MATH. SCAND. 78 (1996), 5–12

ADJOINTS AND DUALS OF MATROIDS LINEARLY REPRESENTABLE OVER A SKEWFIELD

W. HOCHSTÄTTLER and S. KROMBERG

Abstract.

Following an approach suggested by B. Lindström we prove that the dual of a matroid representable over a skewfield is itself representable over the same field. Along the same line we show that any matroid within this class has an adjoint. As an application we derive an adjoint for the dual of the Non-Pappus-Matroid. Furthermore, we reprove a result by Alfter and Hochstättler concerning the existence of an adjoint for a certain eight point configuration and show that this configuration is linearly representable over a field if and only if the field is skew.

1. Introduction.

There is a well understood "set-theoretical" concept of duality in matroid theory derived by a combinatorial abstraction of orthocomplementary pairs of vector spaces. A different "lattice-theoretical" concept, modelling the point-hyperplane duality of classical geometry – we will call it polarity – was first introduced into matroid theory by A.L.C. Cheung in [Che74]. He observed that this concept generalizes polarity in the linear case but does not work for arbitrary matroids. In oriented matroid theory which is closer to a geometric situation in Euclidean space polarity was introduced and studied by Bachem and coworkers ([BaKe86], [BaWa89]). Since an oriented adjoint induces an adjoint for the underlying matroid a standard technique to prove non-existence of an oriented adjoint is to study the underlying matroid.

The two concepts of duality are somewhat incompatible. In [AKW90] M. Alfter, W. Kern and A. Wanka showed that the dual of the Non-Desargues configuration does not admit an adjoint whereas the Non-Desargues configuration itself – as a rank 3 matroid – does. This raises the question about an adjoint for the dual of the Non-Pappus configuration.

In [AlHo95] M. Alfter and the first author showed that any pseudomodular matroid of rank four has an adjoint and presented a non-linear eight point torus as an example. The starting point for the study of that configuration was

Received August 8, 1994.

a question of J. Richter-Gebert who actually was interested in an adjoint for a corresponding oriented matroid. Reexamining the proof of nonlinearity of this torus we observed that commutativity of the field plays an essential role. In fact, the configuration is linearly representable over any skew field. It took an article by Lindström [L88] to remind us of the fact that orthogonality works fine for vectorspaces over skew fields. Since both duality and polarity are abstraction of orthogonality this suggests to transcribe the standard constructions of the dual and the adjoint from commutative to skew fields. This is what we are going to do here. As applications we get an adjoint of the Non-Pappus dual and a new proof for the existence of an adjoint for the eight point torus.

We assume familiarity with matroid theory. A standard reference is [W76], Good sources for linear algebra over skewfields are [B52] and [A57].

2. Definitions and Notation.

Throughout the paper $M = (E, \mathscr{B})$ denotes a matroid where $|E| = n$ and $\mathscr{B}$ denotes the set of bases. We denote the closure operator by cl, the rank function by ρ and the (geometric) lattice of flats associated with M by L. The set of hyperplanes of M, i.e. the set of coatoms of L, is denoted by $\mathscr{H}$. We first recall the two notions of duality in matroid theory.

DEFINITION 1. (i) The *dual* of M is the matroid M^* on E whose set of bases is given by $\mathscr{B}^* = \{E \backslash B \mid B \in \mathscr{B}\}$.

(ii) A geometric lattice L^Δ is called an *adjoint* of L if $\operatorname{rank}(L) = \operatorname{rank}(L^\Delta)$ and there exists an order-reversing injective map $\Phi\colon L \to L^\Delta$, taking the coatoms of L bijectively onto the atoms of L^Δ.

DEFINITION 2. Let K denote a (not necessarily commutative) field. The matroid M of rank s is *linearly representable* over K, if there exists a map $\iota\colon E \to K^s$ into the (right) K-vectorspace K^s such that

$$B \in \mathscr{B} \Leftrightarrow \iota(B) \text{ is a basis of } K^s.$$

As the linear algebra of vectorspaces over skewfields may be not as familiar as that over commutative fields, we now collect the relevant facts from skew linear algebra. Proofs of these results may be found in [A57] and [B52]. In the following the term "field" refers to a not necessarily commutative field while "skewfield" refers to a non-commutative field. For a subset S of a K-vectorspace V we let $\langle S \rangle$ denote the subspace of V spanned by the vectors contained in S.

A map $\phi\colon V \to V'$ between two right K-vectorspaces V, V' is *linear*, if it satisfies $\phi(va + v'a') = \phi(v)a + \phi(v')a'$ for all $a, a' \in K$ and $v, v' \in V$. As usual, the set $\operatorname{Hom}(V, V')$ becomes an Abelian group dy defining $(\phi + \psi)(v) := \phi(v) + \psi(v)$ for $\phi, \psi \in \operatorname{Hom}(V, V')$. Furthermore, the usual dimension formula holds, i.e.

$$\dim(\ker(\phi)) + \dim(\mathrm{im}(\phi)) = \dim(V),$$

and one has $\psi \circ \phi \in \mathrm{Hom}(V, V'')$, for $\phi \in \mathrm{Hom}(V, V')$ and $\psi \in \mathrm{Hom}(V', V'')$. The abelian group $V^* := \mathrm{Hom}(V, K)$ then becomes a left K-vectorspace by defining $(a\varphi)(v) := a\varphi(v)$ for $a \in K$, $v \in V$ and $\varphi \in \mathrm{Hom}(V, K)$. For $\phi \in \mathrm{Hom}(V, V')$ the *dual* map $\phi^* \in \mathrm{Hom}(V'^*, V^*)$ is given by $\phi^*(\varphi) := \varphi \circ \phi$ for $\varphi \in V'^*$. For each basis $\{e_1, \ldots, e_n\}$ of a vectorspace V of finite dimension n there exists the *dual basis* $\{e_1^*, \ldots, e_n^*\}$ of V^* characterized by $e_i^*(e_j) = \delta_{ij}$. In particular one has that $\dim(V) = \dim(V^*)$.

For subspaces $U \subseteq V$ and $U^* \subseteq V^*$ one defines the *orthogonal* subspaces

$$U^\perp := \{\varphi \in V^* \mid \forall u \in U: \varphi(u) = 0\} \subseteq V^*,$$

$$U^{*\perp} := \{v \in V \mid \forall \varphi \in U^*: \varphi(v) = 0\} \subseteq V.$$

Notice that in the finite-dimensional spaces we are considering the inconsistency of this notation vanishes due to the fact that there exists a canonical isomorphism between V and V^{**} given by

$$v \mapsto \begin{Bmatrix} v^{**}\colon V^* & \to & K \\ \varphi & \mapsto & \varphi(v) \end{Bmatrix}.$$

For a subspace $U \subseteq V$ one has

(1) $$\dim(U) + \dim(U^\perp) = \dim(V).$$

If $U' \subseteq V$ is another subspace of V, the following equalities hold:

$$(U + U')^\perp = U^\perp \cap U'^\perp$$
$$(U \cap U')^\perp = U^\perp + U'^\perp.$$

For the rest of the paper let K denote some (skew-)field and $M = (E, \mathscr{B})$ some matroid of rank s that is linearly represented over K. Before giving the proof that M admits an adjoint itself linearly representable over K and that the dual of M is itself linearly representable over K, we state a lemma which is going to be useful.

LEMMA 1. *For the linearly represented matroid M and the corresponding geometric lattice L the following holds*:

$$\forall X \in L: \bigcap_{X \leq H \in \mathscr{H}} \langle H \rangle = \langle X \rangle$$

PROOF. It is clear that

$$\langle X \rangle \subseteq \bigcap_{X \leq H \in \mathscr{H}} \langle H \rangle =: U.$$

Suppose $\langle X \rangle \neq U$. Pick some maximal independent subset I of X and some $v \in U \backslash \langle X \rangle$. Then $I \cup \{v\}$ is a linearly independent subset of V. Extend this set to a basis of V by some $J \subseteq E$. Then $\mathrm{cl}(I \cup J)$ is a hyperplane of M containing X and $v \notin \mathrm{cl}(I \cup J)$. Contradiction.

3. Existence of a linearly representable Adjoint.

After these preparations it is clear how to mimic point-hyperplane duality in vectorspaces over skewfields.

For every hyperplane $H \in \mathscr{H}$ the subspace $\langle H \rangle^{\perp} \subseteq V^*$ is one-dimensional. Let φ_H be a generator of $\langle H \rangle^{\perp}$ and consider the matroid M^{Δ} on $E^{\Delta} := \{\varphi_H \mid H \in \mathscr{H}\}$ defined via linear independence in the left K-vectorspace V^*; ρ^{Δ} denotes its rank-function and L^{Δ} the corresponding geometric lattice.

THEOREM 1. *L^{Δ} is an adjoint of L.*

PROOF. Let

$$\Phi : L \to L^{\Delta}$$
$$X \mapsto \bigvee_{X \leqq H \in \mathscr{H}} \{\varphi_H\}$$

From the above Lemma we deduce

$$\begin{aligned}\rho^{\Delta}(\Phi(X)) &= \dim(\langle \varphi_H \mid X \leqq H \in \mathscr{H} \rangle) = \dim\Bigl(\sum_{X \leqq H \in \mathscr{H}} \langle \varphi_H \rangle\Bigr)\\ &= \dim\Bigl(\sum_{X \leqq H \in \mathscr{H}} \langle H \rangle^{\perp}\Bigr) = \dim\Bigl(\Bigl(\bigcap_{X \leqq H \in \mathscr{H}} \langle H \rangle\Bigr)^{\perp}\Bigr)\\ &= s - \dim\Bigl(\bigcap_{X \leqq H \in \mathscr{H}} \langle H \rangle\Bigr) = s - \rho(X).\end{aligned}$$

It follows that $\langle \Phi(X) \rangle = \langle X \rangle^{\perp}$ and in particular that L^{Δ} has the same rank as L. Furthermore, it is clear that Φ is order-reversing. Since $\langle H \rangle^{\perp\perp} = \langle H \rangle$ and $H = \langle H \rangle \cap E$ for all $H \in \mathscr{H}$ we get that $\varphi_H \neq \varphi_{H'}$, whenever $H \neq H' \in \mathscr{H}$. Hence Φ maps $\mathscr{H}$ bijectively onto E^{Δ}.

Finally, for $X, X' \in L$ such that $\Phi(X) = \Phi(X')$, we have

$$\langle X \rangle = \langle \Phi(X) \rangle^{\perp} = \langle \Phi(X') \rangle^{\perp} = \langle X' \rangle$$

and hence $X = \langle X \rangle \cap E = \langle X' \rangle \cap E = X'$, i.e. Φ is injective.

4. Representability of the Dual.

The following proof of the representability of the dual follows the lines of the proof given in [W76] for the representability of the dual of a matroid representable over a commutative field. The recipe of that proof is as follows: Let M be represented by an $(s \times n)$-matrix A of full rank s. Construct an $((n-s) \times n)$-matrix B of full rank such that $\ker(A) = \operatorname{im}(B^T)$. Then B provides a representation of M^*. The possibility of transcribing this construction to the non-commutative case has already been observed in [L88]. We now give the details.

Suppose the ground set of our matroid M is $[n] = \{1, \ldots, n\}$, the rank of M is s and M is represented via ι in some (right) K-vectorspace V. Let $\iota(i) =: v_i$ for $1 \leqq i \leqq n$. Fix some n-dimensional right K-vectorspace V, a basis $\{e_1, \ldots, e_n\}$ of V and let ϕ: $V \to V'$ be the linear map given by

$$\phi(e_1 a_1 + \ldots + e_n a_n) = v_1 a_1 + \ldots + v_n a_n \quad (a_1, \ldots, a_n \in K).$$

Then $V'' := \ker(\phi)$ is an $(n-s)$-dimensional right K-vectorspace. Denote the inclusion map by ψ: $V'' \to V$. Let $\{e_1^*, \ldots, e_n^*\}$ denote the basis of the left K-vectorspace V^* that is dual to the basis $\{e_1, \ldots, e_n\}$ of V.

THEOREM 2. *The map* $i \mapsto \psi^*(e_i^*)$ $(1 \leqq i \leqq n)$ *provides a linear representation of* M^* *in the left* K*-vectorspace* V''^*.

PROOF. It suffices to show that for any subset I of $[n]$ of cardinality s we have

$$\dim\langle \phi(e_i) \mid i \in I \rangle < s \Leftrightarrow \dim\langle \psi^*(e_i^*) \mid i \in [n] \backslash I \rangle < n - s.$$

But

$$\begin{aligned}
&\{\phi(e_i) \mid i \in I\} \text{ is linearly dependent} \\
&\Leftrightarrow \exists y \in K^I \backslash \{0\}: \sum_{i \in I} e_i y_i \in \ker(\phi) \\
&\Leftrightarrow \exists w \in V'' \backslash \{0\} \forall i \in [n] \backslash I: e_i^*(\psi(w)) = 0 \\
&\Leftrightarrow \exists w \in V'' \backslash \{0\}: w \in \bigcap_{i \in [n] \backslash I} \langle \psi^*(e_i^*) \rangle^\perp \\
&\Leftrightarrow \exists w \in V'' \backslash \{0\}: w \in \langle \psi^*(e_i^*) \mid i \in [n] \backslash I \rangle^\perp \\
&\overset{(1)}{\Leftrightarrow} \{\psi^*(e_i^*) \mid i \in [n] \backslash I\} \text{ is linearly dependent.}
\end{aligned}$$

```
1 - 2 - 3 - 4 - 1
|   |   |   |   |
5 - 6 - 7 - 8 - 5
|   |   |   | / |
3 - 4 - 1 - 2 - 3.
```

Figure 1: The matroid T.

5. Applications.

Combining Theorems 1 and 2 we have the following Corollary 1 which implies that the dual of the Non-Pappus-Matroid admits an adjoint, as it is well-known that the Non-Pappus-Matroid is linearly representable over (all) skewfields.

COROLLARY 1. *If a matroid is linearly representable over a (skew-)field K then its dual admits an adjoint linearly representable over K.*

In [AlHo93] the authors considered the matroid T on $E = \{e_1, \dots, e_8\}$ of rank 4 whose set of bases consists of all 4-element subsets of E with the exception of

$$\{e_1, e_2, e_5, e_6\}, \{e_2, e_3, e_6, e_7\}, \{e_3, e_4, e_7, e_8\}, \{e_1, e_4, e_5, e_8\},$$
$$\{e_3, e_4, e_5, e_6\}, \{e_1, e_4, e_6, e_7\}, \{e_1, e_2, e_7, e_8\}.$$

A representation of T is shown in Figure 1. Being a pseudomodular matroid of rank 4 it follows from the main result of that paper that it admits an adjoint. We reprove that T admits an adjoint by showing that T is linearly representable over any skewfield. The reason for the non-linearity of T over commutative fields is the independence of $\{e_2, e_3, e_5, e_8\}$ in T. We restate this in the following condition which is a reduced version of the sixteen-point theorem of projective geometry as observed by J. Richter-Gebert (see [RG92]).

DEFINITION 3. A projective geometry of rank not less than 4 is said to satisfy the *Torus-Condition* if the following holds:
If $\{p_1, \dots, p_8\}$ are any eight points spanning a flat of dimension 4, any 3-element subset of $\{p_1, \dots, p_8\}$ is independent and the 4-element sets

$$\{p_1, p_2, p_5, p_6\}, \{p_2, p_3, p_6, p_7\}, \{p_3, p_4, p_7, p_8\}, \{p_1, p_4, p_5, p_8\},$$
$$\{p_3, p_4, p_5, p_6\}, \{p_1, p_4, p_6, p_7\} \quad \text{and} \quad \{p_1, p_2, p_7, p_8\}$$

are dependent then $\{p_2, p_3, p_5, p_8\}$ is dependent.

THEOREM 3. *The Torus Condition is satisfied by a projective geometry of rank not less than 4 if and only if the coordinatizing field K is commutative.*

PROOF. It was shown in [AlHo93] that for point configurations in vector-spaces over commutative fields the Torus-Condition is always satisfied. Thus from the fundamental theorem of projective geometry (see e.g. [T69] Kap. V) one direction of the claim follows.

Thus, it suffices to show that the existence of some $\alpha, \beta \in K$ such that $\alpha\beta \neq \beta\alpha$ yields an eight point configuration in K^4 (regarded as a right K-vectorspace) violating the Torus-Condition. Let e_i $(1 \leqq i \leqq 4)$ denote any basis of K^4. Then $p_i := e_i$ for $(1 \leqq i \leqq 4)$, $p_5 := \sum_{i=1}^{4} e_i$, $p_6 := e_1 + e_2 + (e_3 + e_4)\alpha$, $p_7 := e_1\beta +$

$e_2 + e_3\alpha + e_4\alpha\beta$ and $p_8 := e_1\beta\alpha + e_2\alpha + 3_3\alpha + e_4\alpha\beta$ yield an eight point configuration violating the Torus-Condition.

The result from [AlHo95] suggested that T is a not too weird perturbation of a matroid linear over Q and thus its oriented version should admit an adjoint. Here we proved that, more directly, an adjoint can be derived from the skewlinear structure of T. This does not seem to give any hint concerning the question about an oriented adjoint. More generally speaking the following is unclear. Does there exist an orientable matroid which is linearly representable over a skew-field but does not admit an oriented adjoint?

NOTE ADDED IN PROOF. While the present paper was in proof we realized that the standard construction of an *oriented* matroid from a linear subspace $V \subseteq \mathsf{R}^E$ (see [BLSWZ93], pp. 105) can easily be transcribed to a subspace of a vectorspace over any ordered field. Note that ordered skewfields exist ([A57], ch. 9). We were able to prove that the oriented analogues of Theorems 1 and 2 of the present paper hold for oriented matroids the orientation of which is induced from a representation over an ordered field. Furthermore, the *oriented* 8-point torus of Richter-Gebert with underlying matroid T is representable over the ordered skewfield described in [A57], ch. 9. Thus, the oriented 8-point torus does admit an adjoint and Richter-Gebert's original question is finally answered in the affirmative. The interested reader may consult [HK95] or [K95].

REFERENCES

[AlHo95] M. Alfter and W. Hochstättler, *On pseudomodular matroids and adjoints*, Disc. Appl. Math. 60 (1995), 3–11.

[AKW90] M. Alfter, W. Kern and A. Wanka, *On adjoints and dual matroids*, J. Combin, Theory Ser. B 60 (1990), 208–213.

[A57] E. Artin, *Geometric Algebra*, Interscience Publishers, Inc., New York, 1957.

[BaKe86] A. Bachem and W. Kern, *Adjoints of oriented matroids*. Combinatorica 6 (1986), 249–308.

[BaWa89] A. Bachem and A. Wanka, *Matroids without adjoint*, Geom. Dedicata 29 (1989), 311–315.

[B52] R. Baer, *Linear Algebra and Projective Geometry*, Academic Press, New York, 1952.

[BLSWZ93] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, *Oriented Matroids*, Cambridge, 1993.

[Che74] A. L. C. Cheung, *Adjoints of a geometry*, Canad. Math. Bull. 17 (1974), 363–365.

[HK95] W. Hochstättler and S. Kromberg, *Oriented Matroids Representable over Ordered Skewfields*, in preparation.

[K95] S. Kromberg, *Adjoints, Schiefkörper und algebraische Matroide*, Dissertation. Köln, 1995.

[L88] B. Lindström, *On p-polynomial-representations of projective geometries in algebraic combinatorial geometries*, Math. Scand. 63 (1988), 36–42.

[RG92] J. Richter-Gebert, *On the Realizability Problem of Combinatorial Geometries – Decision Methods*, Ph. D. Thesis, TH-Darmstadt, Darmstadt, 1992.

[T69] O. Tamaschke, *Projektive Geometrie* 1, Bibliographisches Institut, Mannheim, 1969.

[W76] D. J. A. Welsh, *Matroid Theory*, Academic Press, New York, London, San Francisco, 1976.

UNIVERSITÄT ZU KÖLN
WEYERTAL 86–90
D-50931 KÖLN
GERMANY
Email: WH@MI.UNI-KOELN.DE,
SK@MI.UNI-KOELN.DE