APPROXIMATELY FINITE C*-ALGEBRAS AND PARTIAL AUTOMORPHISMS

RUY EXEL

Abstract.

We prove that every AF-algebra is isomorphic to a crossed product of a commutative AF-algebra by a partial automorphism.

1. Introduction.

A partial automorphism of a C^*-algebra A is, by definition, a triple $\Theta = (\theta, I, J)$ where I and J are closed two-sided ideals in A and $\theta \colon I \to J$ is a $*$-isomorphism. Given such a partial automorphism, we constructed in [4] a crossed product C^*-algebra $A \times_\Theta \mathsf{Z}$, generalizing the well known construction of crossed products by automorphisms.

The purpose of the present work is to show that the AF-algebras introduced by Bratteli [1] can be obtained, non-trivially, as the crossed product of a commutative AF-algebra (i.e., the algebra of continuous functions on a totally disconnected space) by a partial automorphism.

To achieve this goal we first show that AF-algebras admit somewhat canonical actions of the circle group, whose fixed points are exactly the elements of the standard diagonal subalgebra. In addition we need to prove that this action satisfies the technical hypothesis of being semi-saturated and regular (see below). Our main result, Theorem 2.5, then comes as a consequence of Theorem 4.21 in [4], which states that any C^*-algebra admitting a semi-saturated regular circle action is isomorphic to a crossed product of the fixed point subalgebra by a partial automorphism. As an example, we consider UHF-algebras, obtaining a result closely related to Putnam's [12] study of crossed products of the Cantor set by "odometer" maps.

Our result holds in full generality, applying to all AF-algebras, without further restrictions. This illustrates the power of the concept of partial crossed products

and suggests that there may be large classes of C^*-algebras which profitably fit into that description.

We should mention that some of the techniques used here, including the study of circle actions on AF-algebras, have already been employed by others. Also, special cases of our main theorem have appeared in the literature. Perhaps the first result in that direction is Theorem 7.9 in [5], where circle actions on AF-algebras, satisfying certain maximal properties, are shown to give rise to a description of the AF-algebra as a subalgebra of a (full) crossed product. Although the concept of partial crossed products was not known at the time, that subalgebra is precisely the partial crossed product by a suitable restriction of the automorphism involved. In addition, at the end of section 7 in [5], the circle action we discuss below is shown to exist without the essential simplicity of the ordered Bratteli diagram, thus hinting that Theorem 7.9, mentioned above, survives in a more general seting. See also [9].

A strong relationship also exists between circle actions and integer valued cocycles on the associated AF-groupoid. These have been extensively studied in connection with the theory of triangular AF-algebras (see [7], [10] and [11]). In section 10.20 of [11] it is shown how to construct a one parameter group of automorphisms on an AF-algebra from a real valued cocycle on its AF-groupoid. In the special case of integer valued cocycles, the automorphism group turns out to be periodic, thus yielding an action of S^1. Moreover, when the cocycle is constructed from standard embeddings of the upper triangular matrix algebras, as in section 2 of [7], the action obtained in the process turns out to be the one we use here.

The idea of partial homeomorphisms, the commutative version of partial automorphisms, is also very much present in the recent literature on AF-algebras. For example, in 2.8 of [7], a partial homeomorphism is constructed from an integer valued cocycle. Interestingly enough, that partial homeomorphism is the same one we obtain from the circle action which, in turn, arises from the cocycle. See also section 4 in [10] and section 3 of [5] for alternative descriptions of that partial homeomorphism.

Our main result should also be interpreted as establishing a new link between the class of AF-algebras on one hand and crossed products on the other, a relationship which has been recognized long ago and which has one of its most interesting examples in the famous embedding of the irrational rotation C^*-algebras into AF-algebras, obtained by Pimsner and Voiculescu [8].

The work of Loring [6] as well as that of Elliott and Loring [3] on AF-embeddings of the algebra of continuous functions on the two-torus, and the Theorem of Elliott and Evans [2] showing that the irrational rotation C^*-algebras can be described as inductive limits of circle algebras, give some more striking examples of that relationship.

The notation and terminology used below is taken from [4] where the reader will find the main definitions and results on generalized crossed products, as well as the facts on actions of the circle that we shall need here. As far as notation is concerned, we should warn the reader of a slightly non-standard convention from [4] used here: if X and Y are subsets of a C^*-algebra, then XY denotes the *closed* linear span of the set of products xy with $x \in X$ and $y \in Y$.

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3. The Main Result.

Let A and B be C^*-algebras and let α and β be actions of S^1 on A and B, respectively. If $\phi: A \to B$ is a covariant homomorphism (always assumed to be star preserving), then it is clear that $\phi(A_n) \subseteq B_n$, where A_n and B_n are the corresponding spectral subspaces. This obviously implies that $\phi(A_n^* A_n) \subseteq B_n^* B_n$ for each n.

Let us now suppose that both α and β are regular actions in the sense of 4.4 in [4]. That is, there are maps

  (i) $\phi_A: A_1^* A_1 \to A_1 A_1^*$
  (ii) $\lambda_A: A_1^* \to A_1 A_1^*$
  (iii) $\theta_B: B_1^* B_1 \to B_1 B_1^*$
  (iv) $\lambda_B: B_1^* \to B_1 B_1^*$

satisfying the conditions described in [4].

 2.1. DEFINITION. A covariant homomorphism $\phi: A \to B$ is said to be *regular* (with respect to a given choice of θ_A, λ_A, θ_B and λ_B) if for any x^* in A_1^* and a in $A_1^* A_1$ one has

  (i) $\phi(\lambda_A(x^*)) = \lambda_B(\phi(x^*))$
  (ii) $\phi(\theta_A(a)) = \theta_B(\phi(a))$.

Let us now analyse, in detail, an example of a regular homomorphism in finite dimensions, which will prove to be crucial in our study of AF-algebras.

Denote by M_k, the algebra of $k \times k$ complex matrices. Let A be a finite dimensional C^*-algebra, so that A is isomorphic to a direct sum $A = \oplus_{i=1}^n M_{p_i}$. In order to define a circle action on A, let for each k, α^k denote the action of S^1 on M_k given by

$$\alpha_z^k(a) = \begin{pmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z^{k-1} \end{pmatrix} a \begin{pmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z^{k-1} \end{pmatrix}^{-1}$$

for any a in M_k and z in S^1. Define an action α_A of S^1 on A by $\alpha_A = \oplus_{i=1}^{n} \alpha^{p_i}$. This action will be called the standard action of S^1 on A. It should be noted that the standard action depends on the choice of matrix units for A.

One can easily check that α_A is semi-saturated (see 4.1 in [4]). Let us show that it is also regular. For each k let s_k be the partial isometry in M_k given by

$$
s_k = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}
$$

and put $s_A = \oplus_{i=1}^{n} s_{p_i}$. The maps

$$\lambda_A : x^* \in A_1^* \mapsto s_A x^* \in A_1 A_1^*$$

$$\theta_A : a \in A_1^* A_1 \mapsto s_A a s_A^* \in A_1 A_1^*$$

satisfy the conditions of 4.4 in [4] as the reder may easily verify, which therefore implies the regularity of α_A.

Now let $B = \oplus_{i=1}^{m} M_{q_i}$ be another finite dimensional C^*-algebra and let α_B, s_B, λ_B and θ_B be defined as above.

A standard homomorphism is any homomorphism $\phi : A \to B$ of the form

$$\phi(a_1, \ldots, a_n) = (\phi_1(a_1, \ldots, a_n), \ldots, \phi_m(a_1, \ldots, a_n))$$

where, for each $k = 1, 2, \ldots, m$

$$
\phi_k(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_{i_1} & & & & & & & \\ & a_{i_2} & & & & & & \\ & & \ddots & & & & & \\ & & & a_{i_r} & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 \end{pmatrix}
$$

where r may vary with k, the indices $i_1, i_2 \ldots i_r$ are chosen from the set $\{1, 2, \ldots, n\}$ while each a_i is in M_{p_i}. Standard homomorphisms are well known to play an important role in the theory of AF-algebras [1].

2.2. PROPOSITION. *A standard homomorphism is covariant and regular with respect to the respective standard actions.*

PROOF. Let $\phi : A \to B$ be a standard homomorphism where A and B are direct sum of matrix algebras, as above. That ϕ is covariant follows from the easy fact that spectral subspaces are mapped accordingly. In order to verify regularity, let $x \in A_1$. Then

$$\phi(\lambda_A(x^*)) = \phi(s_A x^*) = \phi(s_A)\phi(x^*).$$

Note that $s_B\phi(s_A)^*\phi(s_A) = \phi(s_A)$, since $\phi(s_A)$ can be thought of as a "restriction" of s_B to a subspace of the initial space of s_B. Thus

$$\phi(\lambda_A(x^*)) = s_B\phi(s_A)^*\phi(s_A)\phi(x^*) = s_B\phi(s_A^* s_A x^*) = s_B\phi(x^*) = \lambda_B(\phi(x^*)).$$

On the other hand, for a in $A_1^* A_1$, of the form $a = x^*y$, with $x, y \in A_1$, we have, by the same reasons given above

$$\phi(\theta_A(a)) = \phi(s_A x^* y s_A^*) = s_B\phi(x^*y)s_B^* = \theta_B(\phi(a)).$$

This completes the proof.

2.3. THEOREM. *Let $(A^k)_{k\in\mathbb{N}}$ be a sequence of C^*-algebras and $\phi^k\colon A^k \to A^{k+1}$ be injective homomorphisms. Suppose each A^k carry a semi-saturated regular action α^k of S^1. Suppose, in addition, that each ϕ^k is covariant and regular with respect to a fixed choice of maps λ^k and θ^k, at each level, as in 2.1). Then the inductive limit C^*-algebra $A = \lim_k A^k$ admits a semi-saturated regular action α of S^1. Moreover the canonical inclusions $\psi^k\colon A^k \to A$ are covariant and regular.*

PROOF. Let A' be the union of all $\psi^k(A^k)$ in A. For each k and each $a \in A^k$ define

$$\alpha_z(\psi^k(a)) = \psi^k(\alpha_z^k(a)) \quad \text{for} \quad z \in S^1.$$

Since the ϕ^k are covariant, it follows that α_z is well defined as an automorphism of A' and that it extends to an action of S^1 on A, with respect to which, each ψ^k is covariant. So it is clear that $\psi^k(A_n^k) \subseteq A_n$, where the subscript n indicates spectral subspace. The fact that α is semi-saturated, thus follows immediately.

We now claim that each A_n is indeed the closure of $\bigcup_{k=1}^{\infty} \psi^k(A_n^k)$. In fact, let P_n^k and P_n be the nth spectral projections corresponding to A^k and A, respectively. Given a in A_n, choose a^k in A^k such that $a = \lim_k \psi^k(a^k)$. Note that $a = P_n(a) = \lim_k \psi^k(P_n^k(a^k))$ which proves our claim. We then clearly have that, for any integer n, $A_n^* A_n$ is the closure of $\bigcup_{k=1}^{\infty} \psi^k(A_n^{k*} A_n^k)$. For each $k \in \mathbb{N}$, $x \in A_1^k$ and $a \in A_1^{k*} A_1^k$ define

$$\lambda(\psi^k(x^*)) = \psi^k(\lambda^k(x^*))$$

$$\theta(\psi^k(a)) = \psi^k(\theta^k(a)).$$

Invoking the regularity of ϕ^k, we see that both λ and θ are well defined, the domain of λ being $\bigcup_{k=1}^{\infty} \psi^k(A_1^{k*})$ and that of θ being $\bigcup_{k=1}^{\infty} \psi^k(A_1^{k*} A_1^k)$. The hypothesis clearly imply that ψ^k, θ^k and λ^k are isometries, so both λ and θ extend to the closure of their current domains, that is, A_1^* in the case of λ and $A_1^* A_1$ in case of θ. One now needs to show that for any $x, y \in A_1$, $a \in A_1^* A_1$ and $b \in A_1 A_1^*$

(i) $\lambda(x^*b) = \lambda(x^*)b$

(ii) $\lambda(ax^*) = \theta(a)\lambda(x^*)$

(iii) $\lambda(x^*)^*\lambda(y^*) = xy^*$

(iv) $\lambda(x^*)\lambda(y^*)^* = \theta(x^*y)$

which are the axioms for regular actions (see 4.4 in [4]). These identities hold on dense sets, by regularity of the ϕ^k, and hence everywhere by continuity. A few remaining details are left to the reader.

A simple, yet crucial consequence of 2.2 and 2.3 is the existence of certain somewhat canonical circular symmetries on AF-algebras. More precisely we have:

2.4. THEOREM. *Every* AF-*algebra posseses a semi-saturated regular action of the circle group, such that the fixed point subalgebra is an* AF-*massa.*

PROOF. For the existence of the action it is enough to note that any AF-algebra can be written as a direct limit of direct sums of matrix algebras, in which the connecting maps are standard homomorphisms in the above sense [1]. The fixed point subalgebra is the inductive limit of the direct sum of the diagonal subalgebras at each stage, hence it is commutative and AF. The maximality follows from (I.1.3) in [13].

Of course the action described in 2.4 is not unique as it depends on the choice of a particular chain of finite dimensional subalgebras, as well as on the choice of matrix units at each stage. Nevertheless, the isomorphism class of the fixed point algebra will always be the same [11]. We shall, nevertheless, refer to this action as the standard action.

Combining the existence of standard actions with Theorem 4.21 in [4], brings us to our main result.

2.5. THEOREM. *Let A be an* AF-*algebra. Then there exists a totally disconnected, locally compact topological space X, a pair of open subsets $U, V \subseteq X$ and a homeomorphism $\varphi \colon V \to U$ such that the covariance algebra for the induced partial automorphism $\theta \colon C_0(U) \to C_0(V)$ is isomorphic to A.*

PROOF. Apply Theorem 4.21 in [4] to the action of S^1 on A provided by 2.4.

3. UHF Algebras.

As an example, let us briefly mention the case of UHF-algebras. For that purpose it is covenient to use the language of ordered Bratteli diagrams [5].

Given a UHF-algebra A, consider its Bratteli diagram, which consists of a single vertex at each stage and edges joining vertex v_{i-1} to v_i (the product of the n_i giving the supernatural number which characterizes A). Now, given that there

is only one vertex at each stage, there is essentially only one way to order such a diagram.

The fixed point subalgebra for the circle action constructed above is then the standard diagonal, which can be identified with the continuous functions on the set X of all infinite paths in the Bratteli diagram.

The order in the Bratteli diagram induces an order on X as follows: two paths are comparable if and only if they agree after some point and then the decision of which is greater is based on the comparison of the last edge where they differ.

This said we can now describe the sets U and V of 2.5 as well as the map φ. Let $\beta_{\min}$ and $\beta_{\max}$ be the unique minimal and maximal paths in X, respectively. Then U is given by $X\backslash\{\beta_{\max}\}$ while $V = X\backslash\{\beta_{\min}\}$. $\varphi: U \to V$ is the map sending each path to its successor. Note that $\beta_{\max}$, which does not belong to the domain of φ, is the only infinite path not possessing a successor.

With this in hand we can easily see that φ can also be described as an odometer map (see [12]). More explicitly, identify X with the infinite product

$$X = \prod \{0, 1, \ldots, n_i - 1\}.$$

Then $\beta_{\min} = (0, 0, \ldots)$ and $\beta_{\max} = (n_1 - 1, n_2 - 1, \ldots)$ while φ is given by formal addition with $(1, 0, 0, \ldots)$ with carry over to the right.

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DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE DE SÃO PAULO
RUA DO MATÃO, 1010
05508-900 SÃO PAULO SP
BRAZIL
email: exel@ime.usp.br