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A BEST COVERING PROBLEM*

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1. The problem and the main theorem.

The problem to be studied in this paper originates from Beurling's notion of lower uniform density, which is defined for uniformly discrete (u.d.) sets of real numbers, i.e., for sets Λ satisfying

$$\inf\{|\lambda - \mu|: \lambda \neq \mu; \lambda, \mu \in \Lambda\} > 0.$$

If $n^-(r)$ denotes the smallest number of points from Λ to be found in an interval of length r, the lower uniform density of Λ is

$$D^-(\Lambda) = \lim_{r \to \infty} \frac{n^-(r)}{r},$$

where the limit exists because $n^-(r_1 + r_2) \geqq n^-(r_1) + n^-(r_2)$. The lower uniform density is a key concept for understanding exponential frames on an interval and sampling of bandlimited functions [4, 5, 3, 6], and it provides a particularly elegant solution to the original problem considered by Beurling in [1].

Previous discussions involving this lower uniform density have focused mainly on function theoretic problems. In this paper we formulate a combinatorial problem, for which the primary issue is to calculate a certain minimal density. Thus our problem is only loosely connected to Beurling's work. Nevertheless, some of the ideas developed in [1] have provided us with powerful tools which are crucial for proving our main result.

We begin by fixing a nonempty set $A \subset \mathsf{R}$. Then a u.d. set $\Lambda \subset \mathsf{R}$ is said to be an A-covering if

$$\mathsf{Z} \subset A - \Lambda = \{a - \lambda: a \in A, \lambda \in \Lambda\}.$$

We define

$$d(A) = \inf\{D^-(\Lambda): \Lambda \text{ an } A\text{-covering}\}$$

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and say that an A-covering Λ is a solution to the A-covering problem provided $D^{-}(\Lambda) = d(A)$. In other words, we wish to find a set Λ of minimal density which meets all the integer translates of A. If A is an unbounded set, it is readily seen that $d(A) = 0$, and this trivial case is therefore excluded from our discussion.

We say that $\Lambda \subset \mathsf{R}$ is periodic if there exists a real number $\alpha \neq 0$ such that $\Lambda = \Lambda + \alpha$. The main result of this paper is the existence of periodic solutions to the A-covering problem for bounded sets A:

THEOREM 1.1. *For every nonempty, bounded set $A \subset \mathsf{R}$ the A-covering problem has a periodic solution.*

The period will turn out to be an integer, and it follows as a corollary that $d(A)$ is a rational number for every nonempty $A \subset \mathsf{R}$.

To clarify the problem and to explain what makes it interesting, we introduce the following notions and transform A into a certain standard form. Any set $K \subset \mathsf{Z}$ is called a blanket. Two blankets K and L are said to be equivalent if there exists an integer k such that $L = K + k$. The equivalence classes of blankets are called blanket types. For a real number $\alpha \in \mathsf{Z} + A$ we let

$$A_\alpha = (A - \alpha) \cap \mathsf{Z}.$$

We say that A_α is an A-blanket located at α. The blanket types to which the A-blankets belong, are the blanket types of A.

We may now deduce the following lemma (its proof is best left to the reader), showing the relevance of our notion of blankets and blanket types.

LEMMA 1.2. *If the set of blanket types of B contains the set of blanket types of A, then $d(B) \leqq d(A)$. In particular, if the blanket types of A coincide with the blanket types of B, then $d(A) = d(B)$.*

If A is bounded, there is only a finite number of blanket types of A. Hence, the following is true.

COROLLARY 1.3. *For every bounded set $A \subset \mathsf{R}$ there is a finite set $A' \subset A$ so that $d(A') = d(A)$.*

The above remarks lead to a visualization of the A-covering problem as the problem of covering the integers by blankets from a finite set of blanket types, using as "few" blankets as possible.

Without loss of generality, we assume from now on that A is a finite set and that

$$A = \bigcup_{j=1}^{n} A_j,$$

where the sets $A_j - \min A_j$ are blankets belonging to distinct blanket types. We assume also that $\min A_j$ are distinct numbers on the interval $[0, 1)$ and that $\min A = 0$. We shall refer to such a set A as standard.

We see that if $a = \sup_j \# A_j \leqq 2$, then trivially $d(A) = 1/a$. However, the problem with $n = 1$ and $\# A = 3$ is nontrivial and difficult, as we will show in our detailed discussion of this case in Section 3. It is a major challenge to obtain good estimates on $d(A)$ for $n = 1$ and arbitrary $\# A$.

It has been brought to our attention that our problem is closely related to certain density problems studied in additive number theory [2], but we have found little in this field directly applicable to the A-covering problem.

We turn now to the main part of the paper, which is the proof of Theorem 1.1.

2. Construction of a periodic solution.

We assume that A is standard. Let $A^{\mathbf{Z}}$ denote the set of functions $f\colon \mathbf{Z} \to A$. For every $f \in A^{\mathbf{Z}}$ we put

$$\Lambda_f = \{k + f(k) \colon k \in \mathbf{Z}\},$$

which is then an A-covering. Conversely, if Λ is an A-covering, we associate with it an element $f_\Lambda \in A^{\mathbf{Z}}$ defined by

$$f_\Lambda(k) = \min(a\colon a \in A,\ k + a \in \Lambda\}$$

for every k. Since $\Lambda_{f_\Lambda} \subset \Lambda$ for every A-covering Λ and every Λ_f is an A-covering, we have

$$d(A) = \inf\{D^-(\Lambda_f)\colon f \in A^{\mathbf{Z}}\}.$$

We say that a function $f \in A^{\mathbf{Z}}$ is a solution to the A-covering problem provided Λ_f is a solution. Note that Λ is periodic if and only if f_Λ is periodic, so proving Theorem 1.1 is equivalent to proving existence of periodic solutions $p \in A^{\mathbf{Z}}$. Sometimes we find it natural to work with elements in $A^{\mathbf{Z}}$ rather than with A-coverings.

We begin by noting two simple and important properties of $A^{\mathbf{Z}}$. Let us define

$$T^l f(k) = f(k - l)$$

for every $l \in \mathbf{Z}$ and observe that $f \in A^{\mathbf{Z}}$ implies $T^l f \in A^{\mathbf{Z}}$, i.e., $A^{\mathbf{Z}}$ is invariant under integer translations. We note that $A^{\mathbf{Z}}$ is compact in the sense that every sequence f_j of functions in $A^{\mathbf{Z}}$ contains a subsequence f_{j_i} converging pointwise to some $f \in A^{\mathbf{Z}}$, i.e.,

$$\lim f_{j_i}(k) = f(k)$$

for every k. We say that f_{j_i} converges weakly to f, or briefly $f_{j_i} \rightharpoonup f$.

Of particular importance are sequences in $A^{\mathbf{Z}}$ obtained by translating one element $f \in A^{\mathbf{Z}}$. Accordingly, we denote by $W(f)$ the collection of all weak limits of translates of f.

It should be noted that this notion of weak limits is a simple adaption to our setting of the same notion used by Beurling in [1].

Before starting the construction of a periodic solution we note the following property of the lower uniform density.

LEMMA 2.1. *For every finite interval* $I = [x, x + M)$, *M an integer, there exists an* $f \in A^{\mathbf{Z}}$ *such that*

$$\#(\Lambda_f \cap I) \leqq d(A)M.$$

PROOF. In what follows, let

$$\lfloor x \rfloor = \sup\{k \in \mathbf{Z}: k \leqq x\}.$$

Let us assume the lemma is false. Then there exist x and M such for every $f \in A^{\mathbf{Z}}$ and every interval $I_j = [x + j, x + j + M)$ we have

$$\#(\Lambda_f \cap I_j) \geqq \lfloor d(A)M \rfloor + 1,$$

because of the translation invariance of $A^{\mathbf{Z}}$. This may be written as

$$\#(\Lambda_f \cap I_j) \geqq d(A)M + \varepsilon,$$

where $\varepsilon > 0$. It follows that for any interval of the form $L = [x + j, x + j + NM)$ we have

$$\#(\Lambda_f \cap L) \geqq N(d(A)M + \varepsilon).$$

This inequality implies that

$$D^-(\Lambda_f) \geqq d(A) + \varepsilon/M$$

for every $f \in A^{\mathbf{Z}}$, which is absurd taking into account the definition of $d(A)$.

We introduce next a sequence of functions needed in our construction. For every positive integer N let f_N be a function in $A^{\mathbf{Z}}$ for which

$$\#(\Lambda_{f_N} \cap [-N, N)) = \min_{f \in A^{\mathbf{Z}}} \#(\Lambda_f \cap [-N, N)).$$

The functions f_N are important because of the following lemma.

LEMMA 2.2. *For every positive integer N and every interval* $[x, x + M) \subset [-N, N)$, *$M$ an integer, we have*

$$\#(\Lambda_{f_N} \cap [x, x + M)) \leqq d(A)M + 2\,\mathrm{diam}(A).$$

PROOF. Let us assume the lemma is false. From Lemma 2.1 we know that there exists an $f \in A^{\mathbf{Z}}$ such that

$$\#((\Lambda_f \cap [x, x + M)) \leqq d(A)M,$$

which we use to define

$$\tilde{f}_N(k) = \begin{cases} f(k), & k \in [\max(-N, x - \operatorname{diam}(A)), x + M) \\ f_N(k), & k \notin [\max(-N, x - \operatorname{diam}(A)), x + M). \end{cases}$$

It follows that

$$\#(\tilde{f}_N(\mathbf{Z}) \cap [-N, N)) < \#(f_N(\mathbf{Z}) \cap [-N, N)),$$

which is a contradiction by the definition of f_N.

By the compactness of $A^{\mathbf{Z}}$, we may now find a sequence N_k of integers and a function $f_\infty \in A^{\mathbf{Z}}$ such that

$$f_{N_k} \rightharpoonup f_\infty.$$

An immediate consequence of Lemma 2.2 is the following.

LEMMA 2.3. *We have $D^-(\Lambda_f) = d(A)$ for every $f \in W(f_\infty)$.*

We shall seek a solution in $W(f_\infty)$, and for that purpose, we introduce the following notion. For every positive integer n we define

$$\Pi(n) = \{\Lambda_f \cap [0, n) \colon f \in A^{\mathbf{Z}}\},$$

which is a finite set since A is finite. The elements π of $\Pi(n)$ are referred to as patterns of length n.

For a given patern π of length n and $f \in A^{\mathbf{Z}}$ we put

$$Z(\pi, f) = \{k \in \mathbf{Z} \colon \Lambda_f \cap [k, k + n) = \pi + k\}.$$

This set can be thought of as the set of "appearances" of π in Λ_f. The following quantity is then the "maximal distance" between two consecutive "appearances" of π in Λ_f:

$$l(\pi, f) = \sup\{M \colon (k, k + M) \cap Z(\pi, f) = \emptyset \text{ for some } k\}.$$

The following simple fact is important for us.

LEMMA 2.4. *For every $f \in A^{\mathbf{Z}}$ and every positive integer n there exists at least one pattern π of length n and a corresponding function $g \in W(f)$ such that*

$$l(\pi, g) < \infty.$$

PROOF. Let $\pi_1 \in \Pi(n)$ be such that $l(\pi_1, f) = \infty$. Then we can find a sequence

m_j of integers and a function $f_1 \in F$ such that $T^{m_j} f \rightharpoonup f_1$ and $Z(\pi_1, f_1) = \emptyset$. This procedure may be iterated so that if $l(\pi_k, f_{k-1}) = \infty$, we can find a sequence m_j and a function $f_k \in F$ such that $T^{m_j} f_{k-1} \rightharpoonup f_k$ and $Z(\pi_k, f_k) = \emptyset$. Since $\Pi(n)$ is a finite set, the iteration must eventually stop so that for some k we have $l(\pi, f_k) < \infty$ for at least one pattern π.

As a final preparation for our construction, it is convenient to note the following fact.

LEMMA 2.5. *Let Λ be a periodic set of integer period $l > 0$. Suppose that Ω is an A-covering and that $\Lambda \cap I = \Omega \cap I$ for some interval $I = [m, m + M)$ of length $M > l + \operatorname{diam}(A)$. Then Λ is an A-covering.*

We are now in position to construct a periodic solution and thus prove the theorem. According to Lemma 2.4, we may pick a $g \in W(f_\infty)$ and a pattern $\pi \in \Pi([\operatorname{diam}(A)] + 1)$ such that

$$l_0 = l(\pi, g) < \infty.$$

The definition of $l(\pi, g)$ implies that we can find an increasing sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ of distinct integers such that

$$\alpha_{j+1} - \alpha_j \leqq l_0$$

and

$$\Lambda_g \cap [\alpha_j, \alpha_j + \lfloor \operatorname{diam}(A) \rfloor + 1) = \pi + \alpha_j$$

for every j.

Consider the sequence of "local densities",

$$\rho_j = \frac{(\Lambda_g \cap [\alpha_j, a_{j+1}))}{\alpha_{j+1} - \alpha_j}.$$

Since $l_0 < \infty$, the numbers ρ_j constitute a finite set, and the infimum

$$\rho = \inf_j \rho_j$$

is obtained for some j, which without loss of generality is ssumed to be $j = 0$, i.e., $\rho_0 = \rho$.

We put $l = \alpha_1 - \alpha_0$ and let Λ be the l-periodic set which satisfies

$$\Lambda \cap [\alpha_0, \alpha_1) = \Lambda_g \cap [\alpha_0, \alpha_1).$$

It follows from Lemma 2.5 that Λ is an A-covering. Since

$$D^-(\Lambda) = \rho \leqq D^-(\Lambda_g) = d(A),$$

we have $D^-(\Lambda) = d(A)$, and we conclude that Λ is a solution to the A-covering problem.

3. Estimates on the minimal density.

We assume again that A is standard. We begin by noting two general and "rough" estimates on $d(A)$.

PROPOSITION 3.1. *Let $A = \bigcup_{j=1}^{n} A_j$ be standard. Then*

$$\frac{1}{M} \leqq d(A) \leqq C\frac{\log M}{M}$$

where $M = \max_{1 \leqq j \leqq N} \# A_j$ and $C > 0$ is an absolute constant.

PROOF. By Theorem 1.1, $d(A) = 1/M$ holds if and only if some of the blankets $A_j - \min_j A$ satisfying $\# A_j = M$ can cover $\mathbf{Z}$ completely without overlap. The right inequality follows from a more general result in additive number theory due to Lorentz (see [2, p. 14]).

In what follows we shall restrict ourselves to $A \subset \mathbf{Z}$. Let us observe, however, that the presence of several blankets makes the problem more involved. For instance, there exist finite sets $A = A_\alpha \cup A_\beta$ for which

$$d(A) < \min\{d(A_\alpha), d(A_\beta)\}:$$

Put $A = \{0, 1, 3, \pi, \pi + 2, \pi + 3\}$; then $A_0 = \{0, 1, 3\}$ and $A_\pi = \{0, 2, 3\}$. The set $\Lambda = \{2 + k, \pi + k \colon k \in \mathbf{Z}\}$ satisfies $D^-(\Lambda) = 1/3$ and $A - \Lambda \supset \mathbf{Z}$. This and the right inequality in Proposition 3.1 yield $d(A) = 1/3$. On the other hand, some calculations show that $d(A_0) = d(A_\pi) = 2/5$.

We assume from now on that $A \subset \mathbf{Z}$. We define

$$\lceil x \rceil = \min\{k \in \mathbf{Z} \colon k \geqq x\},$$

and let (a, b) denote the greatest common divisor of the integers a and b. With this notation, the main result of this section can be stated as follows.

PROPOSITION 3.2. *Let $A = \{0, a\alpha, b\alpha\}$, where $a, b, \alpha \in \mathbf{Z}$, $0 < a < b$, and $(a, b) = 1$. Then*

$$d(A) \leqq \min\left\{\frac{\lceil (a + b)/3 \rceil}{a + b}, \frac{\lceil (2b - a)/3 \rceil}{2b - a}\right\}.$$

It is strongly felt that, in fact,

$$d(A) = \min\left\{\frac{\lceil (a + b)/3 \rceil}{a + b}, \frac{\lceil (2b - a)/3 \rceil}{2b - a}\right\},$$

but we have not been able to prove this equality.

The estimate in Proposition 3.2 is based on some general observations related to our main theorem.

Let $\mathbf{Z}_m$ denote the group $\{0,\dots,m-1\}$ with addition law $n+k \pmod m$. For any blanket Q we denote by Q_m its projection onto $\mathbf{Z}_m$: $Q_m = \{q \bmod m): q \in Q\}$. For a given blanket A we consider the density

$$d_m(A_m) = \min\left\{\frac{\# Q_m}{m}: Q_m \subset \mathbf{Z}_m;\ A_m - Q_m = \mathbf{Z}_m\right\}.$$

LEMMA 3.3. *Let $A \subset \mathbf{Z}$ be a finite set. Then*

(1) $$d(A) = \min_m d_m(A_m),$$

(2) $$d_m(A_m) \leqq d(A)\left(1 + \frac{\operatorname{diam}(A)}{m}\right).$$

PROOF. Let Q be any subset of $\mathbf{Z}_m$ satisfying $A_m - Q_m = \mathbf{Z}_m$. Then the set

$$Q = Q_m + m\mathbf{Z} = \{q + mk: q \in Q_m, k \in \mathbf{Z}\}$$

is m-periodic and satisfies $A - Q = \mathbf{Z}$. Since $D^-(Q) = \# Q_m/m$, we have

$$d(A) \leqq \min_m d_m(A_m).$$

Theorem 1.1 shows that there exist an integer m and an m-periodic set Λ satisfying $A - \Lambda = \mathbf{Z}$. We have $A_m - \Lambda_m = \mathbf{Z}_m$ and therefore

$$d(A) = D^-(\Lambda) = \#\Lambda_m/m \geqq d_m(A_m).$$

Similar arguments prove (2).

Let $A \subset \mathbf{Z}$ and $c, r \in \mathbf{Z}$. We denote by $c + rA$ the blanket $\{c + ra: a \in A\}$ and by $(c + rA_m)$ its projection onto $\mathbf{Z}_m$.

LEMMA 3.4. *Let $(r,m) = 1$. Then for any c and A_m we have*

$$d_m(A_m) = d_m(c + rA_m).$$

PROOF. The lemma follows from the fact that $A_m - Q_m = \mathbf{Z}_m$ implies

$$(c + rA_m) - (c + rQ_m) = \mathbf{Z}_m$$

whenever $(r,m) = 1$.

PROOF OF PROPOSITION 3.2. Lemma 3.4 shows that $d_m(\{0,\alpha a,\alpha b\}) = d_m(\{0,a,b\})$ for any $m > \alpha b$, $(m,\alpha) = 1$. Letting $m \to \infty$ in (2) of Lemma 3.3 we see that $d(\{0,\alpha a,\alpha b\}) = d(\{0,a,b\})$. So we may assume that $\alpha = 1$.

Put $m = a + b$. Then $b = -a \pmod m$ and Lemma 3.4 gives

$$d_m(\{0, a, b\}) = d_m(\{-a, 0, a\}) = d_m(\{0, ra, 2ra\}),$$

where $(r, m) = 1$. Since $(a, b) = 1$, we have $(a, m) = 1$, and so there exists r satisfying $ra = 1 \pmod m$. Thus Lemma 3.3 gives

$$d(\{0, a, b\}) \leqq d_m(\{0, a, b\}) = d_m(\{0, 1, 2\}).$$

But we see that

$$d_m(0, 1, 2) = \frac{\lceil m/3 \rceil}{m},$$

and this proves the first inequality in Proposition 3.2.

The second inequality in Proposition 3.2 follows by a similar argument.

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