MATH. SCAND. 77 (1995), 251–271

# WEIGHTED SOBOLEV AND POINCARÉ INEQUALITIES AND QUASIREGULAR MAPPINGS OF POLYNOMIAL TYPE

JUHA HEINONEN AND PEKKA KOSKELA*

## 1. Introduction.

In this paper, *weight* means a locally integrable positive function in $\mathsf{R}^n$, $n \geqq 2$, and we assume throughout that $1 < p < n$. A *weighted Sobolev inequality* is an inequality of the type

$$(1.1)\qquad \left(\frac{1}{w(B)}\int_B |\psi|^{\kappa p} w\,dx\right)^{1/\kappa p} \leqq C \operatorname{diam} B \left(\frac{1}{w(B)}\int_B |\nabla\psi|^p w\,dx\right)^{1/p},$$

valid for all balls $B$ in $\mathsf{R}^n$ and functions $\psi \in C_0^\infty(B)$, and a *weighted Sobolev-Poincaré inequality* is an inequality

$$(1.2)\qquad \inf_{a\in\mathsf{R}} \left(\frac{1}{w(B)}\int_B |\psi - a|^{\kappa p} w\,dx\right)^{1/\kappa p} \leqq C \operatorname{diam} B \left(\frac{1}{w(B)}\int_B |\nabla\psi|^p w\,dx\right)^{1/p},$$

valid for all balls $B$ and bounded functions $\psi \in C^\infty(B)$; the constants $\kappa > 1$ and $C > 0$ should be independent of $B$ and $\psi$. Here and throughout we use the notation

$$w(E) = \int_E w(x)\,dx, \quad |E| = \int_E dx,$$

for a measurable subset $E$ of $\mathsf{R}^n$; thus $|E|$ *is the Lebesgue* $n$-measure of $E$.

For the constant weight $w \equiv 1$ both inequalities (1.1) and (1.2) hold for $\kappa = n/(n-p)$. This is due to Sobolev. More recently, Fabes, Kenig, and Serapioni [5] verified (1.1) and (1.2) for two important classes of weights: $w$ can be an $A_p$-weight of Muckenhoupt or of the form $w(x) = J(x,f)^{1-p/n}$, where $J(x,f)$ is the Jacobian of a quasiconformal mapping $f\colon \mathsf{R}^n \to \mathsf{R}^n$. The relevance of the second

---

* Both authors supported in part by NSF and the Academy of Finland. The first author is a Sloan Fellow.

Received April 28, 1994.

example comes from two sources. First, composing a harmonic function with a quasiconformal mapping $f$ yields a solution to a linear elliptic equation whose degeneracy is given in terms of the weight $w(x) = J(x, f)^{1-2/n}$, and one can reduce the study of harmonic functions in certain nonsmooth domains to the study of solutions to degenerate equations in their smooth quasiconformal images. Second, the use of quasiconformal mappings provides a relatively easy proof for the fact that (1.1) holds for many interesting weights such as $w(x) = |x|^{\gamma}, \gamma > -n$. We refer to [10] for an expostion of these ideas, and for references to numerous earlier works.

It has been asked in [18], [19] whether the four conditions described in [5] hold for the weight $w(x) = J(x, f)^{1-p/n}$, where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a quasiregular mapping. These conditions are needed in running the familiar Moser iteration (giving Harnack type inequalities for weak solutions to related degenerate elliptic equations) and they are: the weighted inequalities (1.1) and (1.2) (in fact, (1.2) is only needed with $\kappa = 1$), the doubling condition (1.3), and the uniqueness of the gradient in the pertinent weighted Sobolev space. Recall that a weight $w$ is called *doubling* if there is a constant $C > 0$ such that

$$w(2B) \leqq Cw(B) \tag{1.3}$$

for all balls $B$ in $\mathbb{R}^n$; we use the standard notation $\lambda B$ for the ball with same center as $B$ but radius dilated by $\lambda > 0$.

In [10, p. 7–8], a weight is declared $p$-admissible if the above four conditions are satisfied for a fixed $p$, and, again, inequality (1.2) is assumed to hold for $\kappa = 1$ only. We shall show in this paper that for a quasiregular mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the weight $w(x) = J(x, f)^{1-p/n}$ is $p$-admissible if and only if $f$ is of polynomial type (see the definition below). In proving this, our task is to verify (1.1), (1.2), and the doubling condition (1.3). The uniqueness of the gradient was proved earlier by Øksendal [19, Lemma 2.1]. In Section 5 we shall show in fact that the uniqueness of the gradient is a consequence of the Poincaré inequality (1.2) (with $\kappa = 1$). This was pointed out to us by Semmes [23], and we are grateful to him for allowing us to include his proof in this paper.

Recall that a continuous mapping $f\colon \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open, is *quasiregular* if $f$ belongs to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega)$ and satisfies

$$|Df(x)|^n \leqq KJ(x, f) \tag{1.4}$$

for a.e. $x \in \Omega$ and for some $K \geqq 1$. The infimal $K$ for which (1.4) holds will be referred to as the dilatation of $f$ and denoted by $K(f)$. Thus $f$ is quasiconformal if and only if $f$ is quasiregular and injective. If $n = 2$ and $K = 1$ in (1.4), we recover precisely the analytic functions of one complex variable.

CONVENTION. We assume that all quasiregular mappings $f\colon \Omega \to \mathsf{R}^n$ in this paper are nonconstant in any of the components of an open set $\Omega$.

DEFINITION. A quasiregular mapping $f\colon \mathsf{R}^n \to \mathsf{R}^n$ is said to be of *polynomial type* if $|f(x)| \to \infty$ as $|x| \to \infty$.

For a mapping $f\colon A \to \mathsf{R}^n$ and $y \in \mathsf{R}^n$ we define the (crude) multiplicity of $f$ in $A$ by

$$N(y, f, A) = \#\{f^{-1}(y) \cap A\} \in [0, \infty].$$

We also put

$$N(f, A) = \sup_{y \in \mathsf{R}^n} N(y, f, A).$$

Our first result is a characterization of polynomial type quasiregular mappings.

1.5. THEOREM. *Let $f\colon \mathsf{R}^n \to \mathsf{R}^n$ be a quasiregular mapping. Then the following are equivalent:*

(1) *$J(x, f)$ is doubling;*
(2) *$J(x, f)^s$ is doubling for some $0 < s \leqq 1$;*
(3) *$J(x, f)$ is an $A_\infty$-weight in $\mathsf{R}^n$;*
(4) *$f$ is of polynomial type;*
(5) *$N(f, \mathsf{R}^n)$ is finite;*
(6) *$J(x, f)$ is a strong $A_\infty$-weight in $\mathsf{R}^n$.*

The associated constants depend only on each other, and on $n$ and $K(f)$.

The equivalence of (4) and (5) in Theorem 1.5 is well known and it is included in the theorem only to emphasize the connection between the doubling property of $J(x, f)$ and the multiplicity of $f$. For instance, the derivative of the complex polynomial $f(z) = z^N$ is doubling but with a constant depending on $N$.

By an $A_\infty$-weight we mean a weight that belongs to the Muckenhoupt class $A_q(\mathsf{R}^n)$ for some $q > 1$. Recall that $A_q(\mathsf{R}^n)$ consists of all weights $w$ for which there exists a constant $C > 0$ such that

$$\frac{1}{|B|}\int_B w(x)\,dx \leqq C\left(\frac{1}{|B|}\int_B w(x)^{1/(1-q)}\,dx\right)^{1-q}$$

for all balls $B$ in $\mathsf{R}^n$.

David and Semmes in [3] (see also [22]) introduced the class of strong $A_\infty$-weights, which we next describe. For each $A_\infty$-weight $w$ one can associate a quasidistance $\delta_w(x, y)$ by setting

$$\delta_w(x, y) = \left(\int_{B_{x,y}} w(z)\,dz\right)^{1/n},$$

where $B_{x,y}$ is the smallest (closed) ball that contains both $x$ and $y$. Now $\delta_w(\cdot,\cdot)$ is not generally a metric in $\mathbb{R}^n$, and we say that $w$ is a *strong* $A_\infty$*-weight* if $\delta_w$ is equivalent to a distance function, that is, there is a function $\delta'$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\delta'$ defines a metric in $\mathbb{R}^n$ and that

$$C^{-1}\delta_w(x,y) \leqq \delta'(x,y) \leqq C\delta_w(x,y)$$

for all $x, y \in \mathbb{R}^n$ and for some $C > 0$ independent of $x$ and $y$. A fundamental result of Gehring [7] says that the Jacobian $J(x,f)$ of a quasiconformal mapping $f$: $\mathbb{R}^n \to \mathbb{R}^n$ is an $A_\infty$-weight, and by using basic distortion results for quasiconformal mappings it is not difficult to see that $J(x,f)$ is a strong $A_\infty$-weight. David and Semmes arrived at the concept of a strong $A_\infty$-weight in their study of parametrizations of surfaces. They asked whether for a given strong $A_\infty$-weight $w$ there is a quasiconformal self-mapping of $\mathbb{R}^n$ whose Jacobian is comparable in size to $w$. Besides being a beautiful "existence" theorem, an affirmative answer would provide nice parametrizations for a large class of $n$-dimensional surfaces inside $\mathbb{R}^m$, $m > n$.

The answer to this question turned out to be "no". Semmes has recently constructed examples of strong $A_\infty$-weights in $\mathbb{R}^3$ which cannot be comparable to the Jacobian of a quasiconformal mapping. As a consequence of Theorem 1.5, one could try to construct more such examples by exhibiting a quasiregular mapping of polynomial type in $\mathbb{R}^n$ whose Jacobian is not comparable to that of a quasiconformal mapping. Unfortunately, this does not seem to make the task easier, and we believe in fact that no such examples exist. In the plane this nonexistence can easily be proved by invoking the Beltrami equation.

As an application of Theorem 1.5, we prove the following result. Unless otherwise stated, all the balls in this paper are open, and we write $B(x,r)$ for the ball centered at $x$ with radius $r$.

1.6. THEOREM. *Let* $f$: $\mathbb{R}^n \to \mathbb{R}^n$ *be quasiregular mapping of polynomial type. Then the image* $f(B)$ *of each ball* $B = B(x,r)$ *is a John domain with John center* $f(x)$ *and constant depending only on* $n$, $K(f)$, *and* $N(f,\mathbb{R}^n)$.

We recall that a domain $D$ in $\mathbb{R}^n$ is a *John domain* with constant $c \geqq 1$ if there is a point $x_0$ in $D$, called the *John center*, such that each point $x \in D$ can be joined to $x_0$ by an arc (equivalently, by a path) $\gamma$ satisfying

$$\operatorname{diam} \gamma[x,y] \leqq c \operatorname{dist}(y, \partial D) \tag{1.7}$$

for all $y \in \gamma$, where $\gamma[x,y]$ denotes the subpath of $\gamma$ between $x$ and $y$.

It is obvious that Theorem 1.6 cannot be improved much. First, the image of the ball $B((1,0),1) \subset \mathbb{R}^2$ under the complex polynomial $z \mapsto z^2$ is a John domain but not a quasidisk. Second, let $f(z) = e^z$ in the plane. Then the John constant of $f(B(0,\lambda))$, with center $f(0)$, tends to infinity as $\lambda \to \infty$.

David and Semmes established in [3] weighted Sobolev and Poincaré inequalities for strong $A_\infty$-weights; these inequalities involve different powers of weights on different sides. The argument of David and Semmes was used and extended by Franchi, Gutiérrez and Wheeden in [6] to produce more examples of weights satisfying Poincaré type inequalities. We exhibit a simple argument in Proposition 4.1 below which reduces (1.1) to the results in [3] and (1.2) to the results in [6], in case $w$ is an appropriate power of a strong $A_\infty$-weight. Notice that [3] contains a weighted Poincaré inequality that does not seem to be general enough for our purposes. In the case of quasiregular mappings, we provide a different, more elementary approach to the Sobolev inequality.

1.8. THEOREM. *Suppose that $v$ is a strong $A_\infty$-weight. Then* (1.1) *and* (1.2) *hold for $w = v^{1-p/n}$. The constant $C$ depends only on $n$, $p$, and on the constants associated with $v$, and one can choose*

$$\kappa = \frac{p + q(n-p)}{q(n-p)},$$

*where $q > 1$ is such that $v \in A_q(\mathbb{R}^n)$.*

1.9. COROLLARY. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular mapping of polynomial type. There are constants $\kappa > 1$ and $C > 0$, depending only on $n$, $p$, $K(f)$, and $N(f, \mathbb{R}^n)$ such that* (1.1) *and* (1.2) *hold for $w(x) = J(x,f)^{1-p/n}$.*

Observing the uniqueness of the gradient from Section 5, we have yet another corollary:

1.10. COROLLARY. *Let $v$ be a strong $A_\infty$-weight. Then $w = v^{1-p/n}$ is $p$-admissible. In particular, for a quasiregular mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the weight $w(x) = J(x,f)^{1-p/n}$ is $p$-admissible if and only if $f$ is of polynomial type.*

The above theorems should be viewed as the main results of this note. In the following we discuss inequalities which involve two weights; these results are partially known.

1.11. THEOREM. *Let $f\, \Omega \to \mathbb{R}^n$ be a quasiregular mapping. Then there is a constant $C > 0$, depending only on $n$, $p$, and $K(f)$, such that*

$$\int_\Omega |\psi(x)|^p J(x,f)\,dx \leqq C(\operatorname{diam} f(\Omega))^p \int_\Omega |\nabla\psi(x)|^p J(x,f)^{1-p/n}\,dx \tag{1.12}$$

*whenever $\psi \in C_0^\infty(\Omega)$.*

For $p = 2$ Theorem 1.11 was first proved by Øksendal [19] with an argument based on stochastic and Hilbert space methods not available for $p \neq 2$. Recently, Moscariello [15] pointed out that (1.12) easily follows from a basic inequality in the theory of quasiregular mappings [1, (3.4)]. We provide a different, geometric argument which also allows us to verify a weighted Sobolev inequality with "correct" Sobolev exponent on the left hand side, provided that $f$ has finite multiplicity.

1.13. THEOREM. *Let $f\colon \Omega \to \mathsf{R}^n$ be a quasiregular mapping with $N(f, \Omega) = N < \infty$. Then there is a constant $C > 0$, depending only on $n$, $p$, $K(f)$ and $N$, such that*

$$(1.14)\quad \left(\int_\Omega |\psi(x)|^{pn/(n-p)} J(x, f)\, dx\right)^{(n-p)/pn} \leqq C\left(\int_\Omega |\nabla\psi(x)|^p J(x, f)^{1-p/n}\, dx\right)^{1/p}$$

*whenever $\psi \in C_0^\infty(\Omega)$.*

Assuming Theorem 1.5, inequality (1.14) follows from the main theorem in [3] in the case $\Omega = \mathsf{R}^n$. Our argument here is different and more elementary than that in [3]. The crucial observation is that even in the case of noninjective quasiregular mappings $f\colon \Omega \to \mathsf{R}^n$ a function $\psi$ in $\Omega$ can be "pushed" to a function $\psi^*$ in $f(\Omega)$ by an averaging method. Then we can mimic the proof in the case of quasiconformal mappings, where inequality (1.14) readily follows from a change of variables and the usual Sobolev inequality. The push-forward function was apparently first used by Martio in [14]. For later applications in connection with mappings and partial differential equations, see [9], [17], [10].

The results in this paper were obtained for the most part when the second author was visiting the University of Michigan in the fall of 1991. Some of the theorems and methods here found applications in a recent work [12], thus giving us new motivation to write up our observations.

ADDED IN APRIL 1994. Hajłasz and Koskela have recently verified that for a doubling weight the Poincaré inequality (5.1) in section 5 implies both the Sobolev inequality (1.1) and the Sobolev-Poincaré inequality (1.2). Thus, in light of this and Semmes's uniqueness result, we see that two of the four conditions that define $p$-admissibility in the sense of [10] are implied by the other two: the doubling condition and the Poincaré inequality.

## 2. Preliminaries.

Let $f\colon \Omega \to \mathsf{R}^n$ be a quasiregular mapping. According to a well-known theorem of Reshetnyak [20, II 6.3], $f$ is an open and discrete mapping, the latter meaning that $f^{-1}(y)$ consists of isolated points for each $y \in \mathsf{R}^n$. In particular, $N(f, A) < \infty$ whenever $A \subset\subset \Omega$.

An open, connected neighborhood $U \subset\subset \Omega$ of a point $x \in \Omega$ is called a *normal neighborhood* of $x$ if $f(\partial U) = \partial f(U)$ and $U \cap f^{-1}(f(x)) = \{x\}$. Denote by

$U(x, f, r)$ the $x$-component of $f^{-1}(B(f(x), r))$. For the following lemma, see [16, Lemma 2.9] or [21, I 4.9].

2.1. LEMMA. *For each point $x \in \Omega$ there is $\sigma_x > 0$ such that $U(x, f, r)$ is a normal neighborhood of $x$ whenever $0 < r \leqq \sigma_x$. Moreover,* diam $U(x, r, f) \to 0$ *as* $r \to 0$.

Next suppose that $x \in \Omega$ and $0 < r \leqq \sigma_x$, where $\sigma_x$ is as in Lemma 2.1. The local topological index $i(x, f)$ of $f$ at $x$ can be defined by

$$i(x, f) = N(f, U(x, f, r)).$$

Note in particular that $N(f, U(x, f, r))$ is independent of $r \leqq \sigma_x$. Thus, $i(x, f) = 1$ if and only if $x \in \Omega \setminus B_f$, where $B_f$ is the branch set of $f$. We refer the reader to [16, Section 2], [10, Chapter 14], or [21, Chapter I] for this discussion.

A proof of the next lemma can be found in [10, 14.27].

2.2. LEMMA. *Suppose that $U = U(x_0, f, r)$ is a normal neighborhood of $x_0 \in \Omega$. Then*

$$\int_U u(f(x)) J(x, f)\, dx = i(x, f) \int_{f(U)} u(y)\, dy$$

*for each nonnegative measurable function $u$ in $f(U)$.*

We also have a more general change of variables formula for quasiregular mappings $f\colon \Omega \to \mathsf{R}^n$:

$$\int_A J(x, f)\, dx = \int_{\mathsf{R}^n} N(y, f, A)\, dy \tag{2.3}$$

whenever $A \subset \Omega$; see [20, p. 99] or [21, I4.14].

## 3. $A_\infty$ and quasiregular mappings.

In this section we prove Theorems 1.5 and 1.6. So we assume that $f\colon \mathsf{R}^n \to \mathsf{R}^n$ is a quasiregular mapping with dilatation $K = K(f)$.

First recall that the Jacobian of $f$ satisfies the weak reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B J(x, f)^{1+\varepsilon}\, dx\right)^{1/(1+\varepsilon)} \leqq C \frac{1}{|2B|} \int_{2B} J(x, f)\, dx \tag{3.1}$$

whenever $B \subset \mathsf{R}^n$ is a ball. The constants $\varepsilon > 0$ and $C > 0$ depend only on $n$ and $K$. For a proof, see [1, Theorem 5.1]. In fact, 2 can be replaced with any $\lambda > 1$ in which case $\varepsilon$ and $C$ depend also on $\lambda$. It is well known that inequality (3.1) improves itself to

(3.2) $$\left(\frac{1}{|B|}\int_B J(x,f)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leqq C(n,K,s)\left(\frac{1}{|2B|}\int_{2B} J(x,f)^s\,dx\right)^{1/s}$$

for any $s > 0$, where $\varepsilon > 0$ is the same as in (3.1). See [13, Theorem 2], or [10, 3.38].

PROOF OF THEOREM 1.5. We let $C$ denote various constants depending only on the parameters in the statement of the theorem.

The implication (1) $\Rightarrow$ (2) is trivial, and to prove the implication (2) $\Rightarrow$ (3) we use (3.2), the assumption, and Hölder's inequality to obtain

$$\left(\frac{1}{|B|}\int_B J(x,f)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leqq C\left(\frac{1}{|B|}\int_{2B} J(x,f)^s\,dx\right)^{1/s}$$
$$\leqq C\left(\frac{1}{|B|}\int_B J(x,f)^s\,dx\right)^{1/s} \leqq C\frac{1}{|B|}\int_B J(x,f)\,dx$$

for all balls $B \subset \mathsf{R}^n$. It is well known that this implies that $J(x,f)$ is an $A_\infty$-weight in $\mathsf{R}^n$; see [2].

Next we prove the implication (3) $\Rightarrow$ (4): We may assume that $f(0) = 0$. The derivative $Df$ belongs locally to $L^q$ for some $q = q(n,K) > n$ by the weak reverse Hölder inequality (3.1) and quasiregularity of $f$. Hence Sobolev's embedding theorem together with (3.1) yields

$$|f(x)| = |f(x) - f(0)| \leqq C|x|^{1-n/q}\left(\int_B |Df(y)|^q\,dy\right)^{1/q}$$
$$\leqq C\,\mathrm{diam}\,B\left(\frac{1}{|B|}\int_B |Df(y)|^q\,dy\right)^{1/q}$$
$$\leqq C\,\mathrm{diam}\,B\left(\frac{1}{|2B|}\int_{2B} J(y,f)\,dy\right)^{1/n},$$

where $B$ is any ball centered at the origin and containing $x$. This combined with the $A_\infty$-condition

(3.3) $$\frac{|B(0,1)|}{|B(0,R)|} \leqq C\left(\frac{w(B(0,1))}{w(B(0,R))}\right)^\alpha, \quad a > 0,$$

where $w = J(x,f)$ (see [2, Lemma 5] or [10, 15.5]), implies

$$\max_{|x|\leqq R} |f(x)| \leqq Cw(B(0,2R))^{1/n} \leqq CR^{1/\alpha}.$$

We conclude that $f$ grows at most polynomially:

$$|f(x)| \leqq C|x|^{1/\alpha}, \quad |x| \to \infty.$$

Now we invoke a result of Väisälä [25, Theorem 4.2] which implies that $\infty$ cannot be an essential singularity of $f$, and, in particular, $|f(x)| \to \infty$ as $|x| \to \infty$.

The implication (4) $\Rightarrow$ (5) is well known; see [21, I.4].

The implication (5) $\Rightarrow$ (6) is immediate in view of the following sufficient condition for $w$ to be a strong $A_\infty$-weight: there is a mapping $\rho$: $\mathsf{R}^n \to \mathsf{R}^n$ such that $\rho \in W^{1,1}_{\mathrm{loc}}(\mathsf{R}^n)$, that $|D\rho(x)|^n \leqq w(x)$ for a.e. $x$, and that

$$w(\rho^{-1}(B(x,R))) \leqq CR^n \tag{3.4}$$

for some constant $C > 0$ independent of $x$ and $R > 0$. This is proved in [3, 1.8]. Therefore, we only need to check the validity of (3.4) for $\rho = f$. But this readily follows from the change of variables formula (2.3) because $N(f, \mathsf{R}^n) < \infty$.

Because (6) trivially implies (3), and hence (5), it remains to establish (5) $\Rightarrow$ (1). We have by (2.3)

$$\int_{2B} J(x,f)\,dx = \int_{f(2B)} N(y,f,2B)\,dy \leqq N(f,\mathsf{R}^n)\,|f(2B)|$$

and

$$\int_B J(x,f)\,dx = \int_{f(B)} N(y,f,B)\,dy \geqq |f(B)|,$$

and therefore it suffices to show that

$$|f(2B)| \leqq C\,|f(B)| \tag{3.5}$$

whenever $B \subset \mathsf{R}^n$ is a ball.

To this end, fix a ball $B$, centered at $x$. Since $f$ is open, $f(x)$ is an interior point of the open connected set $U = f(B)$. Let $d = \operatorname{dist}(f(x), \partial U)$ and set

$$r = \max_{y \in \partial U'} |f(x) - y|,$$

where $U' = f(2B)$. Then $2r \geqq \operatorname{diam}(U')$. Next let $L_0$ be a line segment of length $d$ joining $f(x)$ to $\partial U$ and let $L_1$ be a continuum in $\overline{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ joining a point in $\partial B(f(x), r) \cap \partial U'$ to $\infty$ in $\overline{\mathsf{R}}^n \setminus B(f(x), r)$. Then some lifts of $L_0$ and $L_1$ join $x$ to $\partial B$, and $\partial 2B$ to $\infty$, respectively, and hence

$$\operatorname{mod}(f^{-1}(L_0), f^{-1}(L_1); 3B) \geqq C > 0,$$

where $\operatorname{mod}(A, B; D)$ refers to the usual conformal modulus between sets $A$ and $B$ in $D$. On the other hand, the quasi-invariance of modulus (see e.g. [26, 10.13]) gives

$$\operatorname{mod}(f^{-1}(L_0), f^{-1}(L_1); 3B) \leqq KN(f, \mathbf{R}^n)\operatorname{mod}(L_0, L_1; \mathbf{R}^n) \leqq C\left(\log\frac{r}{d}\right)^{1-n}$$

We obtain $r \leqq Cd$, which shows that $f(B)$ and $f(2B)$ have comparable volumes, as desired.

3.6. REMARKS. (a) The proof of the implication "(5) $\Rightarrow$ (1)" above shows the following stronger conclusion: Let $f\colon \lambda B \to \mathbf{R}^n$, $\lambda > 1$, be a quasiregular mapping with $N = N(f, \lambda B) < \infty$. Then there is an exponent $\varepsilon > 0$ and a constant $C > 0$, depending only on $n$, $K(f)$, $N$, and $\lambda$, such that

$$\left(\frac{1}{|B|}\int_B J(x,f)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leqq C\frac{1}{|B|}\int_B J(x,f)\,dx.$$

Moreover, the doubling (3.5) holds in this case with 2 replaced with $(1+\lambda)/2$, and we have that

$$C^{-1}\operatorname{diam} f(B) \leqq \left(\int_B J(x,f)\,dx\right)^{1/n} \leqq C\operatorname{diam} f(B), \tag{3.7}$$

where, as above, $C > 0$ depends only on $n$, $K(f)$, $N$, and $\lambda$.

(b) If $f\colon \Omega \to \mathbf{R}^n$ is a quasiregular maping, defined in a proper subset $\Omega$ of $\mathbf{R}^n$, then the question when $J(x,f) \in A_\infty(\Omega)$ is much more delicate even for injective mappings. By $J(x,f) \in A_\infty(\Omega)$ we mean that a reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B J(x,f)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leqq C\frac{1}{|B|}\int_B J(x,f)\,dx$$

holds for each ball $B \subset \Omega$. (The answer here also depends whether we use cubes instead of balls.) By using [16, Lemma 4.3] and [14] one can show that $J(x,f) \in A_\infty(\Omega)$ only if there is a uniform bound for the local index; that is $\sup_{x\in\Omega} i(x,f) \leqq N$ for some $N < \infty$. On the other hand, such a bound does not in general imply $J(x,f) \in A_\infty(\Omega)$; consider, for instance, the analytic function $f(z) = e^z$ in the plane. This problem for homeomorphic quasiregular mappings was studied in [11].

PROOF OF THEOREM 1.6. Let $B \subset \mathbf{R}^n$ be a ball centered at $x_0$, and write $U = f(B)$, $y_0 = f(x_0)$. Pick a point $y \in U$ and let $x \in B$ be such that $f(x) = y$. The proof of the implication (5) $\Rightarrow$ (1) in Theorem 1.5 gives that

$$\operatorname{diam} f([x, x_0]) \leqq C\operatorname{dist}(f(x_0), \partial f(B_0)), \tag{3.8}$$

where $B_0 = B(x_0, |x - x_0|) \not\subset B$ and $C > 0$ depends only on $n$, $K(f)$, and $N(f, \mathbf{R}^n)$. In fact, (3.8) remains valid equally well when we replace $x_0$ with any point $z$ on the radius $[x, x_0]$ and $B_0$ with the ball $B_z = B(z, |x - z|) \subset B$. Then

$$\operatorname{diam} f([x,z]) \leqq C \operatorname{dist}(f(z), \partial f(B_z)) \leqq C \operatorname{dist}(f(z), \partial U),$$

and we infer that the path $f([x, x_0])$ will satisfy (1.7). The theorem follows.

## 4. Weighted Sobolev and Sobolev-Poincaré inequalities.

We first prove Theorem 1.8. The following simple proposition is crucial.

4.1. PROPOSITION. *Suppose that $v \in A_q(\mathsf{R}^n)$ and suppose that*

$$\left(\int_B |g(x)|^{np/(n-p)} v(x)\,dx\right)^{(n-p)/np} \leqq C_1 \left(\int_B |h(x)|^p v(x)^{1-p/n}\,dx\right)^{1/p} \tag{4.2}$$

*for some ball $B$ in $\mathsf{R}^n$ and for some $g$ and $h$ in $L^\infty(B)$. Then there is a constant $C_2 > 0$, depending only on $n$, $p$, $C_1$, and the $A_\infty$ constants of $v$, such that*

$$\left(\frac{1}{w(B)} \int_B |g(x)|^{\kappa p} w(x)\,dx\right)^{1/\kappa p} \leqq C_2 \operatorname{diam} B \left(\frac{1}{w(B)} \int_B |h(x)|^p w(x)\,dx\right)^{1/p}, \tag{4.3}$$

*where $w(x) = v(x)^{1-p/n}$ and*

$$\kappa = \frac{p + q(n-p)}{q(n-p)}. \tag{4.4}$$

PROOF. Let $\kappa > 1$ be as in (4.4). Then

$$\int_B |g|^{\kappa p} v^{1-p/n}\,dx \leqq \left(\int_B |g|^{np/(n-p)} v\,dx\right)^{\kappa(n-p)/n} \times \left(\int_B v^s\,dx\right)^{(\kappa p - (\kappa-1)n)/n}, \tag{4.5}$$

where

$$s = \frac{(1-\kappa)(n-p)}{\kappa p - (\kappa - 1)n} = \frac{1}{1-q} < 0.$$

Because

$$\left(\frac{1}{|B|}\int_B v^{1-p/n}\,dx\right)^{n/(n-p)} \leqq \frac{1}{|B|}\int_B v\,dx \leqq C\left(\frac{1}{|B|}\int_B v^s\,dx\right)^{1/s},$$

we obtain

$$\left(\int_B v^s\,dx\right)^{(\kappa p-(\kappa-1)n)/n} \leqq C|B|^{\kappa p/n}\left(\int_B v^{1-p/n}\,dx\right)^{1-\kappa} \leqq C(\operatorname{diam} B)^{\kappa p} w(B)^{1-\kappa}.$$

Therefore (4.5) and the assumption imply

$$\left(\int_B |g|^{\kappa p} v^{1-p/n}\,dx\right)^{1/\kappa p} \leqq C\left(\int_B |g|^{np/(n-p)} v\,dx\right)^{(n-p)/np} \operatorname{diam} B w(B)^{1/\kappa p - 1/p}$$

$$\leqq C\left(\int_B |h|^p v^{1-p/n}\,dx\right)^{1/p} \operatorname{diam} B w(B)^{1/\kappa p - 1/p},$$

from which the assertion follows.

PROOF OF THEOREM 1.8. We apply Proposition 4.1 and results in [3] and [6]. Thus, let $v$ be a strong $A_\infty$-weight. For $\psi \in C_0^\infty(B)$ inequality (4.2) with $g = \psi$ and $h = \nabla\psi$ was established in the main theorem of [3] (with constant $C_1$ independent of $\psi$ and $B$, of course). In the case $v$ is the Jacobian of a polynomial type quasiregular mapping, this is Theorem 1.13. Hence (1.1) follows from Proposition 4.1.

Next, we show that for a given $B$ and bounded $\psi \in C^\infty(B)$, there is $a \in \mathsf{R}$ such that (4.2) holds with $g = \psi - a$ and $h = \nabla\psi$, where, again, $C_1$ is independent of $\psi$ and $B$.

To this end, consider first the inequality

$$\left(\int_B |\psi - \psi_B|^{n/(n-1)} v\,dx\right)^{(n-1)/n} \leqq C\int_B |\nabla\psi|\, v^{1-1/n}\,dx, \tag{4.6}$$

where $\psi_B$ is the ($v$-)weighed average of $\psi$ in $B$. This inequality follows from Theorem I in [6] with the choices $q = n/(n-1)$ and $p = 1$, provided the following two conditions hold:

$$\frac{\operatorname{diam} B}{\operatorname{diam} B_0}\left(\frac{\int_B v}{\int_{B_0} v}\right)^{(n-1)/n} \leqq C\frac{\int_B v^{1-1/n}}{\int_{B_0} v^{1-1/n}} \tag{4.7}$$

For all balls $B$ and $B_0$ with $B \subset B_0$, and there is a strong $A_\infty$-weight $w$ such that

$$v^{1-1/n} w^{-(1-1/n)} \in A_1(w^{1-1/n}\,dx). \tag{4.8}$$

Here the notation is that of [6]. Now (4.8) is trivially true, as we can choose $w = v$, and (4.7) follows upon observing that

$$\frac{1}{|B|}\int_B v^s \approx \left(\frac{1}{|B|}\int_B v\right)^s$$

for all $0 < s \leqq 1$ whenever $v$ is an $A_\infty$-weight. Thus (4.6) holds.

The proof is now finished by standard arguments. For convenience, we provide the details. Write

$$\phi = \max(\psi - a, 0)^s - \max(a - \psi, 0)^s,$$

where $s = p(n-1)/(n-p) > 1$ and $a \in \mathsf{R}$ is chosen such that

$$\int_B \phi v \, dx = 0.$$

Then by (4.6) and Hölder's inequality, we have

$$\left(\int_B |\psi - a|^{np/(n-p)} v \, dx\right)^{(n-1)/n} = \left(\int_B |\phi|^{n/(n-1)} v \, dx\right)^{(n-1)/n}$$

$$\leqq C \int_B |\nabla\psi| \, |\psi - a|^{s-1} v^{1-1/n} \, dx$$

$$\leqq C \left(\int_B |\nabla\psi|^p v^{(n-p)/n} \, dx\right)^{1/p} \left(\int_B |\psi - a|^{np/(n-p)} v \, dx\right)^{(p-1)/p},$$

from which (4.2) follows by dividing.

This gives us (1.2) by Proposition 4.1, and the proof of Theorem 1.8 is complete.

It remains to establish Theorems 1.11 and 1.13. As cited in the introduction, these results are fairly easily proved for quasiconformal mappings via a change of variables argument. We intend to mimic this idea in the case of non-injective mappings.

Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a quasiregular mapping with dilatation $K$. For a function $\psi \in C_0(\Omega)$ we define

$$\psi^*(y) = \sum_{x \in f^{-1}(y)} i(x, f)\psi(x).$$

Because $f$ is discrete and because $\operatorname{spt}(\psi)$ is compact, the sum contains only a finite number of terms.

4.9. Lemma. *If $\psi \in C_0(\Omega)$ is Lipschitz continuous, then $\psi^* \in C_0(f(\Omega)) \cap W_0^{1,n}(f(\Omega))$.*

Proof. This is proved in [14] and in [10, 14.30, 14.31]. The careful reader notices that in these sources it is assumed that $\psi \in C_0^\infty(\Omega)$. An examination of the proof reveals, however, that only the Lipschitz character of $\psi$ is needed.

4.10. Lemma. *If $\psi \in C_0(\Omega)$ is nonnegative and Lipschitz continuous, then*

$$\int_\Omega \psi(x) J(x, f) \, dx \leqq C(n, K) \operatorname{diam} f(\Omega) \int_\Omega |\nabla\psi(x)| \, J(x, f)^{1-1/n} \, dx.$$

Proof. Since $f$ is open and since $\psi^* \in W_0^{1,1}(f(\Omega))$ by Lemma 4.9, the usual Sobolev inequality implies

$$\int_{f(\Omega)} \psi^*(y)\,dy \leqq C(n)\operatorname{diam} f(\Omega) \int_{f(\Omega)} |\nabla\psi^*(y)|\,dy. \tag{4.11}$$

Fix $y_0 \in \operatorname{spt}(\psi^*)\backslash f(B_f \cap \operatorname{spt}(\psi))$ and let $\{x_1,\ldots,x_k\} = f^{-1}(y_0) \cap \operatorname{spt}(\psi)$. By Lemma 2.1 there exists $r_0 > 0$ such that

$$f^{-1}(B(y_0,r_0)) \cap \operatorname{spt}(\psi) \subset \bigcup_{j=1}^{k} U_j,$$

where $U_j = U(x_j, f, r_0)$ are pairwise disjoint normal neighborhoods of $x_j$ with $i(x_j, f) = 1$. Thus $f_j = f \mid U_j$ is a $K$-quasiconformal mapping of $U_j$ onto $B(y_0, r_0)$, and for each $y \in B(y_0, r_0)$,

$$\psi^*(y) = \sum_{j=1}^{k} \psi(f_j^{-1}(y)).$$

Because the inverse of a $K$-quasiconformal mapping is $K^{n-1}$-quasiconformal, we now have

$$\begin{aligned}
\int_{B(y_0,r_0)} |\nabla\psi^*(y)|\,dy &\leqq \sum_{j=1}^{k} \int_{B(y_0,r_0)} |\nabla(\psi(f_j^{-1}(y))|\,dy \\
&\leqq \sum_{j=1}^{k} \int_{B(y_0,r_0)} |\nabla\psi(f_j^{-1}(y))|\,|Df_j^{-1}(y)|\,dy \\
&\leqq K^{(n-1)/n} \sum_{j=1}^{k} \int_{B(y_0,\,r_0)} |\nabla\psi(f_j^{-1}(y))|\,J(y, f_j^{-1})J(y, f_j^{-1})^{1/n-1}\,dy \\
&= K^{(n-1)/n} \sum_{j=1}^{k} \int_{U_j} |\nabla\psi(x)|\,J(x, f)^{1-1/n}\,dx \\
&= K^{(n-1)/n} \int_{f^{-1}(B(y_0,r_0))} |\nabla\psi(c)|\,J(x, f)^{1-1/n}\,dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{f^{-1}(B(y_0,r_0))} \psi(x)J(x,f)\,dx &= \sum_{j=1}^{k} \int_{B(y_0,r_0)} \psi(f_j^{-1}(y))\,dy \\
&= \int_{B(y_0,r_0)} \psi^*(y)\,dy.
\end{aligned}$$

Now Lemma 2.1 and the Vitali covering theorem permit us to decompose the set $\operatorname{spt}(\psi^*)\backslash f(B_f \cap \operatorname{spt}(\psi))$ to a pairwise disjoint union of balls $\{B(y_1, r_1), B(y_2, r_2), \ldots\}$ such that each $B(y_i, r_i)$ has the properties of $B(y_0, r_0)$ as indicated above and that

$$\operatorname{spt}(\psi^*)\backslash f(B_f \cap \operatorname{spt}(\psi)) \subset \cup_{i=1}^{\infty} B(y_i, r_i) \cup A,$$

where $|A| = 0$. Consequently, since $|B_f| = 0$ and since for nonconstant quasiregular mappings $|f(E)| = 0$ if and only if $|E| = 0$, we have that

$$\operatorname{spt}(\psi^*) \subset \cup_{i=1}^{\infty} B(y_i, r_i) \cup A',$$

where $|A'| = 0$.

The proof is now finished by collecting the facts proved above. Indeed,

$$\begin{aligned}
\int_{\Omega} \psi(x) J(x, f)\, dx &= \sum_{i=1}^{\infty} \int_{f^{-1}(B(y_i, r_i))} \psi(x) J(x, f)\, dx \\
&= \sum_{i=1}^{\infty} \int_{B(y_i, r_i)} \psi^*(y)\, dy = \int_{f(\Omega)} \psi^*(y)\, dy \\
&\leqq C \operatorname{diam} f(\Omega) \int_{f(\Omega)} |\nabla \psi^*(y)|\, dy = \operatorname{diam} f(\Omega) \sum_{i=1}^{\infty} \int_{B(y_i, r_i)} |\nabla \psi^*(y)|\, dy \\
&\leqq C \operatorname{diam} f(\Omega) \sum_{i=1}^{\infty} \int_{f^{-1}(B(y_i, r_i))} |\nabla \psi(x)|\, J(x, f)^{1-1/n}\, dx \\
&= C \operatorname{diam} f(\Omega) \int_{\Omega} |\nabla \psi(x)|\, J(x, f)^{1-1/n}\, dx.
\end{aligned}$$

The lemma follows.

PROOF OF THEOREM 1.11. Given $\psi \in C_0^{\infty}(\Omega)$ we apply Lemma 4.10 to the Lipschitz continuous function $\psi = |\psi|^p$ to obtain

$$\int_{\Omega} |\psi(x)|^p J(x, f)\, dx \leqq C \operatorname{diam} f(\Omega) \int_{\Omega} |\psi(x)|^{p-1} |\nabla \psi(x)|\, J(x, f)^{1-1/n}\, dx.$$

Hölder's inequality yields

$$\begin{aligned}
&\int_{\Omega} |\psi(x)|^p J(x, f)\, dx \leqq C \operatorname{diam}(f(\Omega)) \\
&\quad \times \left( \left( \int_{\Omega} |\psi(x)|^p\, |J(x, f)|\, dx \right)^{(p-1)/p} \left( \int_{\Omega} |\nabla \psi(x)|^p J(x, f)^{1-p/n}\, dx \right) \right)^{1/p}
\end{aligned}$$

and the claim follows after dividing; note that

$$\int_{\Omega} |\psi|^p J(x, f)\, dx = \int_{\operatorname{spt}(\psi)} |\psi|^p J(x, f)\, dx < \infty.$$

PROOF OF THEOREM 1.13. Given $\psi \in C_0^{\infty}(\Omega)$, it clearly suffices to establish the asserted inequality separately for $\psi_+ = \max\{\psi, 0\}$ and $\psi_- = -\min\{\psi, 0\}$. Both of these functions are nonnegative, Lipschitz continuous and compactly supported in $\Omega$. Thus we assume without loss of generality that $\psi \in C_0(\Omega)$ is nonnegative and Lipschitz continuous. Then $\psi^* \in W_0^{1,p}(f(\Omega))$ by Lemma 4.9, and the Sobolev inequality gives

$$\left(\int_{f(\Omega)} |\psi^*|^{np/(n-p)}\, dx\right)^{(n-p)/np} \leqq C\left(\int_{f(\Omega)} |\nabla\psi^*|^p\, dx\right)^{1/p},$$

where $C = C(p, n)$. Reasoning as in the proof of Lemma 4.10 we conclude that we only need to verify

$$\int_B |\nabla\psi^*|^p\, dy \leqq C\int_{f^{-1}(B)} |\nabla\psi|^p J(x,f)^{1-p/n}\, dx \tag{4.12}$$

and

$$\int_B |\psi^*|^{np/(n-p)}\, dy \geqq \int_{f^{-1}(B)} |\psi|^{np/(n-p)} J(x,f)\, dx \tag{4.13}$$

whenever $B \subset f(\Omega)$ is an open ball with $f^{-1}(B) \cap \operatorname{spt}\psi \subset \cup_1^k U_j$ (disjoint union) and $f_j = f \mid U_j$ is injective for each $j = 1, \ldots, k$.

To this end, we estimate

$$\begin{aligned}
\int_B |\nabla\psi^*|^p\, dy &\leqq \int_B \left(\sum_{j=1}^k |\nabla(\psi(f_j^{-1}(y)))|\right)^p dy \\
&\leqq k^{p-1}\sum_{j=1}^k \int_B |\nabla\psi(f_j^{-1}(y))|^p\, |Df_j^{-1}(y)|^p\, dy \\
&\leqq N^{p-1}K^{p(n-1)/n}\sum_{j=1}^k \int_B |\nabla\psi(f_j^{-1}(y))|^p J(y,f_j^{-1}) J(y,f_j^{-1})^{p/n-1}\, dy \\
&\leqq N^{p-1}K^{p(n-1)/n}\sum_{j=1}^k \int_{U_j} |\nabla\psi(x)|^p J(x,f)^{1-p/n}\, dx \\
&\leqq N^{p-1}K^{p(n-1)/n}\int_{\cup_{j=1}^k U_j} |\nabla\psi(x)|^p J(x,f)^{1-p/n}\, dx,
\end{aligned}$$

and (4.12) follows. Moreover,

$$\begin{aligned}
\int_B |\psi^*(y)|^{np/(n-p)}\, dx &= \int_B \left(\sum_{j=1}^k \psi(f_j^{-1}(y))\right)^{np/(n-p)} dy \\
&\geqq \sum_{j=1}^k \int_B |\psi(f_j^{-1}(y))|^{np/(n-p)}\, dy = \sum_{j=1}^k \int_{f_j^{-1}(B)} |\psi(x)|^{np/(n-p)} J(x,f)\, dx
\end{aligned}$$

establishing (4.13). The theorem follows.

## 5. Uniqueness of the gradient.

Let $w$ be a weight. We assume that the Poincaré inequality

$$\int_B |\psi - \psi_B|^p w\,dx \leqq C(\operatorname{diam} B)^p \int_B |\nabla\psi|^p w\,dx, \tag{5.1}$$

holds for all balls $B$ and bounded functions $\psi \in C^\infty(B)$; the constant $C > 0$ naturally should be independent of $B$ and $\psi$. This weighted Poincaré inequality is clearly implied by (1.2) via Hölder's inequality; it is easily checked that the number $a$ can be replaced by the weighted average $\psi_B$.

In this section we verify the following uniqueness property, a result which is due to Semmes [23].

5.2. THEOREM. *Fix an open set $\Omega \subset \mathbb{R}^n$. Let $(\psi_j)$ be a sequence of functions in $C^\infty(\Omega)$ such that*

$$\psi_j \to 0 \quad \text{in} \quad L^p_w(\Omega), \tag{5.3}$$

*and*

$$\nabla\psi_j \to g \quad \text{in} \quad L^p_w(\Omega). \tag{5.4}$$

*Then $g = 0$ almost everywhere in $\Omega$.*

We assume for simplicity that $\Omega = \mathbb{R}^n$. We say that a compact set $K$ is *helpful* if there is a constant $C = C_K$ such that

$$|u(x) - u(y)| \leqq C\,|x - y|\,(M_{\nabla u}(x) + M_{\nabla u}(y))$$

whenever $u \in C^\infty(\mathbb{R}^n)$ and $x, y \in K$ satisfy $|x - y| \leqq 1$. Here $M_{\nabla u}$ is the maximal function defined by

$$M_{\nabla u}(y) = \sup_{r>0}\left(r^{-n}\int_{B(y,r)} |\nabla u|^p w\,dx\right)^{1/p}.$$

Theorem 5.2 is an immediate consequence of the following two lemmas.

5.5. LEMMA. *Let $K$ be helpful. If $(\psi_j)$, $g$ are as in* (5.3) *and* (5.4), *then $g = 0$ almost everywhere in $K$.*

5.6. LEMMA. *There is a sequence $(K_j)$ of helpful compact sets with $|\mathbb{R}^n \setminus \cup K_j| = 0$.*

PROOF OF LEMMA 5.5. We may assume that the diameter of $K$ does not exceed 1. By passing to a subsequence we may also assume that

$$\left(\int_{\mathbb{R}^n} |\psi_{j+1} - \psi_j|^p w\,dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |\nabla\psi_{j+1} - \nabla\psi_j|^p w\,dx\right)^{1/p} \leqq 10^{-j}.$$

Let $\lambda > 0$ be large. Set $g_j = \psi_{j+1} - \psi_j$. Pick an open set $V_j$ with

(5.7) $$|g_j(x)|^p w(x) \leqq (2^{-j}\lambda)^p \quad \text{on} \quad \mathsf{R}^n \backslash V_j$$

and

$$|V_j| \leqq 2(5^j\lambda)^{-p}.$$

Define

(5.8) $$\Omega_j = \{x\colon M_{\nabla g_j}(x) > 2^{-j}\lambda\}.$$

Then $\Omega_j$ is open, and since the Hardy-Littlewood maximal operator is of weak type one-one, we have

$$|\Omega_j| \leqq C(2^{-j}\lambda)^{-p}10^{-jp} = C(5^j\lambda)^{-p}.$$

Next, since $w > 0$ almost everywhere, we can pick an open set $U_\lambda$ such that

(5.9) $$w(x) \geqq \lambda^{-p} \quad \text{on} \quad K\backslash U_\lambda$$

and we may further require that

$$\lim_{\lambda\to\infty} |U_j| = 0.$$

Define

(5.10) $$K_\lambda = K\backslash(\cup_j V_j \cup \cup_j \Omega_j \cup U_\lambda).$$

Then

$$|K\backslash K_\lambda| \leqq \sum_j (|V_j| + |\Omega_j|) + |U_\lambda| \leqq C\lambda^{-p} + |U_\lambda|,$$

and so $\lim_{\lambda\to\infty} |K\backslash K_\lambda| = 0$.

By (5.7) and (5.10),

$$|g_j(x)|^p w(x) \leqq C(2^{-j}\lambda)^{-p} \quad \text{on} \quad K_\lambda$$

and since $w \geqq \lambda^{-p}$ on $K_\lambda$ by (5.9), we have that

$$|g_j(x)| \leqq C2^{-j} \quad \text{on} \quad K_\lambda.$$

Also,

$$|g_j(x) - g_j(y)| \leqq C2^{-j}\lambda\,|x - y|$$

by (5.8) and (5.10) whenever $x, y \in K_\lambda$ since $K$ is helpful. Using the Whitney extension theorem (see [24, Chapter VI]), we find Lipschitz functions $u_j$ in $\mathsf{R}^n$ such that

$$|u_j(x) - u_j(y)| \leqq C2^{-j}\lambda\,|x - y|$$

for all $x, y$ in $\mathsf{R}^n$,

$$|u_j(x)| \leqq C2^{-j}$$

for all $x$ in $\mathbb{R}^n$ and

$$u_j = g_j \quad \text{on} \quad K_\lambda.$$

Set

$$v_j = \psi_1 + \sum_{i=1}^{j-1} u_i.$$

Then each $v_j$ is (locally) Lipschitz with

$$|v_j - v_{j+1}| \leqq C2^{-j},$$

and with

$$|\nabla(v_j - v_{j+1})| \leqq C2^{-j}\lambda$$

almost everywhere. Thus $(v_j)$ converges (locally) uniformly to some Lipschitz function $v$ on $\mathbb{R}^n$; also, by standard reasoning, $(\nabla v_j)$ converges to $\nabla v$ almost everywhere. Moreover, $v = 0$ almost everywhere in $K_\lambda$ since $v_j = \psi_j$ *on* $K_\lambda$. Thus $\nabla v = 0$ almost everywhere in $K_\lambda$ and we conclude from (5.4) that $g = 0$ almost everywhere on $K_\lambda$. The claim follows by letting $\lambda \to \infty$.

PROOF OF LEMMA 5.6. For $j = 1, 2, \ldots$ define

$$\Omega_j = \{x : j^{-1}r^n \leqq w(B(x,r)) \leqq jr^n\}$$

whenever $0 < r < 10$. Since $w > 0$ almost everywhere and $w$ is locally integrable, it follows that

$$|\mathbb{R}^n \backslash \cup_j \Omega_j| = 0.$$

Fix $j$. It suffices to show that each compact subset $K$ of $\Omega_j$ is helpful. Let $K$ be a compact subset of $\Omega_j$, and let $u \in C^\infty(\mathbb{R}^n)$. Fix $x, y \in K$ with $|x - y| \leqq 1$. For $k = 1, 2, \ldots$ write $B_k = B(x, 2^{-k})$ and set $a_k = w(B_k)^{-1} \int_{B_k} u(z)w(z)\,dz$. Then

$$|a_k - a_{k+1}| \leqq w(B_{k+1})^{-1} \int_{B_{k+1}} |u - a_k|\, w\, dz \leqq C2^{kn} \int_{B_k} |u - a_k|\, w\, dz$$

because $x \in \Omega_j$. Applying the Hölder inequality and the Poincaré inequality (5.1), we deduce that

$$(5.11) \qquad |a_k - a_{k+1}| \leqq C2^{-k}\left(2^{kn} \int_{B_k} |\nabla u|^p w\, dz\right)^{1/p} \leqq C2^{-k} M_{\nabla u}(x).$$

Since $u$ is continuous, $a_k$ tends to $u(x)$ as $k \to \infty$. Fix $k_0$ such that $2^{-k_0-1} \leqq |x-y| \leqq 2^{-k_0}$. Then, summing over $k$ and using symmetry and (5.11) we conclude that

$$|u(x) - u(y)| \leqq C2^{-k_0}(M_{\nabla u}(x) + (M_{\nabla u}(y)) + |a_{k_0-1}(x) - a_{k_0+1}(y)|,$$

where

$$a_k(z) = w(B(z, 2^{-k}))^{-1} \int_{B(z,2^{-k})} uw\,dx.$$

As in (5.11), we then conclude that the latter term is bounded by $C|x-y|\,M_{\nabla u}(x)$. The claim follows.

## REFERENCES

1. B. Bojarski and T. Iwaniec, *Analytical foundation of the theory of quasiconformal mappings in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 8 (19983), 257–324.
2. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.
3. G. David and S. Semmes, *Strong $A_\infty$-weights, Sobolev inequalities, and quasiconformal mappings*, Analysis and partial differential equations. Lecture Notes in Pure and Applied Mathematics, Marcel Dekker . 122 (1990).
4. E. B. Fabes, D. Jerison and C. E. Kenig, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) 32 (1982), 151–182.
5. E. B. Fabes, C. E. Kenig and R. Serapioni, *The local regularity of solutions to degenerate elliptic equations*, Comm. Partial Differential Equations, 7 (1982), 77–116.
6. B. Franchi, C. E. Gutiérrez and R. L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations 19 (1994), 523–604.
7. F. W. Gering, *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–277.
8. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Second edition), Springer-Verlag, Berlin-Heidelberg, 1983.
9. S. Granlund, P. Lindqvist and O. Martio, *Conformally invariant variational inequalities*, Trans. Amer. Math. Soc. 227 (1983), 43–73.
10. J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, 1993.
11. J. Heinonen and P. Koskela, *$A_\infty$-condition for the Jacobian of a quasiconformal mapping*, Proc. Amer. Math. Soc. 120 (1994), 535–543.
12. J. Heinonen and S. Rohde, *Koenigs functions, and* BMO *quasicircles*, Duke Math. J. 78 (1995), 301–313.
13. T. Iwaniec and C. A. Nolder, *Hardy-Littlewood inequality for quasiregular mappings in certain domains in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 267–282.
14. O. Martio, *A capacity inequality for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. 474 (1970), 1–18.
15. G. Moscariello, *Weighted Sobolev inequalities associated with quasiregular mappings*, Nonlinearity 6 (1993), 49–55.
16. O. Martio, S. Rickmann and J. Väisälä, Definitions for quasiregular mappings, Ann. Acad. Sci. Fenn. Ser. A I Math. 448 (1969), 1–40.

17. O. Martio and J. Väisälä, *Elliptic equations and maps of bounded length distortion*, Math. Ann. 282 (1988), 423–443.
18. B. Øksendal, *Dirichlet forms, quasiregular functions and Brownian motion*, Invent. Math. 91 (1988), 273–297.
19. B. Øksendal, *Weighted Sobolev inequalities and harmonic measure associated with quasiregular functions*, Comm. Partial Differential Equations 15 (1990), 1447–1459.
20. Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, Translation of Mathematical Monographs 73, American Mathematical Society, Providence, 1989.
21. S. Rickman, *Quasiregular Mappings*, Springer-Verlag, 1993.
22. S. Semmes, *Bilipschitz mappings and strong $A_\infty$-weights*, Ann. Acad. Sci. Fenn. Ser. A I Math. 18 (1993), 211–248.
23. S. Semmes, *Personal communication.*
24. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.
25. J. Väisälä, *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. (1972), 1–14.
26. M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319 (1988).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR MI 48109
U.S.A.