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HIGHER ORDER COMMUTATORS IN INTERPOLATION THEORY

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Abstract.

Recently, estimates for higher order commutators of interpolation theory have been obtained for the complex and for the real method (cf. [Ro] and [CCMS], and [Mi], respectively). The analysis of cancellation properties allows us to obtain a new commutator theorem which extends the previous results. An application to the boundedness of higher order commutators for singular operators between weighted L^p spaces is also given.

1. Introduction.

In [RW], R. Rochberg and G. Weiss extended the proof of the Riesz-Thorin theorem on complex interpolation of bounded linear operators to give some interesting estimates for non-linear commutators of bounded linear operators and certain operator Ω, generally unbounded and nonlinear, and in [JRW], B. Jawerth, R. Rochberg and G. Weiss obtained the corresponding commutator theorem for the real interpolation theory.

Both kinds of results can be derived from a general simple construction based on the abstract method of V. Williams [Wi], as it was shown in [CCS] by the authors. This construction applies also to other interpolation methods, like the Lions-Schechter complex method and the real methods with function parameters.

Recently, R. Rochberg in [Ro], M. Milman in [Mi], and the authors with M. Milman in [CCMS], have obtained higher order commutator estimates for the complex method and the real method.

In this paper we present these type of results in the frame of [CCS] giving the precise role that cancellation plays in the boundedness of higher order commutators, with the use of twisted direct sums and we show how new estimates can be derived.

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We start by recalling very briefly the above mentioned construction.

DEFINITION 1.1. By an *interpolator* Φ over H we mean a functor $H_\Phi = H$ from compatible couples $\bar{A} = (A_0, A_1)$ of Banach spaces to normed spaces $H(\bar{A})$ with the property that there exists a bounded linear operator

$$\Phi_{\bar{A}}: H(\bar{A}) \to \Sigma(\bar{A}) = A_0 + A_1,$$

for every couple $\bar{A}$, such that

$$T \circ \Phi_{\bar{A}} = \Phi_{\bar{B}} \circ H(T), \tag{1}$$

for every interpolation operator $T: \bar{A} \to \bar{B}$, i.e., a linear operator $T: A_0 + A_1 \to B_0 + B_1$ which is bounded from A_j to B_j $(j = 0, 1)$.

We usually set $\bar{A}_\Phi = \Phi_{\bar{A}}(H(\bar{A}))$, with the norm

$$\|a\|_\Phi = \inf\{\|f\|_{H(\bar{A})};\ \Phi_{\bar{A}}(f) = a\},$$

so that $\bar{A}_\Phi \hookrightarrow \Sigma(\bar{A})$, with norm $\leqq \|\Phi_{\bar{A}}\|$. If $H(\bar{A})$ is complete, $\bar{A}_\Phi$ is a Banach space.

Property (1) implies that $\bar{A} \rightsquigarrow \bar{A}_\Phi$ is an interpolation method such that

$$\|T\|_{\bar{A}_\Phi, \bar{B}_\Phi} \leqq \|H(T)\|_{H(\bar{A}), H(\bar{B})}$$

for any interpolation operator $T: \bar{A} \to \bar{B}$.

Many interesting examples in interpolation theory, say the real J and K methods and the complex methods (cf. [BL]), are constructed under this scheme:

(a) The J-method is associated to the interpolator

$$\Phi_{J,\bar{A}}(u) = \int_0^\infty u(t)\frac{dt}{t} \in \Sigma(\bar{A}),$$

on the spaces

$$H_J(\bar{A}) = \{u: \mathrm{R}^+ \to \Delta(\bar{A}) \text{ measurable};\ \Phi_{\theta,p}(J(t, u(t))) < \infty\},$$

with the norm $\|u\|_{H(\bar{A})} = \Phi_{\theta,p}(J(t, u(t)))$, where $J(t, a) = \max(\|a\|_{A_0}, t\|a\|_{A_1})$, if $a \in \Delta(\bar{A})$ and

$$\Phi_{\theta,p}(\gamma(t)) = \left(\int_0^\infty (t^{-\theta}\gamma(t))^p \frac{dt}{t}\right)^{1/p},$$

$0 < \theta < 1$, $1 \leqq p \leqq \infty$. For every $T: \bar{A} \to \bar{B}$, $H_J(T)u = T \circ u$.

Then

$$\bar{A}_\Phi = (A_0, A_1)_{\theta,p;J}.$$

(b) In the case of the K-method, $H_K(\bar{A})$ is the vector space of all measurable functions (a_0, a_1): $\mathrm{R}^+ \to A_0 \times A_1$ such that $a_0(t) + a_1(t)$ is constant and

$$\|(a_0, a_1)\|_{H(\bar{A})} = \left(\int_0^\infty \left(\frac{\|a_0(t)\|_0 + t\,\|a_1(t)\|_1}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p} < \infty,$$

with $0 < \theta < 1$ and $1 \leqq p \leqq \infty$. Now $H_K(T)(a_0, a_1) = (T \circ a_0, T \circ a_1)$.

It is easily seen that if $\Phi_{K,\bar{A}}(a_0, a_1) = a_0 + a_1$ then $\bar{A}_\Phi = \bar{A}_{\theta,p;K}$, with equality of norms.

(c) The first complex method of Calderón $\bar{A}_\theta = [A_0, A_1]_\theta$ is associated to

$$\Phi_{C,\bar{A}}(f) = \delta_\theta(f) = f(\theta),$$

with $H_C(\bar{A}) = \mathscr{F}(\bar{A})$, the Banach space of vector-valued analytic functions on the strip S considered by Calderón [Ca], and $H_C(T)f = T \circ f$.

(d) The Schechter complex interpolation methods of derivatives arise also in this way, with $H_C(\bar{A}) = \mathscr{F}(\bar{A})$ and $H_C(T)f = T \circ f$, but now $\Phi_{C,\bar{A}}(f) = f^{(n)}(\theta)$. In this case we write

$$\bar{A}_{\delta^{(n)}(\theta)} = [A_0, A_1]_{\delta^{(n)}(\theta)}.$$

The paper is organized as follows. In Section 2 we define the notion of compatible system of interpolators. With this definition we isolate the cancellation properties that, together with conditions (2), are needed to prove the higher order commutator theorem (Theorem 2.6), which is the main result of this paper. Section 3 is devoted to showing that the commutator theorems for classical examples, i.e. the complex method considered in [Ro] and the real method studied in [Mi], follow easily from Theorem 2.6. In Section 4 we see how new results can be derived by obtaining the higher order version of a result contained in [CCMS], about the boundedness of commutators of linear operators with multiplication by a BMO function, in the case of weighted L^p spaces and Schechter interpolation method.

2. Higher order commutators.

When dealing with commutators of order n for the complex method, as in [Ro], the system of derivatives

$$\Phi_C^k(f) = \frac{1}{(\mathrm{k}-1)!} f^{(k-1)}(\theta) \qquad (f \in H_C(\bar{A})),$$

with $k = 1, \ldots, n$, appears in a natural way. For the real K-method, in the computations of [Mi] for the n-commutator, the use of the system of n interpolators

$$\Phi^k_K(a_0, a_1) = c_k\left(\int_0^1 (\log t)^{k-2} a_0(t)\frac{dt}{t} - \int_1^\infty (\log t)^{k-2} a_1(t)\frac{dt}{t}\right),$$

with suitable factors c_k, on $H_K(\bar{A})$ is implicit.

The properties of these interpolators in connection with the commutator theorem suggest the following definition, where we omit the use of the subscript $\bar{A}$.

Definition 2.1. We say that a system of n interpolators, $\vec{\Phi} = (\Phi^1, \Phi^2, \ldots, \Phi^n)$, defined in the same spaces $H(\bar{A})$, is *compatible* if the following condition holds:

For every $k = 2, \ldots, n$,

$$\Phi^k(\operatorname{Ker}\Phi^1 \cap \cdots \cap \operatorname{Ker}\Phi^{k-1}) = \operatorname{Im}\Phi^1,$$

in the sense that there exists a constant $C = C(\bar{A}) > 0$ such that

(a) if $g \in \operatorname{Ker}\Phi^1 \cap \cdots \cap \operatorname{Ker}\Phi^{k-1}$, then $\Phi^k(g) = \Phi^1(f)$, for some $f \in H(\bar{A})$ such that $\|f\|_{H(\bar{A})} \leqq C\|g\|_{H(\bar{A})}$, and

(b) if $f \in H(\bar{A})$, then $\Phi^1(f) = \Phi^k(g)$, for some $g \in \operatorname{Ker}\Phi^1 \cap \cdots \cap \operatorname{Ker}\Phi^{k-1}$ such that $\|g\|_{H(\bar{A})} \leqq C\|f\|_{H(\bar{A})}$.

Set $\bar{A}_{\Phi^j} = \Phi^j(H(\bar{A}))$ and let us consider

$$\mathscr{E}_k = \{\vec{a} = (a_1, \ldots, a_k);\ a_j = \Phi^j(f),\ f \in H(\bar{A})\}$$

with the quotient norm

$$\|\vec{a}\|_{\mathscr{E}_k} = \inf\{\|f\|_{H(\bar{A})};\ \vec{\Phi}(f) = \vec{a}\}.$$

Let $c > 1$, a fixed constant, and for every $\vec{a} = \vec{\Phi}(f) \in \mathscr{E}_{n-1}$, let us take an almost optimal election $h_{\vec{a}}$, which is an element $h_{\vec{a}} \in H(\bar{A})$ such that $\vec{\Phi}(h_{\vec{a}}) = \vec{a}$ and $\|h_{\vec{a}}\|_{H(\bar{A})} \leqq c\|\vec{a}\|_{\mathscr{E}_{n-1}}$.

Now we define, for any $\vec{a} = (\Phi^1(f), \ldots, \Phi^{n-1}(f)) \in \mathscr{E}_{n-1}$,

$$\Omega_{n-1}\vec{a} = \Omega_{n-1}(\Phi^1(f), \ldots, \Phi^{n-1}(f)) = \Phi^n(h_{\vec{a}}),$$

with $\Phi^j(h_{\vec{a}}) = \Phi^j(f), j = 1, \ldots, (n-1)$ and $\|h\|_{H(\bar{A})} \leqq C\|f\|_{H(\bar{A})}$. We shall write $\Omega = \Omega_1$.

Definition 2.2. We define the *twisted direct sum of order n*, $\overset{n}{\oplus}\bar{A}_\Phi$, to be the set of all elements $\vec{a} = (a_1, \ldots, a_n) \in (A_0 + A_1)^n$ such that

$$\|\vec{a}\|_{\overset{n}{\oplus}\bar{A}_\Phi} = \|(a_1, \ldots, a_{n-1})\|_{\overset{n}{\oplus}\bar{A}_\Phi} + \|\Omega_{n-1}(a_1, \ldots, a_{n-1}) - a_n\|_{\bar{A}_\Phi} < +\infty,$$

where, if $n = 1$, $\overset{1}{\oplus}\bar{A}_\Phi = \bar{A}_{\Phi^1}$.

In the case $n = 2$, with the notation of [CCS] we have $\vec{\Phi} = (\Phi, \Psi)$, $\bar{A}_{\Phi^1} = \bar{A}_\Phi$, $\Omega_1 = \Omega$ and $\overset{2}{\oplus}\bar{A}_\Phi = \bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi$.

Theorem 2.3. *If $\vec{\Phi} = (\Phi^1, \ldots, \Phi^n)$ is compatible, then $\overset{n}{\oplus}\bar{A}_\Phi = \mathscr{E}_n$ with equivalent norms.*

PROOF. In the case $n = 2$, choose $(a,b) \in \bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi$. Then $\Omega a = \Phi^2(h_a)$ with $\Phi^1(h_a) = a$, $\|h_a\|_{H(\bar{A})} \leqq C\|a\|_{\bar{A}_\Phi}$, $b - \Omega a = \Phi^1(g) = \Phi^2(h)$ with $\Phi^1(h) = 0$ and $\|h\|_{H(\bar{A})} \leqq C\|b - \Omega a\|_{\bar{A}_\Phi}$.

Therefore, $a = \Phi^1(h_a + h)$, $b = b - \Omega a + \Omega a = \Phi^2(h_a + h)$; that is, $(a,b) \in \mathscr{E}_2$ and

$$\|(a,b)\|_{\mathscr{E}_2} \leqq \|h_a + h\|_{H(\bar{A})} \leqq C(\|a\|_{\bar{A}_\Phi} + \|b - \Omega a\|_{\bar{A}_\Phi}) = C\|(a,b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}.$$

Let now $(a,b) \in \mathscr{E}_2$ and set $a = \Phi^1(h)$, $b = \Phi^2(h)$ and $\|h\|_{H(\bar{A})} \leqq C\|(a,b)\|_{\mathscr{E}_2}$. Then $a \in \bar{A}_\Phi$, $\Omega a = \Phi^2(h_a)$ and $\Omega a - b = \Phi^2(h_a - h)$, with $\Phi^1(h_a - h) = 0$. Therefore, $\Omega a - b \in \bar{A}_\Phi$ and

$$\|(a,b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi} = \|a\|_{\bar{A}_\Phi} + \|\Omega a - b\|_{\bar{A}_\Phi} \leqq C(\|h\|_{H(\bar{A})} + \|h_a - h\|_{H(\bar{A})})$$
$$\leqq C\|h\|_{H(\bar{A})} \leqq C\|(a,b)\|_{\mathscr{E}_2}.$$

Now let us assume that the equality holds for $n - 1$ and let us prove it for n.

Let $\vec{a} = (a_1, \ldots, a_n) \in \overset{n}{\oplus} \bar{A}_\Phi$. Then $(a_1, \ldots, a_{n-1}) \in \overset{n-1}{\oplus} \bar{A}_\Phi$ and, by induction hypothesis, $(a_1, \ldots, a_{n-1}) \in \mathscr{E}_{n-1}$. That is, for any $\mathscr{E} > 0$ there exists $f \in H(\bar{A})$ such that

$$\Phi^j(f) = a_j \ (1 \leqq j \leqq n-1) \text{ and } \|f\|_{H(\bar{A})} \leqq C\|(a_1, \ldots, a_{n-1})\|_{\overset{n-1}{\oplus} \bar{A}_\Phi}$$

Moreover,

$$\Omega_{n-1}(a_1, \ldots, a_{n-1}) = \Phi^n(h) \text{ and } \Phi^j(h) = \Phi^j(f) \quad (1 \leqq j \leqq n-1).$$

Then $\Omega_{n-1}(a_1, \ldots, a_{n-1}) - a_n = \Phi^n(h) - a_n = \Phi^n(g)$ with $\Phi^1(g) = \cdots = \Phi^{n-1}(g) = 0$, and

$$\|h\|_{H(\bar{A})} \leqq \|\Omega_{n-1}(a_1, \ldots, a_{n-1}) - a_n\|_{\overset{n-1}{\oplus} \bar{A}_\Phi}.$$

Therefore, $a_n = \Phi^n(h - g)$ and $a_j = \Phi^j(f) = \Phi^j(h - g)$. That is $\vec{a} \in \mathscr{E}_n$ and

$$\|\vec{a}\|_{\mathscr{E}_n} \leqq \|h - g\|_{H(\bar{A})} \leqq C\|\vec{a}\|_{\overset{n}{\oplus} \bar{A}_\Phi}.$$

Conversely, if $\vec{a} = (a_1, \ldots, a_n) \in \mathscr{E}_n$, for any $\varepsilon > 0$ there exists $f \in H(\bar{A})$ such that $a_j = \Phi^j(f)$ and $\|f\|_{H(\bar{A})} \leqq (1 + \varepsilon)\|\vec{a}\|_{\mathscr{E}_n}$. Then $(a_1, \ldots, a_{n-1}) \in \mathscr{E}_{n-1} = \overset{n-1}{\oplus} \bar{A}_\Phi$ and

$$\Omega_{n-1}(a_1, \ldots, a_{n-1}) - a_n = \Phi^n(h) - \Phi^n(f) = \Phi^n(h - f)$$

with

$$\Phi^j(h - f) = 0 \quad (1 \leqq j \leqq n-1).$$

Therefore $\Omega_{n-1}(a_1, \ldots, a_{n-1}) - a_n \in \bar{A}_\Phi$ and

$$\|\Omega_{n-1}(a_1,\dots,a_{n-1}) - a_n\|_{\bar{A}_\Phi} \leqq C\,\|h - f\|_{H(\bar{A})} \leqq C\,\|(a_1,\dots,a_{n-1})\|_{\overset{n-1}{\oplus}\bar{A}_\Phi}$$
$$\leqq C\,\|f\|_{H(\bar{A})} \leqq C(1+\varepsilon)\,\|\vec{a}\|_{\mathscr{E}_n}.$$

As a consequence, Ω_n is well defined on $\overset{n}{\oplus}\bar{A}_\Phi$ and $\Omega_n\colon \overset{n}{\oplus}\bar{A}_\Phi \to \bar{A}_{\Phi^{n+1}}$ is bounded. Now, if we define $T_n(a_1,\dots,a_n) = (Ta_1,\dots,Ta_n)$, where $T\colon \bar{A}\to\bar{B}$ is a given interpolation operator, we get the following result.

THEOREM 2.4. *If $(\Phi^1,\dots,\Phi^n,\Phi^{n+1})$ is compatible and*

$$[T_n,\Omega_n] = T\Omega_{n,\bar{A}} - \Omega_{n,\bar{B}}T_n,$$

then

$$[T_n,\Omega_n]\colon \overset{n}{\oplus}\bar{A}_\Phi \to \bar{B}_{\Phi^1}$$

is bounded.

PROOF. Let $\vec{a} = (a_1,\dots,a_n)\in\overset{n}{\oplus}\bar{A}_\Phi$. Then, there exists $f\in H(\bar{A})$ such that $\Phi^j(f) = a_j$, $j = 1,\dots,n$ and $\|f\|_{H(\bar{A})} \leqq C\,\|\vec{a}\|_{\overset{n}{\oplus}\bar{A}_\Phi}$. By definition, we have $\Omega_n(\vec{a}) = \Phi^{n+1}(h)$ with $\Phi^j(h) = \Phi^j(f)$ $(1\leqq j\leqq n)$ and $\|h\|_{H(\bar{A})} \leqq C\,\|\vec{a}\|_{\overset{n}{\oplus}\bar{A}_\Phi}$. Then,

$$T\Omega_n(\vec{a}) = T\Phi^{n+1}(h) = \Phi^{n+1}(H(T)h).$$

On the other hand,

$$\Omega_n(Ta_1,\dots,Ta_n) = \Omega_n(T\Phi^1(f),\dots,T\Phi^n(f))$$
$$= \Omega_n(\Phi^1(H(T)f),\dots,\Phi^n(H(T)f)) = \Phi^{n+1}(g),$$

with $\Phi^j(g) = \Phi^j(H(T)f)$, $\|g\|_{H(\bar{B})} \leqq G\|f\|_{H(\bar{A})}$. Hence,

$$[T_n,\Omega_n](\vec{a}) = \Phi^{n+1}(H(T)h - g),$$

with $\Phi^j(H(T)h - g) = 0, j = 1,\dots,n$. Therefore, $[T_n,\Omega_n](\vec{a})\in\bar{B}_{\Phi^1}$ and

$$\|[T_n,\Omega_n](\vec{a})\|_{\bar{B}_{\Phi^1}} \leqq C\,\|H(T)h - g\|_{H(\bar{B})} \leqq C\,\|f\|_{H(\bar{A})} \leqq C\,\|\vec{a}\|_{\overset{n}{\oplus}\bar{A}_\Phi}.$$

Let us now consider the following subset of $\overset{n}{\oplus}\bar{A}_\Phi$:

$$\mathscr{B} = \{(a,\Omega_1 a,\Omega_2(a,\Omega_1 a),\dots,\Omega_{n-1}(a,\Omega_1 a,\dots));\, a\in\bar{A}_{\Phi^1}\}.$$

Then, $\mathscr{B}$ is isometric to $\bar{A}_{\Phi^1}$ since, if $J_n(a) = (a,\Omega_1 a,\dots,\Omega_{n-1}(a,\dots))$, then $\|J_n(a)\|_{\overset{n}{\oplus}\bar{A}_\Phi} = \|a\|_{\bar{A}_{\Phi^1}}$.

Hence, we can define, for $a\in\bar{A}_{\Phi^1}$,

$$\Omega_n a = \Omega_n(J_n(a)).$$

With this notation the elements of $\mathscr{B}$ can be written as $(a,\Omega_1 a,\dots,\Omega_{n-1}a)$ and we can consider the commutator

$$[T, \Omega_n]a = T\Omega_n a - \Omega_n Ta = T\Omega_n(J_n(a)) - \Omega_n(J_n(Ta)).$$

Now, by Theorem 2.4, we get that, for every $a \in \bar{A}_{\Phi^1}$

$$\|[T_n, \Omega_n](J_n(a))\|_{\bar{B}_{\Phi^1}} \leqq C \|a\|_{\bar{A}_{\Phi^1}},$$

from which we will obtain Theorem 2.6, which in particular gives the estimates for higher order commutators of [Ro], [CCMS] and [Mi].

First we set some notation, similar to that of [Ro] and [Mi] for the higher order commutator estimates in the case of the complex and the real method, respectively.

If $T: \bar{A} \to \bar{B}$ is an interpolation operator, in order to study now the boundedness of

$$C_n = \begin{cases} [T, \Omega_1], & \text{if } n = 1 \\ [T, \Omega_2] - \Omega_1[T, \Omega_1], & \text{if } n = 2 \\ \vdots & \vdots \\ [T, \Omega_n] - \sum_{k=1}^{n-1} \Omega_k C_{n-k}, & \text{if } n > 2, \end{cases}$$

we say that the compatible system of interpolators $(\Phi^1, \ldots, \Phi^{n+1})$ satisfies the condition

(2) $\quad (\Phi^j, \ldots, \Phi^{n+1})(\operatorname{Ker} \Phi^1 \cap \cdots \cap \operatorname{Ker} \Phi^{j-1}) \subset \operatorname{Im}(\Phi^1, \ldots, \Phi^{(n+1)-(j-1)}),$

for every index $j = 2, \ldots, n+1$, when $\Phi^1(f) = \cdots = \Phi^{j-1}(f) = 0$ implies that there exists $h \in H(\bar{A})$ such that

$$\Phi^p(h) = \Phi^{p+j-1}(f),\ p = 1, \ldots, (n+1) - (j-1),$$

and $\|h\|_{H(\bar{A})} \leqq C \|f\|_{H(\bar{A})}$.

To give by induction a complete proof of our theorem we first prove a lemma.

Lemma 2.5. *If (Φ^1, Φ^2, Φ^3) is compatible and $(\Phi^2, \Phi^3)(\operatorname{Ker} \Phi^1) \subset \operatorname{Im}(\Phi^1, \Phi^2)$, then, for every interpolation operator $T: \bar{A} \to \bar{B}$,*

$$\Omega_2(Ta, \Omega_1 Ta) - \Omega_2(Ta, T\Omega_1 a) = -\Omega_1[T, \Omega_1]a + Sa \quad (a \in \bar{A}_{\Phi^1})$$

with $\|Sa\|_{\bar{B}_{\Phi^1}} \leqq C \|a\|_{\bar{A}_{\Phi^1}}$.

Proof. Let $B = \Omega_2(Ta, \Omega_1 Ta) - \Omega_2(Ta, T\Omega_1 a)$. We have that $\Omega_2(Ta, T\Omega_1 a) = \Phi^3(f)$ with $\Phi^1(f) = Ta$, $\Phi^2(f) = T\Omega_1 a$ and $\|f\|_{H(\bar{B})} \leqq C \|a\|_{\bar{A}_{\Phi^1}}$. Similarly, $\Omega_2(Ta. \Omega_1 Ta) = \Phi^3(g)$ with $\Phi^1(g) = Ta$, $\Phi^2(g) = \Omega_1 Ta$ and $\|g\|_{H(\bar{B})} \leqq C \|a\|_{\bar{A}_{\Phi^1}}$. Then, $B = \Phi^3(g - f) = \Phi^3(h)$, with $\Phi^1(h) = 0$, $\Phi^2(h) = -[T, \Omega_1]a$ and $\|h\|_{H(\bar{B})} \leqq C \|a\|_{\bar{A}_{\Phi^1}}$.

Therefore, we have that $B = \Phi^2(h^*)$ with $\Phi^1(h^*) = -[T, \Omega_1]a$ and $\|h^*\|_{H(\bar{B})} \leqq C \|a\|_{\bar{A}_{\Phi^1}}$.

Now, $\Omega_1\Phi^1(h^*) = \Phi^2(\bar{h})$ with $\Phi^1(h^*) = \Phi^1(\bar{h})$, and $\|\bar{h}\|_{H(\bar{B})} \leqq C\,\|a\|_{\bar{A}_{\Phi^1}}$. Hence, $B = -\Omega_1[T,\Omega_1]a + \Phi^2(h^* - \bar{h})$ with $\Phi^1(h^* - \bar{h}) = 0$ and thus, $B = -\Omega_1[T,\Omega_1]a + Sa$ with $\|Sa\|_{\bar{B}_{\Phi^1}} = \|\Phi^2(h^* - \bar{h})\|_{\bar{B}_{\Phi^1}} \leqq C\,\|a\|_{\bar{A}_{\Phi^1}}$.

THEOREM 2.6. *Let* $(\Phi^1,\dots,\Phi^{n+1})$ *be compatible with property* (2), *for every index* $j = 2,\dots,n+1$, *and* $T\colon \bar{A} \to \bar{B}$ *an interpolation operator.*

Then $C_n = [T,\Omega_n] - \sum_{k=1}^{n-1}\Omega_k C_{n-k}$ $(C_1 = [T,\Omega_1])$ *is a bounded operator*

$$C_n\colon \bar{A}_{\Phi^1} \to \bar{B}_{\Phi^1}.$$

PROOF. We know that $[T_n,\Omega_n](J_n(a)) \in \bar{B}_{\Phi^1}$. Now,

$$\begin{aligned}[T_n,\Omega_n](J_n(a)) &= T\Omega_n a - \Omega_n(Ta, T\Omega_1 a,\dots,T\Omega_{n-1}a)\\ &= T\Omega_n a - \Omega_n Ta + \Omega_n Ta - \Omega_n(Ta, T\Omega_1 a,\dots,T\Omega_{n-1}a)\\ &= [T,\Omega_n]a + \mathcal{R}_n,\end{aligned}$$

where $\mathcal{R}_n = \Omega_n Ta - \Omega_n(Ta, T\Omega_1 a,\dots,T\Omega_{n-1}a)$. The following step is to show that

$$\mathcal{R}_n = -\sum_{k=1}^{n-1}\Omega_k C_{n-k}a + S_n a,$$

where $S_n\colon \bar{A}_{\Phi^1} \to \bar{B}_{\Phi^1}$ is a bounded operator.

If $A_n^k = \Omega_{n-k}C_k a$,

$$\begin{aligned}\mathcal{R}_n = {} & \left[\Omega_n Ta - \Omega_n\left(Ta,\dots,\Omega_{n-2}Ta, T\Omega_{n-1}a - \sum_{k=1}^{n-2}A_{n-1}^k\right)\right]\\ & + \left[\Omega_n\left(Ta,\dots,\Omega_{n-2}Ta, T\Omega_{n-1}a - \sum_{k=1}^{n-2}A_{n-1}^k\right)\right.\\ & \quad \left. - \Omega_n\left(Ta,\dots,\Omega_{n-3}Ta, T\Omega_{n-2}a - \sum_{k=1}^{n-3}A_{n-2}^k, T\Omega_{n-1}a - \sum_{k=1}^{n-3}A_{n-1}^k\right)\right]\\ & + \left[\Omega_n\left(Ta,\dots,\Omega_{n-3}Ta, T\Omega_{n-2}a - \sum_{k=1}^{n-3}A_{n-2}^k, T\Omega_{n-1}a - \sum_{k=1}^{n-3}A_{n-1}^k\right)\right.\\ & \quad \left. -\Omega_n\left(Ta,\dots,\Omega_{n-4}Ta, T\Omega_{n-3}a - \sum_{k=1}^{n-4}A_{n-3}^k,\dots, T\Omega_{n-1}a - \sum_{k=1}^{n-4}A_{n-1}^k\right)\right]\\ & + \cdots\\ & + [\Omega_n(Ta, \Omega_1 Ta, T\Omega_2 a - \Omega_1 C_1 a,\dots, T\Omega_{n-1}a - \Omega_{n-2}C_1 a)\\ & \quad -\Omega_n(Ta, T\Omega_1 a,\dots, T\Omega_{n-1}a)]\\ = {} & \sum_{k=2}^{n} B_k^n,\end{aligned}$$

where

$$B_k^n = \Omega_n\left(J_k(Ta), T\Omega_k a - \sum_{p=1}^{k-1} \Omega_{k-p}C_p a, \ldots, T\Omega_{n-1}a - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a\right)$$
$$- \Omega_n\left(J_{k-1}(Ta), T\Omega_{k-1}a - \sum_{p=1}^{k-2} \Omega_{k-1-p}C_p a, \ldots, T\Omega_{n-1}a - \sum_{p=1}^{k-2} \Omega_{n-1-p}C_p a\right)$$

We now prove by induction in k and n that, if $2 \leqq k \leqq n$,

(i) B_k^n is well defined, in the sense that

$$b_k^n = \left(J_k(Ta), T\Omega_k a - \sum_{p=1}^{k-1} \Omega_{k-p}C_p a, \ldots, T\Omega_{n-1}a - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a\right) \in \overset{n}{\oplus} \bar{B}_\Phi,$$

and

(ii) $B_k^n = -\Omega_{n-k+1}C_{k-1}a + T_k^n$, where $T_k^n\colon \bar{A}_{\Phi^1} \to \bar{B}_{\Phi^1}$ is bounded.

If $n = 2$ then $k = 2$ and $b_1^2 = (Ta, T\Omega_1 a) \in \overset{2}{\oplus} \bar{B}_{\Phi^1}, b_2^2 = (Ta, \Omega_1 Ta) \in \overset{2}{\oplus} \bar{B}_{\Phi^1}$ and

$$B_2^2 = \Omega_2(Ta, \Omega_1 Ta) - \Omega_2(Ta, T\Omega_1 a).$$

By Lemma 2.5, B_2^2 satisfies (ii). Let us assume now that (i) and (ii) hold for B_k^{n-1} for $k = 2, \ldots, (n-1)$, and prove that they also hold for B_k^n $(k = 2, \ldots, n)$. Let $k = 2, \ldots, (n-1)$. Then,

$$b_k^n = \left(b_k^{n-1}, T\Omega_{n-1}a - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a\right).$$

To show that $b_k^n \in \overset{n}{\oplus} \bar{A}_\Phi$ we have to see that

(a) $b_k^{n-1} \in \overset{n-1}{\oplus} \bar{B}_\Phi$, which is true by induction, and

(b) $A = T\Omega_{n-1}a - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a - \Omega_{n-1}b_k^{n-1} \in \bar{B}_{\Phi^1}$.

Now, as we did for $\mathscr{R}_n$, one can show that

$$\begin{aligned} A &= [T\Omega_{n-1}a - \Omega_{n-1}(Ta, T\Omega_1 a, \ldots, T\Omega_{n-2}a)] \\ &\quad + \Bigg[\Omega_{n-1}(Ta, T\Omega_1 a, \ldots, T\Omega_{n-2}a) \\ &\quad - \Omega_{n-1}\left(Ta, \Omega_1 Ta, \ldots, \Omega_{k-1}Ta, T\Omega_k a - \sum_{p=1}^{k-1} \Omega_{k-p}C_p a, \ldots\right)\Bigg] \\ &\quad - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a \\ &= [T_{n-1}, \Omega_{n-1}]a - \sum_{j=2}^{k} B_j^{n-1} - \sum_{p=1}^{k-1} \Omega_{n-1-p}C_p a \\ &= [T_{n-1}, \Omega_{n-1}]a - \sum_{p=1}^{k-1} (B_{p+1}^{n-1} + \Omega_{n-1-p}C_p a). \end{aligned}$$

Since, by induction,

$$-B_{p+1}^{n-1} = \Omega_{n-1-p} C_p a + T_{p+1}^{n-1} a,$$

with $T_{p+1}^{n-1}\colon \bar{A}_{\Phi^1} \to \bar{B}_{\Phi^1}$ bounded, we get that $A \in \bar{B}_{\Phi^1}$ and $\|A\|_{\bar{B}_{\Phi^1}} \leqq C\,\|a\|_{\bar{A}_{\Phi^1}}$. Now, if $k = n$, we have that $b_n^n = (J_n(Ta)) \in \overset{n}{\oplus}\, \bar{B}_\Phi$ and

$$b_{n-1}^n = \left(b_{n-1}^{n-1}, T\Omega_{n-1}a - \sum_{p=1}^{n-2} \Omega_{n-1-p} C_p a\right) \in \overset{n}{\oplus}\, \bar{B}_\Phi,$$

since, by induction, $b_{n-1}^{n-1} \in \overset{n-1}{\oplus}\, \bar{B}_\Phi$ and $T\Omega_{n-1}a - \sum_{p=1}^{n-2} \Omega_{n-1-p} C_p a - \Omega_{n-1} b_{n-1}^{n-1} = C_{n-1} a \in \bar{B}_{\Phi^1}$. Finally, $B_k^n = \Phi^{n+1}(f)$ with $\Phi^1(f) = \cdots = \Phi^{k-1}(f) = 0$, $\Phi^k(f) = -C_{k-1}a$, $\Phi^{k+1}(f) = -\Omega_1 C_{k-1} a, \ldots, \Phi^n(f) = -\Omega_{n-k} C_{k-1} a$. Then, using (2), we get that there exists $h \in H(\bar{B})$ such that $\Phi^j(h) = \Phi^{j+k-1}(f) = -\Omega_{j-1} C_{k-1} a$, $j = 1, \ldots, n-k+1$, and $\|h\|_{H(\bar{B})} \leqq C\,\|f\|_{H(\bar{B})}$. On the other hand, $\Omega_{n-k+1} C_{k-1} a = \Phi^{n-k+2}(g)$, with $\Phi^1(g) = C_{k-1}a$, and $\Phi^j(g) = \Omega_{j-1} C_{k-1} a$ for $j = 2, \ldots, n-k+1$. Thus, $\Phi^j(h+g) = 0$ $(j = 1, \ldots, n-k+1)$ and therefore

$$\Phi^{n-k+2}(h+g) = \Phi^{n-k+2}(h) + \Omega_{n-k+1} C_{k-1} a = \Phi^{n+1}(f) + \Omega_{n-k+1} C_{k-1} a \in \bar{B}_{\Phi^1},$$

with norm less than or equal to $C\,\|a\|_{\bar{A}_{\Phi^1}}$. Hence, $B_k^n = -\Omega_{n-k+1} C_{k-1} a + T_k^n a$, with $T_k^n\colon \bar{A}_{\Phi^1} \to \bar{B}_{\Phi^1}$ bounded.

3. The classical interpolation methods.

We shall now show that the complex method and the real J and K methods are associated to systems of interpolators Φ^k which satisfy the compatibility conditions 2.1 and property (2). We have to see that, for every $j = 2, \ldots, n+1$,

(3) $$\operatorname{Im} \Phi^1 \subset \Phi^j(\operatorname{Ker} \Phi^1 \cap \cdots \cap \operatorname{Ker} \Phi^{j-1}),$$

and,

(4) $$(\Phi^j, \ldots, \Phi^{n+1})(\operatorname{Ker} \Phi^1 \cap \cdots \cap \operatorname{Ker} \Phi^{j-1}) \subset \operatorname{Im}(\Phi^1, \ldots, \Phi^{(n+1)-(j-1)}).$$

(I) *Complex method.*

For simplicity, we shall work in the unit disk D and $\theta = 0$ (see [FK]). Let us define

$$\Phi_C^k(f) = \frac{f^{(k-1)}(0)}{(k-1)!}$$

on $H_C(\bar{A})$.

Then, if $a = f(0) \in \bar{A}_{\Phi^1}$ and $g(z) = z^{j-1} f(z)$, we get that $\|g\|_{H(\bar{A})} = \|f\|_{H(\bar{A})}$, $\Phi^k(g) = 0$, $k = 1, \ldots, j-1$ and $\Phi_C^j(g) = f(0) = a$; that is, (3) holds.

Now, if $f(0) = \cdots = f^{(j-1)}(0) = 0$ and we consider $g(z) = z^{-j+1} f(z)$, we have that $\|g\|_{H(\bar{A})} = \|f\|_{H(\bar{A})}$ and $\Phi_C^p(f) = \Phi_C^{(p-j)}(g)$ for any $p = j, \ldots, n+1$ and we obtain (4).

(II) *Real J-method.*
Set

$$\Phi_J^k(u) = \frac{1}{(k-1)!}\int_0^\infty (\log t)^{k-1} u(t)\frac{dt}{t} \quad (u \in H_J(\bar{A})).$$

Let $a = \int_0^\infty u(t)\frac{dt}{t}$ and let us define, for $p = 1,\dots,k-1$

$$v_0(t) = u(t),$$
$$v_p(t) = v_{p-1}(t) - v_{p-1}(te).$$

PROPOSITION 3.1. (i) $\frac{1}{k!}\int_0^\infty (\log t)^k v_{j-1}(t)\frac{dt}{t} = 0$, *for every* $k < j-1$; *that is,* $\Phi_J^k(v_{j-1}) = 0$, *for every* $k < j$.

(ii) $\Phi_J^j(u) = \frac{1}{(j-1)!}\int_0^\infty (\log t)^{j-1} v_{j-1}(t)\frac{dt}{t} = \int_0^\infty u(t)\frac{dt}{t} = a.$

PROOF. We prove it by induction in k and j. If $j = 2$,

$$\int_0^\infty v_1(t)\frac{dt}{t} = \int_0^\infty (u(t) - u(te))\frac{dt}{t} = 0$$

and

$$\int_0^\infty (\log t) v_1(t)\frac{dt}{t} = \int_0^\infty (\log t) u(t)\frac{dt}{t} - \int_0^\infty \left(\log\frac{t}{e}\right) u(t)\frac{dt}{t} = a.$$

If $j > 2$, let us assume (i) and (ii) for $j-1$ and prove them for j. Now

$$\begin{aligned}\int_0^\infty (\log t)^k v_j(t)\frac{dt}{t} &= \int_0^\infty (\log t)^k v_{j-1}(t)\frac{dt}{t} - \int_0^\infty (\log t)^k v_{j-1}(te)\frac{dt}{t} \\ &= \int_0^\infty (\log t)^k v_{j-1}(t)\frac{dt}{t} - \int_0^\infty \left(\log\frac{t}{e}\right)^k v_{j-1}(t)\frac{dt}{t} \\ &= -\sum_{p=0}^{k-1}\binom{k}{p}\int_0^\infty (\log t)^p (-1)^{k-p} v_{j-1}(t)\frac{dt}{t}.\end{aligned}$$

Now, using the induction hypothesis,we get that, if $k < j$, all the terms in the above sum are zero, while for $k = j$

$$\int_0^\infty (\log t)^j v_j(t)\frac{dt}{t} = j\int_0^\infty (\log t)^{j-1} v_{j-1}(t)\frac{dt}{t} = \cdots = j!\int_0^\infty u(t)\frac{dt}{t}.$$

To show now that (4) holds, let us consider $u \in H_J(\bar{A})$ such that

(5) $$\int_0^\infty u(t)\frac{dt}{t} = \int_0^\infty \log t\, u(t)\frac{dt}{t} = \cdots = \int_0^\infty (\log t)^{j-1}u(t)\frac{dt}{t} = 0,$$

and let us define

$$v_0(t) = u(t)$$
$$v_p(t) = -\int_0^t v_{p-1}(s)\frac{ds}{s}.$$

Then we know that $v_p \in H_J(\bar{A})$ and $\|v_p\|_{H_J(\bar{A})} \leqq C\|u\|_{H_J(\bar{A})}$ (see [JRW]). Moreover, for every k,

$$\Phi_J^{k+j-1}(u) = \Phi_J^k(v_{j-1}).$$

To show this, we use integration by parts and the two following lemmae.

LEMMA 3.2. *If $u \in H_J(\bar{A})$ satisfies that $\int_0^\infty u(t)\frac{dt}{t} = 0$ then, for every $k \in \mathbb{N}$,*

$$\lim_{t\to 0,\infty} (\log t)^k \int_0^t u(s)\frac{ds}{s} = 0,$$

in the $A_0 + A_1$-norm.

PROOF. We have that

$$\left\|(\log t)^k \int_0^t u(s)\frac{ds}{s}\right\|_{A_0+A_1} \leqq |\log t|^k \int_0^t \frac{\|u(s)\|_0}{s^\theta} s^\theta \frac{ds}{s}$$
$$\leqq |\log t|^k \|u\|_{H_J(\bar{A})} t^\theta,$$

where the last estimate follows by Hölder's inequality. Hence, the above expression goes to zero when t tends to zero.

Now,

$$\left\|(\log t)^k \int_0^t u(s)\frac{ds}{s}\right\|_{A_0+A_1} = \left\|(\log t)^k \int_t^\infty u(s)\frac{ds}{s}\right\|_{A_0+A_1}$$
$$\leqq |\log t|^k \int_t^\infty \frac{\|u(s)\|_1}{s^\theta} s^\theta \frac{ds}{s}$$
$$\leqq |\log t|^k \int_t^\infty \frac{J(s, u(s))}{s^\theta} s^{\theta-1} \frac{ds}{s}$$
$$= |\log t|^k \|u\|_{H_J(\bar{A})} t^{\theta-1},$$

which tends to zero when t tends to infinity.

LEMMA 3.3. *If* (5) *holds, then, for every* $p < j - 1$, $\int_0^\infty v_p(t)\frac{dt}{t} = 0$.

PROOF. This follows by using integration by parts, induction and the previous lemma, since

$$\int_0^\infty v_p(t)\frac{dt}{t}$$

$$= \lim_{t\to 0}(\log t)\int_0^t v_{p-1}(s)\frac{ds}{s} - \lim_{t\to\infty}(\log t)\int_0^t v_{p-1}(s)\frac{ds}{s} + \int_0^\infty v_{p-1}(t)\log t\frac{dt}{t}$$

$$= \int_0^\infty v_{p-1}(t)\log t\frac{dt}{t} = \int_0^\infty v_{p-2}(t)(\log t)^2\frac{dt}{t} = \cdots =$$

$$= \int_0^\infty u(t)(\log t)^p\frac{dt}{t} = 0.$$

(III) *Real K-method.*

Let $(a_0, a_1) \in H_K(\bar{A})$ and set

$$\Phi_K^1(a_0, a_1) = a_0(t) + a_1(t)$$

$$\Phi_K^2(a_0, a_1) = -\left(\int_0^1 a_0(t)\frac{dt}{t} - \int_1^\infty a_1(t)\frac{dt}{t}\right)$$

$$\Phi_K^k(a_0, a_1) = (-1)^{k-1}\frac{1}{(k-2)!}\left(\int_0^1 (\log t)^{k-2}a_0(t)\frac{dt}{t} - \int_1^\infty (\log t)^{k-2}a_1(t)\frac{dt}{t}\right).$$

Then, to show that (3) holds, we just have to observe that $\operatorname{Im}\Phi_K^1 = \operatorname{Im}\Phi_J^1$ and if $u \in H_J(\bar{A})$ with $\int_0^\infty u(t)\frac{dt}{t} = 0$ then,

$$(b_0(t), b_1(t)) = \left(\int_0^t u(s)\frac{ds}{s}, \int_t^\infty u(s)\frac{ds}{s}\right) \in H_K(\bar{A}),$$

with $\|(b_0, b_1)\|_{K_K(\bar{A})} \leqq \|u\|_{H_J(\bar{A})}$, $\Phi_K^1(b_0, b_1) = 0$ and, for $j \geqq 2$,

$$\Phi_K^j(b_0, b_1) = (-1)^j\Phi_J^j(u).$$

Now, to see that (4) holds, we proceed as in the J-method. Let $(a_0, a_1) \in H_K(\bar{A})$ such that $a_0(t) + a_1(t) = 0$ and, for every $k = 0, \ldots, j-3$,

$$\int_0^1 (\log t)^k a_0(t)\frac{dt}{t} - \int_1^\infty (\log t)^k a_1(t)\frac{dt}{t} = 0.$$

Let us define

$$b_0^0(t) = a_0(t), \quad b_0^1(t) = a_1(t),$$

$$b_p^0(t) = \int_0^t b_{p-1}^0(s)\frac{ds}{s}, \quad b_p^1(t) = -\int_t^\infty b_{p-1}^1(s)\frac{ds}{s}.$$

.Then, one can show that

$$b_p^0 + b_p^1 = \int_0^\infty b_{p-1}^0(t)\frac{dt}{t} = 0, \quad p = 1,\ldots,(j-2),$$

and, using integration by parts and Lemma 3.2, we get that, for $k \geqq 1$,

$$\Phi^k(b_{j-2}^0, b_{j-2}^1) = \Phi^{k+j-1}(a_0, a_1).$$

Finally, we want to remark that, if we define Φ_K^k without the factor $(-1)^k$, then condition (4) does not hold and we have to reformulate Theorem 2.6 as it is done in [Mi].

4. One example.

Let $b \in \mathrm{BMO}(\mathrm{R}^n)$ and let M_b be the operator defined by $M_b f = bf$, which will also be denoted by b, and T a Calderón-Zygmund operator. In [CCMS], the authors together with M. Milman gave a proof of the boundedness of the first order operator

$$[T, b]\colon L^p\left(\frac{1}{1+|b|}\right) \to L^p\left(\frac{1}{1+|b|}\right) \quad (1 < p < \infty).$$

In this section we use Theorem 2.6 to show that the same happens for the second order commutator

$$(6) \qquad [[T, b], b]\colon L^p\left(\frac{1}{1+|b|}\right) \to L^p\left(\frac{1}{1+|b|}\right).$$

Similar results can be obtained for the n-order commutator and for the space $L^p(1/(1+|b|)^\alpha)$, for any $\alpha > 0$.

To prove (6) we shall use the following results about interpolation of L^p spaces with change of measures:

$$[L^p(\omega_0), L^p(\omega_1)]_{\delta^{(n)}(\theta)} = \left\{f; \left(\int\left[\frac{|f(x)|}{(1+|\log(\omega_0/\omega_1)|)^n}\right]^p \omega_0^{1-\theta}\omega_1^\theta\, dx\right)^{1/p} < \infty\right\}$$

(cf. [CC1]), and that for the "upper" Schechter method (cf. [CC2]), defined by

$$[L^p(\omega_0), L^p(\omega_1)]^{\delta^{(n)}(\theta)} \doteq \{a = f(\theta); f'(\theta) = \cdots = f^{(n)}(\theta) = 0, f \in \mathscr{F}\}$$

and endowed with the natural quotient norm,

$$[L^p(\omega_0), L^p(\omega_1)]^{\delta^{(n)}(\theta)} = \left\{f; \left(\int |f(x)|^p \left(1 + \left|\log\frac{\omega_0}{\omega_1}\right|\right)^{np} \omega_0^{1-\theta}\omega_1^{\theta}\,dx\right)^{1/p} < \infty\right\}.$$

Now we first show that, for the Banach couples $\bar{A} = (L^p(\omega_0), L^p(\omega_1))$, over the usual Banach space $H_C(\bar{A}) = \mathscr{F}(\bar{A})$ of analytic functions on the strip S of the complex interpolation method, the system of interpolators

$$\Phi^1(f) = f'(\theta), \quad \Phi^2(f) = f''(\theta), \quad \Phi^3(f) = \tfrac{1}{2}f'''(\theta)$$

is compatible and satisfies conditions (3) and (4), i.e.,

(i) $\operatorname{Im}\Phi^1 \subset \Phi^2(\operatorname{Ker}\Phi^1)$,

(ii) $\operatorname{Im}\Phi^1 \subset \Phi^3(\operatorname{Ker}\Phi^1 \cap \operatorname{Ker}\Phi^2)$,

(iii) $\Phi^3(\operatorname{Ker}\Phi^1 \cap \operatorname{Ker}\Phi^2) \subset \operatorname{Im}\Phi^1$, and

(iv) $(\Phi^2, \Phi^3)(\operatorname{Ker}\Phi^1) \subset \operatorname{Im}(\Phi^1, \Phi^2)$.

We will use the notation

$$E_f(z) = \left(\frac{\omega_0}{\omega_1}\right)^{z-\theta} f(\theta), \quad b = \log\frac{\omega_0}{\omega_1},$$

and φ will denote a suitable analytic function on S such that $\varphi^{-1}(0) = \theta$.

(i) To see that $\operatorname{Im}\Phi^1 \subset \Phi^2(\operatorname{Ker}\Phi^1)$, for a given $f \in \mathscr{F}(\bar{A})$ we define

$$G = \frac{f - E_f}{\varphi}$$

with $\varphi(\theta) = \varphi''(\theta) = 0$ and $\varphi'(\theta) = 1$. Then $f = \varphi G + E_f$ and

$$f'(\theta) = (\varphi G)'(\theta) + bf(\theta).$$

If

$$H_1(z) = E_f(z)\frac{\operatorname{sgn} b + \varphi(z)}{1 + |b|} - f\varphi(z),$$

an easy calculation shows that $H_1'(\theta) = 0$,

$$H_1''(\theta) = bf(\theta) + \frac{bf(\theta)}{1 + |b|} - 2f'(\theta),$$

and

$$f'(\theta) = \left(\frac{b}{1 + |b|} f(\theta) - (\varphi G)'(\theta)\right) - H_1''(\theta) = H_2''(\theta) - H_1''(\theta)$$

with $H_2'(\theta) = 0$, since it follows from $\varphi(\theta) = 0$, $(\varphi G)'(\theta) \in \bar{A}_\theta$ and $(b/(1 + |b|)f(\theta) \in \bar{A}_\theta$, that $\dfrac{b}{1 + |b|} f(\theta) - (\varphi G)'(\theta) \in \bar{A}_\theta$. Thus,

$$f'(\theta) = (H_2 - H_1)''(\theta), \quad (H_2 - H_1)'(\theta) = 0$$

and

$$\|H_2 - H_1\|_{\mathscr{F}} \leqq C\,\|f(\theta)\|_\theta.$$

(ii) To show that $\operatorname{Im}\Phi^1 \subset \Phi^3(\operatorname{Ker}\Phi^1 \cap \operatorname{Ker}\Phi^2)$ we write, as before,

$$f'(\theta) = (\varphi G)'(\theta) + bf(\theta)$$

and let φ_1 and φ_2 be as φ, but with the conditions

$$\varphi_1(\theta) = \varphi_1''(\theta) = \varphi_1'''(\theta) = 0, \quad \varphi_1'(\theta) = 1$$

and

$$\varphi_2(\theta) = \varphi_2'(\theta) = \varphi_2'''(\theta) = 0, \quad \varphi_2''(\theta) = 1.$$

If we define

$$H_3(z) = E_f(z)\frac{\operatorname{sgn} b + \varphi_1(z)}{(1 + |b|)^2} - \frac{\varphi_1(z)E_f(z)}{1 + |b|} + \frac{b\,|b|\,\varphi_2(z)fP}{(1 + |b|)^2},$$

then $H_3'(\theta) = H_3''(\theta) = 0$ and

$$H_3'''(\theta) = \frac{b\,|b|}{(1 + |b|)^2}(-2bf(\theta) + 3f'(\theta)).$$

Now, if we let

$$H_4(z, x) = H_3(z, x)\chi_{\{x;\, b(x) > 0\}} - H_3(z, x)\chi_{\{x;\, b(x) < 0\}},$$

we obtain $H_4'(\theta) = H_4''(\theta) = 0$ and

$$H_4'''(\theta) = \frac{|b|^2}{(1 + |b|^2)}[-2bf(\theta) + 3f'(\theta)],$$

with

$$\frac{-2bf(\theta) + 3f'(\theta)}{(1 + |b|)^2}(1 + 2\,|b|) = g(\theta) \in A_\theta^-.$$

Hence, there exists $H_5 \in \mathscr{F}(A)^-$ such that $H_5'''(\theta) = g(\theta)$ and $H_5(\theta) = H_5'(\theta) = H_5''(\theta) = 0$ and we find that

$$-2bf(\theta) + 3f'(\theta) = H_4'''(\theta) + H_5'''(\theta) = H_6'''(\theta)$$

with $H_6'(\theta) = H_6''(\theta) = 0$. It follows that

$$f'(\theta) = (\varphi G)'(\theta) + \tfrac{3}{2}f'(\theta) - \tfrac{1}{2}H_6'''(\theta)$$

and $f'(\theta) = H_7'''(\theta)$ with $H_7'(\theta) = H_7''(\theta) = 0$.

(iii) Let $f'''(\theta) \in \Phi^3(\operatorname{Ker}\Phi^1 \cap \operatorname{Ker}\Phi^2))$ with $f'(\theta) = f''(\theta) = 0$, thus $f(\theta) \in \bar{A}^{\delta''}(\theta)$ and hence $b^2 f(\theta) \in \bar{A}_\theta$ and $b^3 f(\theta) \in \operatorname{Im}\Phi^1$.

We define

$$G = \frac{f - E_f + \varphi E_f b}{\varphi^2}$$

with $\varphi(\theta) = \varphi''(\theta) = \varphi'''(\theta) = 0$ and $\varphi'(\theta) = 1$. Then $f'''(\theta) = (\varphi^2 G)'''(\theta) + b^3 f(\theta) - 3b^3 f(\theta) \in \operatorname{Im}\Phi^1$.

(iv) To prove that $(\Phi^2, \Phi^3)(\operatorname{Ker}\Phi^1) \subset \operatorname{Im}(\Phi^1, \Phi^2)$, let $f'(\theta) = 0$, so that $bf(\theta) = g(\theta) \in \bar{A}_\theta$. We define

$$H = \frac{f - E_f + \varphi E_g}{\varphi^2} \quad (\varphi \text{ as before}),$$

and an easy calculation shows that

$$\Phi^2(f) = f''(\theta) = (\varphi^2 H)''(\theta) - b^2 f(\theta)$$

and

$$\Phi^3(f) = \tfrac{1}{2} f'''(\theta) = \tfrac{1}{2}(\varphi^2 H)'''(\theta) - b^3 f(\theta)$$

with $(\varphi^2 H)'''(\theta) \in \operatorname{Im}\Phi^1$. Thus,

$$\Phi^3(f) = G'(\theta) - b^3 f(\theta) = F''(\theta) - b^3 f(\theta) = (F - E_g)''(\theta) \quad (F(\theta) = 0)$$

and

$$\Phi^2(f) = (F - E_g)'(\theta) + (\varphi^2 H)''(\theta) - F'(\theta),$$

where $(\varphi^2 H)''(\theta) - F'(\theta) \in \bar{A}_\theta$.

Now we use the following

Claim. $\bar{A}_\theta \hookrightarrow \{x = f'(\theta);\ f \in \mathscr{F}(\bar{A}),\ f''(\theta) = 0\}$.

Thus there exists $F_2 \in \mathscr{F}(\bar{A})$ such that $F_2''(\theta) = 0$ and

$$\Phi^2(f) = (F - E_g + F_2)'(\theta) = \Phi^1(F - E_g + F_2).$$

Moreover, $\Phi^3(f) = \Phi^2(F - E_g + F_2)$.

To prove the claim, for any $f(\theta) \in \bar{A}_\theta$ let us consider $g(\theta) = f(\theta)/(1 + |b|)$ and

$$h = \tfrac{1}{2}\varphi E_f - E_g(\operatorname{sgn} b + \varphi) + \varphi_2 f \frac{b}{1 + |b|} \in \mathscr{F}(\bar{A}),$$

with $\varphi(\theta) = \varphi''(\theta) = \varphi_2(\theta) = \varphi_2''(\theta) = 0$ and $\varphi'(\theta) = \varphi_2'(\theta) = 1$.

Then $\|h\|_{\mathscr{F}} \leqq C\,\|f(\theta)\|_\theta$, $h'(\theta) = -(1/2)f(\theta)$ and

$$h''(\theta) = -bf(\theta) + b\,|b|\,g(\theta) + 2bg(\theta) - f(\theta)\frac{b}{1 + |b|} = 0.$$

For every $u \in (L^p(\omega_0), L^p(\omega_1))_{\delta'_\theta} = \operatorname{Im} \Phi^1$, the function

$$h_u(z) = \left(\frac{\omega_0}{\omega_1}\right)^{z-\theta} \frac{\operatorname{sgn} b + \varphi(z)}{1 + |b|} u,$$

with $\varphi(\theta) = \varphi''(\theta) = \varphi'''(\theta) = 0$ and $\varphi'(\theta) = 1$ has the following properties:

(a) $\|h\|_{\mathscr{F}} \leqq C \|u\|_{\delta'_\theta}$.

(b) $h'_u(\theta) = u$.

(c) $h''_u(\theta) = bu \dfrac{2 + |b|}{1 + |b|}$,

(d) $h'''_u(\theta) = b^2 u \dfrac{3 + |b|}{1 + |b|}$.

Hence, h_u is almost optimal, the associated Ω-operators are defined by

$$\Omega_1 u = bu \frac{2 + |b|}{1 + |b|}$$

and

$$\Omega_2 u = \tfrac{1}{2} b^2 u \frac{3 + |b|}{1 + |b|},$$

and an easy computation shows that

$$([T, \Omega_2] - \Omega_1 [T, \Omega_1])u = [[T, b], b]u + \frac{b^2}{(1 + |b|)^2} Tu - \frac{b}{1 + |b|} T\left(\frac{b}{1 + |b|} u\right)$$
$$- \frac{b}{1 + |b|} [T, b]u + [T, b]\left(\frac{b}{1 + |b|} u\right).$$

We have seen that $[T, \Omega_2] - \Omega_1 [T, \Omega_1]$ is a bounded operator in $\bar{A}_{\delta'_\theta}$ and it is known that the same is true for $[T, b]$ [cf. [CCMS]) and for T. Since

$$\left\| \frac{b}{1 + |b|} T\left(\frac{b}{1 + |b|} u\right) \right\|_{\delta'_\theta} \leqq C \|u\|_{\delta'_\theta}$$

and

$$\left\| \frac{b^2}{(1 + |b|)^2} Tu \right\|_{\delta'_\theta} \leqq C \|u\|_{\delta'_\theta},$$

it follows that $[[T, b], b]$ is a bounded operator in $\bar{A}_{\delta'_\theta}$. Therefore, by taking $\theta = 1/2$, $\omega_0 = \exp(-b/2)$ and $\omega_1 = \exp(b/2)$,

$$[[T, b], b] \colon L^p\left(\frac{1}{1 + |b|}\right) \to L^p\left(\frac{1}{1 + |b|}\right).$$

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