coexponential basis of $\bar{\mathfrak{g}}$ modulo $\bar{\mathfrak{k}}'$, since $\tilde{\mathfrak{g}}$ is an ideal of $\bar{\mathfrak{g}}$ with $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} + \bar{\mathfrak{k}}'$ and $\tilde{\mathfrak{g}} \cap \bar{\mathfrak{k}}' = \mathfrak{k}'$. Similarly, the coexponential basis Ξ' of $\mathfrak{g}$ modulo $\mathfrak{k}'$ is also a coexponential basis of $\bar{\mathfrak{g}}$ modulo $\bar{\mathfrak{k}}'$, since $\mathfrak{g}$ is an ideal of $\bar{\mathfrak{g}}$ with $\bar{\mathfrak{g}} = \mathfrak{g} + \bar{\mathfrak{k}}'$ and $\mathfrak{g} \cap \bar{\mathfrak{k}}' = \mathfrak{k}'$.

Let $\bar{\alpha} \in \mathbb{C}\bar{\mathfrak{g}}^*$ denote the extension of $\alpha \in \mathbb{C}\mathfrak{g}^*$ given by $\bar{\alpha}(W_i^{(n)}) = 0$ for all $i = q+1, \ldots, p$, and set $\tilde{\alpha} := \bar{\alpha}|_{\tilde{\mathfrak{g}}}$. Clearly $\mathfrak{k}' \in S(\tilde{\alpha}, \tilde{\mathfrak{g}})$ and $\bar{\mathfrak{k}}' \in S(\bar{\alpha}, \bar{\mathfrak{g}})$.

Claim: $\mathfrak{k}' \in P(\tilde{\alpha}, \tilde{\mathfrak{g}})$. Proof: Let $\tilde{U} \in \tilde{\mathfrak{g}} \backslash \mathfrak{k}'$ and write $\tilde{U} = \sum_{i=q+1}^{p} a_i W_i^{(n)} + U_0$, where $U_0 \in \mathfrak{g}_1$. Then $U := \sum_{i=q+1}^{p} a_i W_i + U_0 \in \mathfrak{g} \backslash \mathfrak{k}'$. Since $\mathfrak{k}' \in P(\alpha, \mathfrak{g})$, there exists by Lemma 3.3 $V \in \mathfrak{k}'$ such that $[U, V] \in \mathfrak{k}' \backslash \ker \alpha$. Since $[W_i^{(s)}, \mathfrak{k}'] \subseteq [\bar{\mathfrak{k}}', \bar{\mathfrak{k}}'] \subseteq \mathfrak{k}_0 \cap \ker \alpha$, we also have $[\tilde{U}, V] \in \mathfrak{k}' \backslash \ker \alpha = \mathfrak{k}' \backslash \ker \tilde{\alpha}$. By Lemma 3.3 this proves the claim.

Put $\tilde{\lambda} := \lambda_{\bar{\alpha}, \bar{\mathfrak{k}}', \tilde{\Xi}}$, $\lambda' := \lambda_{\bar{\alpha}, \bar{\mathfrak{k}}', \Xi'}$ and $\lambda := \lambda_{\alpha, \mathfrak{k}, \Xi}$. Then

$$
\begin{aligned}
& d\tilde{\lambda}(V) = d\lambda_{\tilde{\alpha}, \mathfrak{k}', \tilde{\Xi}}(V) && \text{for all} \quad V \in \tilde{\mathfrak{g}} \subseteq \bar{\mathfrak{g}}, \\
(5.9) \qquad & d\lambda'(V) = d\lambda_{\alpha, \mathfrak{k}', \Xi'}(V) && \text{for all} \quad V \in \mathfrak{g} \subseteq \bar{\mathfrak{g}}, \\
& d\tilde{\lambda}(V) = d\lambda'(V) && \text{for all} \quad V \in \bar{\mathfrak{g}}.
\end{aligned}
$$

Indeed, the first two relations are trivial. To prove the third set $\bar{\mathfrak{g}}_0 := \mathfrak{g}_0 + \operatorname{span}\{W_{q+1}^{(n)}, \ldots, W_p^{(n)}\}$ and $\bar{\mathfrak{k}}_0 := \mathfrak{k}_0 + \operatorname{span}\{W_{q+1}^{(s)}, \ldots, W_p^{(s)}\}$. Then $\bar{\mathfrak{g}}_0 = \mathfrak{a} + \bar{\mathfrak{k}}$ and $[\bar{\mathfrak{k}}_0, \mathfrak{a}] = \{0\}$, so that $[\bar{\mathfrak{g}}_0, \bar{\mathfrak{k}}_0] \subseteq [\bar{\mathfrak{k}}, \bar{\mathfrak{k}}_0] \subseteq [\bar{\mathfrak{k}}, \bar{\mathfrak{k}}] \subseteq \mathfrak{k}_0 \cap \ker \alpha$. In particular $\bar{\mathfrak{l}}_0 := \bar{\mathfrak{k}}_0 \cap \ker \bar{\alpha}$ is an ideal of $\bar{\mathfrak{g}}_0$ containing $W_{q+1}^{(s)}, \ldots, W_p^{(s)}$, so for every $(w_{q+1}, \ldots, w_p) \in \mathsf{R}^{p-q}$,

$$\exp(w_{q+1} W_{q+1}) \ldots \exp(w_p W_p) \in \exp(w_{q+1} W_{q+1}^{(n)}) \ldots \exp(w_p W_p^{(n)}) \exp(\bar{\mathfrak{l}}_0).$$

This implies by Lemma 2.2 (1) that the canonical equivalence between $\tilde{\lambda}$ and λ' is the identity map.

Combining (5.9) with Claim (*) we finally obtain

$$(5.10) \qquad d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = d\lambda'(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = d\lambda_{\alpha, \mathfrak{k}', \Xi'}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = \Phi(d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))).$$

Now, $\mathfrak{a} \subseteq \mathfrak{k}' \subseteq \mathfrak{a}^{\tilde{\alpha}}$ with $\mathfrak{k}' \in P(\tilde{\alpha}, \tilde{\mathfrak{g}})$ and $\dim \tilde{\mathfrak{g}} = \dim \mathfrak{g}$. Also, $\mathfrak{a}$ is a minimal non-central ideal of $\tilde{\mathfrak{g}}$. Hence by (I) (or the prior cases) the conclusion of the theorem holds for the representation $\lambda_{\tilde{\alpha}, \mathfrak{k}'}$:

(1) In the case $\dim \tilde{\mathfrak{g}}/(\tilde{\mathfrak{n}} + \mathfrak{k}') = k = 0$ this means that

$$(5.11) \qquad d\tilde{\lambda}(\mathfrak{U}(\tilde{\mathfrak{g}}^{\mathsf{C}})) = d\lambda_{\tilde{\alpha}, \mathfrak{k}', \tilde{\Xi}}(\mathfrak{U}(\tilde{\mathfrak{g}}^{\mathsf{C}})) = \mathrm{DP}(\mathsf{R}^n).$$

Moreover, $q = 0$ and $p = 1$ in this case. Indeed, $k = 0$ implies that $\mathfrak{g} = \mathfrak{n} + \mathfrak{k}$, so as $\mathfrak{n} \subseteq \mathfrak{i}$ and $\mathfrak{k} \subseteq \mathfrak{i}$, we have $\mathfrak{g} = \mathfrak{i}$. Hence by Lemma 3.5 (i) and Lemma 3.1 (iii), $p = \dim(\mathfrak{k}/\mathfrak{k}_0) = \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{k}) \leqq \dim(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{z}) \leqq 1$. Since $0 \leqq q < p$, the claim follows. This will simplify the notation below.

Using $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_1 = \operatorname{span}\{X_1, \ldots, X_{n-1}\} + \mathfrak{k}'$ we find, denoting the coordinates on R^n by $x_1, \ldots, x_{n-1}, w_1$, that

$$d\tilde{\lambda}(V) \in DP(\mathsf{R}^{n-1}) \otimes \mathrm{Pol}(w_1) \quad \text{for all} \quad V \in \mathsf{R}W_1^{(s)} + \mathfrak{g}_1,$$

and

$$d\tilde{\lambda}(W_1^{(n)}) \in -\frac{\partial}{\partial w_1} + \mathrm{DP}(\mathsf{R}^{n-1}) \otimes \mathrm{Pol}(w_1),$$

where $\mathrm{Pol}(w_1)$ denotes the algebra of polynomials in w_1.

Since $\tilde{\mathfrak{g}} = \mathsf{R}W_1^{(n)} + \mathfrak{g}_1$, we conclude from this via (5.11) that $\mathrm{DP}(\mathsf{R}^{n-1}) \otimes \mathrm{Pol}(w_1)$ is generated by $d\tilde{\lambda}(\mathfrak{g}_1)$, so

$$d\tilde{\lambda}(W_1^{(s)}) \in \mathrm{DP}(\mathsf{R}^{n-1}) \otimes \mathrm{Pol}(w_1) = d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}_1^{\mathsf{C}})).$$

Since $\mathfrak{g} = \mathsf{R}W_1 + \mathfrak{g}_1$, where $W_1 = W_1^{(n)} + W_1^{(s)}$, it follows that

$$d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = d\tilde{\lambda}(\mathfrak{U}(\tilde{\mathfrak{g}}^{\mathsf{C}})) = \mathrm{DP}(\mathsf{R}^n),$$

which by (5.10) finishes the proof in case $k = 0$:

$$d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})) = \Phi^{-1}(d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))) = \Phi^{-1}(\mathrm{DP}(\mathsf{R}^n)) = \mathrm{DP}(\mathsf{R}^n).$$

(2) In the case $\dim \tilde{\mathfrak{g}}/(\tilde{\mathfrak{n}} + \mathfrak{k}') = k > 0$ there exists a basis $(\tilde{X}_1, \ldots, \tilde{X}_k)$ of $\tilde{\mathfrak{g}}$ modulo $\tilde{\mathfrak{n}} + \mathfrak{k}'$ such that for $\tilde{\Xi}$ replaced by $(\tilde{X}_1, \ldots, \tilde{X}_k) \oplus \tilde{\Xi}_0$ we have

$$\psi(x_1) \in d\lambda_{\tilde{\alpha}, \mathfrak{k}', \tilde{\Xi}}(\mathfrak{U}(\tilde{\mathfrak{g}}^{\mathsf{C}})) = d\tilde{\lambda}(\mathfrak{U}(\tilde{\mathfrak{g}}^{\mathsf{C}})), \tag{5.12}$$

where either $\psi(x_1) = x_1$ or $\psi(x_1) = e^{x_1}$.

By Lemma 2.2 (3) the conclusion (5.12) still holds if we to each $\tilde{X}_i$ add elements from $\tilde{\mathfrak{n}} + \mathfrak{k}'$. Hence we may assume that $\mathrm{span}\{\tilde{X}_1, \ldots, \tilde{X}_k\} = \mathrm{span}\{X_1, \ldots, X_k\}$ and then as well that $\tilde{X}_i = X_i$ for $i = 1, \ldots, k$.

From $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n} \subseteq \mathrm{span}\{X_{k+1}, \ldots, X_{n-p}, W_1, \ldots, W_q\} + \mathfrak{k}'$ we get, denoting the coordinates on R^n by $x_1, \ldots, x_{n-p}, w_1, \ldots, w_p$, that

$$d\tilde{\lambda}(\mathfrak{g}_1) \subseteq \mathrm{DO}(\mathsf{R}^{n-p+q}) \otimes C^\infty(\mathsf{R}^{p-q}),$$

and

$$d\tilde{\lambda}(W_i^{(n)}) \in -\frac{\partial}{\partial w_i} + \mathrm{DO}(\mathsf{R}^{n-p+q}) \otimes C^\infty(\mathsf{R}^{p-q}), \quad i = q+1, \ldots, p.$$

Since $\tilde{\mathfrak{g}} = \mathrm{span}\{W_{q+1}^{(n)}, \ldots, W_p^{(n)}\} + \mathfrak{g}_1$, we thus conclude that (5.12) may be strengthened to

$$\psi(x_1) \in d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}_1^{\mathsf{C}})) \subseteq d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})).$$

Since $\psi(x_1)$ is fixed under Φ^{-1}, it follows by (5.10) that

$$\psi(x_1) \in \Phi^{-1}(d\tilde{\lambda}(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}}))) = d\lambda(\mathfrak{U}(\mathfrak{g}^{\mathsf{C}})),$$

which was to be proved in the case $k > 0$.

This finishes the proof of Theorem 4.1.

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