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ON THE HOMOLOGICAL DIMENSION OF A DER-FREE HYPERSURFACE

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Abstract.

Let Ω be a germ of an analytic hypersurface in $\mathbf{C}^n$ at the origin. Let $D(\Omega)$ denote the left $\mathcal{O}_n$-module of holomorphic vector fields which are tangent to Ω. We say that Ω is a Der-free hypersurface if $D(\Omega)$ is a free $\mathcal{O}_n$-module. See [Saito]. We define $R(\Omega)$ to be the subalgebra of $\mathcal{D}_n$ generated by $\mathcal{O}_n$ and $D(\Omega)$. In this paper we calculate the homological dimension of this ring when Ω is a Der-free hypersurface.

In the following module resp. ideal means left module resp. left ideal. Global homological dimension means left dimension. However, since we only consider noetherian rings, the left and right homological dimensions coincide.

Der-free hypersurfaces and rings of tangential operators.

Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at the origin in $\mathbf{C}^n$ and $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_n)$ the set of $\mathbf{C}$-linear derivations on $\mathcal{O}_n$. $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_n)$ has a natural $\mathcal{O}_n$-module structure and a Lie algebra structure. For $h \in \mathcal{O}_n$, let $\Omega = \{h(z) = 0\}$ be the corresponding (germ of) a hypersurface in $\mathbf{C}^n$. Let $D(\Omega)$ denote the set of tangential vector fields to Ω. Thus $D(\Omega) = \{\Theta;\ \Theta \in \mathrm{Der}_{\mathbf{C}}(\mathcal{O}_n) \text{ and } \Theta h \in \mathcal{O}_n h\}$. $D(\Omega)$ has a natural $\mathcal{O}_n$-module structure and is a Lie sub algebra of $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_n)$. We shall say that Ω is a *Der-free* hypersurface if $D(\Omega)$ is a free $\mathcal{O}_n$-module.

Note that if $\Omega = \{h(z) = 0\}$ is Der-free, then $\mathrm{rank}_{\mathcal{O}_n}(D(\Omega)) = n$. This follows easily from the inclusions (where $\partial_i = \partial/\partial x_i$)

$$\mathcal{O}_n h\partial_1 + \ldots + \mathcal{O}_n h\partial_n \subset D(\Omega) \subset \mathcal{O}_n\partial_1 + \ldots + \mathcal{O}_n\partial_n.$$

For a submodule D of $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_n)$, denote by $R(D)$ the subring of $\mathcal{D}_n$ generated by $\mathcal{O}_n$ and D. Here $\mathcal{D}_n$ is the ring of differential operators with coefficients in $\mathcal{O}_n$. If $D = D(\Omega)$ for some (not necessarily Der-free) hypersurface Ω, we shall write $R(\Omega)$ instead of $R(D)$. It is called the ring of tangential differential operators to Ω.

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$\mathcal{O}_n$ is identified with the ring $\mathbf{C}\{x_1,\dots,x_n\}$ of convergent power series and we shall often use notations such as $\mathbf{C}\{x_1,\dots,x_n; \Theta_1,\dots,\Theta_k\}$ (or $\mathcal{O}_n\{\Theta_1,\dots,\Theta_k\}$) for $R(D)$ when $\Theta_1,\dots,\Theta_k$ generates D over $\mathcal{O}_n$.

REMARKS. Every smooth hypersurface is Der-free, being analytically equivalent to $\{x_n = 0\}$. Every irreducible curve is Der-free. An interesting example of a singular Der-free surface in $\mathbf{C}^3$ is constructed as follows: Regard $\mathbf{C}^3$ as the set $\{(x_1, x_2, x_3, x_4); x_1 + \dots + x_4 = 0\}$. The group Sym(4) acts on $\mathbf{C}^3$ by permuting the coordinates. Let $Y \subset X$ be the union of the hyperplanes stabilized by Sym(4), i.e. $Y = \bigcup_{1 \leqq i < j \leqq 4}\{x_i = x_j\}$. Let $\pi: \mathbf{C}^3 \to \mathbf{C}^3/\mathrm{Sym}(4)$ be the quotient map. By Chevalley's theorem $\mathbf{C}^3/\mathrm{Sym}(4) \cong \mathbf{C}^3$. The singular locus of π is Y. $\pi(Y)$ is the Der-free hypersurface called the swallowtail.

If an irreducible hypersurface is Der-free it is called a *free divisor*. These sets are studied by [Saito], they occur in singularity theory and have been studied in several papers.

If a Der free hypersurface $\Omega = \{h = 0\}$ is a hyperplane arrangement, i.e. $h = \alpha_1 \cdot \ldots \cdot \alpha_k$ where the α_i's are linear forms, Ω is called a *free arrangement*. For instance, the arrangement $\{x_1 \cdot \ldots x_k = 0\}$ is free and so are all *reflection arrangements* (e.g. the arrangement Y above). Most arrangements are not free: If Ω is an arrangment of k hyperplanes in $\mathbf{C}^n$, $k > n \geqq 3$, Ω will be non-free with probability 1. See the book [Orlik-Terao] for the theory of hyperplane arrangements.

Our goal is to calculate the homological dimension of $R(\Omega)$. In [Björk 3] the homological dimensions of several rings of differential operators are determined. However, only rings which can be generated by a finite set of commuting vector fields are treated there. This example shows that theorem 1.1 below does not follow from the results in [Björk 3]:

Let $h = x_1^2 - x_2^3, \Omega = \{h = 0\} \subset \mathbf{C}^2$. Then $D(\Omega) = \mathcal{O}_n(3x_1\partial_1 + 2x_2\partial_2) + \mathcal{O}_n(3x_2^2\partial_1 + 2x_1\partial_2)$, so Ω is a der-free curve. I claim that if $\Theta_1 = f_1\partial_1 + f_2\partial_2$, $\Theta_2 = g_1\partial_1 + g_2\partial_2$ is any $\mathcal{O}_n$-basis of $D(\Omega)$, then $[\Theta_1, \Theta_2] \neq 0$.

Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_n$. It is easy to see that $f_1, f_2, g_1, g_2 \in \mathfrak{m}$. By Saitos criterium [Saito] we have $f_1g_2 - f_2g_1 = e \cdot h$, where e is a unit in $\mathcal{O}_n$. Since $h \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$, either $f_1g_2 \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$ or $f_2g_1 \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. WLOG $f_1g_2 \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. Thus $f_1, g_2 \in \mathfrak{m} \setminus \mathfrak{m}^2$.

A straightforward calculation using the conditions $\Theta_i h \in (h)$, $i = 1, 2$, shows that (WLOG) $f_1 = x_1 + \tilde{f}_1$, $f_2 = (2/3)x_2 + ax_1 + \tilde{f}_2$, $g_1 = (3c/2)x_1 + \tilde{g}_1$, $g_2 = bx_1 + cx_2 + \tilde{g}_2$, where a, b and $c \in \mathbf{C}$, $b - 3ac/2 \neq 0$, $\tilde{f}_1, \tilde{f}_2, \tilde{g}_1$ and $\tilde{g}_2 \in \mathfrak{m}^2$. Then $[\Theta_1, \Theta_2] = (1/3)(b - 3ac/2)x_1\partial_2 +$ "some vector field with coefficients in $\mathfrak{m}^2$". Thus $[\Theta_1, \Theta_2] \neq 0$.

1.1 THEOREM. *If Ω is a Der-free hypersurface, then*

$$\mathrm{gl.dim}\,(R(\Omega)) = 2n - m(\Omega),$$

where $m(\Omega)$ is the dimension of the vector space $\{\Theta(0); \Theta \in \mathrm{Der}_{\mathbf{C}}(\Omega)\}$. Here $\Theta(0)$ is the tangent vector at the origin of the given vector field Θ.

REMARK. If Ω is smooth, we can assume $\Omega = \{x_n = 0\}$. Then $D(\Omega)$ has the $\mathcal{O}_n$-basis $\partial_1, \ldots, \partial_{n-1}, x_n\partial_n$. Thus, $m(\Omega) = n - 1$ and gl.dim $(R(\Omega)) = n + 1$ by theorem 1.1. If Ω is not free I believe that gl.dim $(R(\Omega)) = \infty$, but I cannot prove this.

LEMMA. *Let $\Omega := \{h(z) = 0\}$ be a Der-free hypersurface in $\mathbf{C}^n$ and $m = m(\Omega)$ the integer defined in Theorem 1.1. There exists an $\mathcal{O}_n$basis $\Theta_1, \ldots, \Theta_n$ for $D(\Omega)$ satisfying (in an appropriate system of coordinates):*

(1) $\Theta_i = \partial_i;\ 1 \leqq i \leqq m$.

(2) $\Theta_i = f_{i,m+1}\partial_{m+1} + \ldots + f_{i,n}\partial_n;\ m + 1 \leqq i \leqq n$.

(3) $f_{ij} \in \mathcal{O}_n$ *is independent of* $x_1, \ldots, x_m$ *and* $f_{ij}(0) = 0$ *for every double index in (2).*

PROOF. The proof goes by induction on n. If $n = 0$ there is nothing to prove. Suppose the lemma holds for all Der-free hypersurfaces in $\mathbf{C}^k$, for $k < n$.

We can assume $m \geqq 1$, since the lemma trivially holds when $m = 0$. Pick $\delta \in D(\Omega)$ such that $\delta(0) \neq 0$. After a biholomorphic map we may assume that $\delta = \partial_1$. Then $\dfrac{\partial^k h}{\partial {x_1}^k} \in \mathcal{O}_n h$ for every k which implies that the germ h is independent on x_1. If $h = h(x_2, \ldots, x_n)$ then it folows that

$$D(\Omega) = \mathcal{O}_n\delta_1 + \mathcal{O}_n \otimes_{\mathcal{O}_{n-1}} D(\Omega'),$$

where $\mathcal{O}_{n-1} = \mathbf{C}\{x_2, \ldots, x_n\}$, Ω' is the Der-free hypersurface $h^{-1}(0) \cap \mathbf{C}^{n-1}$ in $\mathbf{C}^{n-1}$. By the induction hypothesis $(1 - 3)$ holds for $D(\Omega')$ and the result follows.

Filtered rings.

Let us recall a few basic facts from the theory of filtered rings. Proofs and details may be found in [Björk 2, Appendix 3]. Let R be a ring equipped with a positive (increasing) filtration $\{\Gamma_i\}$. If the associated graded ring $\mathrm{gr}_\Gamma(R)$ is noetherian we shall say that R is filtered noetherian. This implies that R is noetherian. From now on R denotes a filtered noetherian ring.

Let M be a finitely generated and filtered R-module. The filtration $\{M_k\}$ on M is said to be good if the associated graded module $\mathrm{gr}(M)$ is a finitely generated $\mathrm{gr}_\Gamma(R)$ module. The grade number of M is defined by

$$J_R(M) = \inf\{i : \mathrm{Ext}^i_R(M, R) \neq 0\}.$$

(If all these Ext groups vanish we put $J_R(M) = -\infty$.) Obviously, $J_R(M) \leqq$ gl.dim(R) for any R-module M.

1.3 PROPOSITION. *We have* gl.dim(R) $\leqq$ gl.dim($\mathrm{gr}_\Gamma(R)$). *Further, if* $\mathrm{gr}_\Gamma(R)$ *is commutative and has finite homological dimension, then for every finitely generated* R*-module* M *and any good filtration* $\{M_k\}$ *on* M *one has:*

$$J_R(M) = J_{\mathrm{gr}_\Gamma(R)}(\mathrm{gr}(M)).$$

The two following lemmas generalizes well-known properties of $\mathscr{D}_n$ [Björk 2; chapter 1].

1.4 LEMMA. *Let* Ω *be a Der-free hypersurface and put* $D = D(\Omega)$, $R = R(\Omega)$. *Let* $\Theta_1, \dots, \Theta_n$ *be an* $\mathcal{O}_n$*-basis of* D. *Then the elements* $\{\Theta_\bullet^\alpha = \Theta_1^{\alpha_1} \cdot \ldots \cdot \Theta_n^{\alpha_n}\}_{|\alpha| \geqq 0}$ *form a basis of the* $\mathcal{O}_n$*-module* R.

PROOF. We shall prove that the elements $\{\Theta_\bullet^\alpha\}$ are independent over Θ_n. An element in P is considered as an element in $\mathscr{D}_n$. Write $P = \sum_{|\beta| \leqq r} f_\beta \partial^\beta$. Then, $\sigma(P)$, ($f_\beta \neq 0$ for some $|\beta| = k$) the principal symbol of P is defined to be the element $\sum_{|\beta| = r} f_\beta T^\beta \in \mathcal{O}_n[T_1, \dots, T_n] \cong \mathrm{gr}(\mathscr{D}_n)$, where $T_i = \sigma(\partial_i)$.

It is not hard to see an $\mathcal{O}_n$-linear dependence $\sum_{|\alpha| \leqq k} f_\alpha \Theta_\bullet^\alpha = 0$, implies that $\sum_{|\alpha| = k} f_\alpha \sigma(\Theta_\bullet^\alpha) = 0$.

Now, if $\Theta_i = \sum f_{ij} \partial_j$, then $\sigma(\Theta_i) = \sum f_{ij} T_j$ and $\sigma(\Theta_\bullet^\alpha) = (\sum f_{1j} T_j)^{\alpha_1} \cdot \ldots \cdot (\sum f_{nj} T_j)^{\alpha_n}$. By the assumption, D is free, so $\det(f_{ij}) \neq 0$. Thus, by means of elementary algebra, the polynomials $\{(\sum f_{1j} T_j)^{\alpha_1} \cdot \ldots \cdot (\sum f_{nj} T_j)^{\alpha_n}\}_{|\alpha| = k}$ are linearly independent over $\mathcal{O}_n$. Thus $g_\alpha = 0$ for all multi-indices α of length k. An induction over k shows that $g_\alpha = 0$ for all α. Thus, the elements $\{\Theta_\bullet^\alpha\}$ are independent over $\mathcal{O}_n$.

It is left to the reader to verify (using the fact that D is closed under the Lie bracket) that the elements $\{\Theta_\bullet^\alpha\}$ generate R.

1.5 LEMMA. *Suppose* $D := D(\Omega)$ *has a basis* $\Theta_\bullet$ *such that (1–3) in 1.4 holds. Put* $R = R(\Omega)$. *Then*

(i) $\Gamma_i = \sum_{|\alpha| \leqq i} f_\alpha \Theta_1^{\alpha_1} \cdot \ldots \cdot \Theta_n^{\alpha_n}$, $f_\alpha \in \mathcal{O}_n$, *defines a filtration on* R *such that*

$$\mathrm{gr}_\Gamma(R) \cong \mathcal{O}_n[\xi_1, \dots, \xi_n].$$

(ii) $\Sigma_i = \sum_{|\alpha| \leqq i} f_\alpha \Theta_{m+1}^{\alpha_{m+1}} \cdot \ldots \cdot \Theta_n^{\alpha_n}$, $f_\alpha \in \mathcal{O}_n\{\partial_1, \dots, \partial_m\}$, *defines a filtration on* R *such that*

$$\mathrm{gr}_\Sigma(R) \cong \mathcal{O}_n\{\partial_1, \dots, \partial_m\}[\xi_{m+1}, \dots, \xi_n].$$

Here's the ξ_i*'s are commutative variables. By definition,* ξ_i *corresponds to the image of* Θ_i *in* Γ_1/Γ_0 *under these isomorphisms.*

PROOF. Let us prove (ii). $\{\Sigma_i\}$ defines a filtration on R. Because, by 1.4, we have $\bigcup \Sigma_i = R$ and it follows from standard rules of commutations that $\Sigma_i \Sigma_j \subset \Sigma_{i+j}$.

To avoid confusion we shall denote the Lie brackets on R, resp. on $\mathrm{gr}_\Sigma(R)$, by $[\ ,\ ]_R$ resp. $[\ ,\ ]_{\mathrm{gr}_\Sigma(R)}$.

Put $\xi_i = (\Theta_i + \Gamma_0)/\Gamma_0$, $m+1 \leqq i \leqq n$. The fact that D is closed under $[\ ,\ ]_R$, implies that $[\xi_i, \xi_j]_{\mathrm{gr}_\Sigma(R)} = 0$

Also, it follows from our assumption that Θ_i and ∂_j commutes when $i > m$ and $j \leqq m$. This implies that, for $g_\alpha \in \mathcal{O}_n$, $[g_\alpha \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_m^{\alpha_m}, \Theta_i]_R \subset \mathcal{O}_n\{\partial_1, \ldots, \partial_m\}$. Therefore, $[\Gamma[0], \xi_i]_{\mathrm{gr}_\Sigma(R)} = 0$, and it follows that the $\xi_i's$ lie in the center of $\mathrm{gr}_\Sigma(R)$,

It is left to the reader to verify (use 1.4) that the $\xi_i's$ are independent commutative variables over $\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}$.

The proof of (i) is similar.

PROOF OF THEOREM 1.1. We keep the assumptions in 1.5.

Proof of $\mathrm{gl.dim}(R) \geqq 2n - m$. Define the ideal

$$I = Rx_{m+1} + \ldots + Rx_n + R\partial_1 + \ldots + R\partial_m + R\Theta_{m+1} + \ldots + R\Theta_n.$$

It is left to the reader to verify that I is a proper ideal in R.

Give R the filtration $\{\Gamma_i\}$ in 1.5 (i). Note that $\mathrm{gl.dim}(\mathrm{gr}_\Gamma(R)) = 2n$. Give R/I the induced (good) filtration $\{(R/I)_k := (\Gamma_k + I)/I\}$.

Then $\mathrm{gr}(R/I)$ is isomorphic to a non zero quotient of $\mathcal{O}_m$, as a $\mathrm{gr}_\Gamma(R)$-module, because it is generated as a complex algebra by the image of $\mathcal{O}_m$ in $\mathrm{gr}_\Gamma(R/I)$. This implies that the Krull dimension $d_{\mathrm{gr}_\Gamma(R)}(\mathrm{gr}(R/I)) \leqq m$.

Since $\mathrm{gr}_\Gamma(R)$ is a commutative regular and noetherian ring the following formula is well known [Matsumura, page 135]:

$$J_{\mathrm{gr}_\Gamma(R)}(\mathrm{gr}(R/I)) = \mathrm{gl.dim}(\mathrm{gr}_\Gamma(R)) - d_{\mathrm{gr}_\Gamma}(R)) - d_{\mathrm{gr}_\Gamma(R)}(\mathrm{gr}(R/I)).$$

Thus $J_{\mathrm{gr}_\Gamma(R)}(\mathrm{gr}(R/I)) \geqq 2n - m$. By 1.3, $J_R(R/I) \geqq 2n - m$, and so $\mathrm{gl.dim}(R) \geqq 2n - m$. *Proof of* $\mathrm{gl.dim}(R) \leqq 2n - m$. Give R the filtration $\{\Sigma_i\}$ in 1.5 (ii).

We claim that the homological dimension of $\mathrm{gr}_{\mathrm{gr}_\Sigma}(R)$ equals $2n - m$. This follows from a familiar theorem of Rees [Rotman, 248–249] if we can prove that $\mathrm{gl.dim}\, \mathcal{O}_n\{\partial_1, \ldots, \delta_m\} = n$.

The fact that $\mathrm{gl.dim}(\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}) = n$ is proved by induction on n. If $n = m$, $\mathrm{gl.dim}(\mathcal{O}_n\{\partial_1, \ldots, \partial_m\} = \mathcal{D}_m) = m$, by a theorem of Björk [Björk 1, 88–92]. Thus, for an induction step, assume that $n > m$ and that $\mathrm{gl.dim}(\mathcal{O}_{n-1}\{\partial_1, \ldots, \partial_m\}) = n - 1$.

Notice that $\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}$ has finite homological dimension (the standard filtration and 1.3). Moreover, x_n is a central element which is a non-unit and a non zero divisor in $\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}$ and we have

$$\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}/(x_n) \cong \mathcal{O}_{n-1}\{\partial_1, \ldots, \partial_m\}.$$

We can now apply the theorem of Rees to establish

$$\mathrm{gl.dim}(\mathcal{O}_n\{\partial_1, \ldots, \partial_m\}) = \mathrm{gl.dim}(\{\partial_1, \ldots, \partial_m\}) + 1.$$

This finishes the induction. Thus, $\mathrm{gl.dim}(\mathrm{gr}_\Sigma(R)) = 2n - m$ and so, by 1.3, $\mathrm{gl.dim}(R) \leqq 2n - m$.

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