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SHEAVES ON FIXED POINT SETS AND EQUIVARIANT COHOMOLOGY

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Abstract.

Let G be a finite group. In this paper we interprete the ordinary equivariant cohomology groups of a paracompact G-space X with coefficients in a contravariant coefficient system in terms of the cohomology of a suitable Grothendieck topos. The objects of this topos are certain families of sheaves on the fixed point sets X^K for all subgroups K of G. As an application we obtain a spectral sequence associated to an equivariant map $f: X \to Y$, relating the equivariant cohomology of X to that of Y.

Introduction.

Let G be a finite group. The most natural choice for an ordinary equivariant cohomology theory to be used on a paracompact G-space is the equivariant Alexander-Spanier cohomology, constructed in [9] for all G-spaces. The usefulness of this theory is due to its close connection with sheaf cohomology: If X is a paracompact G-space and m is a contravariant coefficient system, the equivariant cohomology groups $\bar{H}^n_G(X; m)$ are isomorphic to the ordinary cohomology groups of the orbit space X/G, with coefficients in a (non-constant) sheaf. If the G-space X is locally sufficiently nice, the groups $\bar{H}^n_G(X; m)$ are isomorphic to the equivariant singular cohomology groups of X, constructed by Illman in [12]. The construction of $\bar{H}^n_G(X; m)$ and the above results were generalized to the case of a compact Lie group G in [10].

In this paper we consider, instead of sheaves on X/G, families of sheaves on all fixed point sets X^K for subgroups K of G, equipped with suitable structure morphisms (the precise formulation is given in section 1 below). To a contravariant coefficient system m we associate a particularly simple example of such a family, denoted m/X: On X^K we take all constant sheaves with stalks $m(G/H)$, for H subconjugate to K.

One of the basic constructions in [14, ch. I], implies that the families of sheaves described in the preceding paragraph form a Grothendieck topos. Our main

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result, therem 1.3, then states that the cohomology groups of this topos with coefficients m/X are isomorphic to the equivariant cohomology groups $\bar{H}^n_G(X; m)$, provided that the G-space X is paracompact. The proof of this result is rather algebraic in nature, and it occupies the sections 2–6 of the paper.

In the final section 7 we apply the cohomology theory of topoi (see [16]) to obtain a spectral sequence associated to a G-map $f: X \to Y$. In case Y is a point this gives a spectral sequence whose E_2-term depends on the non-equivariant cohomology groups $\bar{H}^q(X^K; m(G/H))$, converging to the equivariant cohomology $\bar{H}^n_G(X; m)$. In particular, if G acts freely on X, this spectral sequence reduces to the Cartan-Leray spectral sequence of the covering space $X \to X\backslash G$.

If f is a G-fibration and Y is a G − CW-complex, we note that the E_2-term of the spectral sequence of f is the cohomology of the topos associated to Y with coefficients in a family of locally constant sheaves whose stalks are isomorphic to the cohomology of the fixed point setsof the fiber of f. Thus the spectral sequence can be regarded as an equivariant Serre spectral sequence. Another construction, using singular cohomology, of such a spectral sequence is given in [13].

As for possible generalizations to the case of a compact Lie group G, a direct description of the relevant topos in terms of families of ordinary sheaves as in this paper, does not seem sufficient.

Some results of this paper are used in [11], which also contains an application of equivariant Alexander-Spanier cohomology.

Finally, I am grateful to Kalevi Suominen for explaining [14] to me, as well as for many suggestions to improvements on an earlier version of this paper.

1. Formulation of main result.

Let G be a finite group. We denote by $\mathrm{Or}(G)$ the orbit category of G, with the G-sets G/H for all subgroups $H \leq G$ as objects and all G-maps between them as morphisms. Fundamental to this paper is the following category $\mathscr{D}$: The objects of $\mathscr{D}$ are the morphisms $u: G/H \to G/K$ of $\mathrm{Or}(G)$, and a morphism $u \to u'$ in $\mathscr{D}$ is a pair (α, β) of morphisms of $\mathrm{Or}(G)$ making the square

(1.1)
$$\begin{array}{ccc} G/H & \xleftarrow{\beta} & G/H' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ G/K & \xrightarrow{\alpha} & G/K' \end{array}$$

commutative.

We remark that in the notation of [7, p. 228], our category $\mathscr{D}$ is $\mathrm{Sub}(\mathrm{Or}(G))$, the subdivision of $\mathrm{Or}(G)$. This subdivision construction is of interest in the homotopy theory of categories, in particular in Quillen's higher algebraic K-theory. In the terminology of [1], $\mathscr{D}$ is the category of factorizations in $\mathrm{Or}(G)$.

Let X be a (Hausdorff) G-space. It determines the functor $X\colon \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Top}$ (= category of topological spaces) given by $G/H \mapsto X^H \cong \mathrm{Map}_G(G/H, X)$ on objects. We compose this with the "target" functor $T\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Or}(G)^{\mathrm{op}}$, $u \mapsto G/K$, $(\alpha, \beta) \mapsto \alpha$ (u, α, β as in 1.1) to obtain the functor

$$X \circ T\colon \mathscr{D}^{\mathrm{op}} \to \mathbf{Top};\ u \mapsto X^K,\ (\alpha, \beta) \mapsto [X(\alpha)\colon X^{K'} \to X^K].$$

Now we start to use the terminology of [14, ch. I]. Let $\mathscr{E} \to \mathbf{Top}$ be the category bifibred in duals of topoi over **Top** consisting of sheaves on various spaces. To be more precise, the objects of $\mathscr{E}$ are the pairs $(R, \mathscr{F})$, where R is a space and $\mathscr{F}$ is a sheaf on R, and a morphism $(R, \mathscr{F}) \to (S, \mathscr{G})$ is a pair (f, φ), where $f\colon R \to S$ is continuous and $\varphi\colon \mathscr{G} \to f_*\mathscr{F}$ is a morphism of sheaves on S.

The fibre product of $X \circ \mathrm{T}\colon \mathscr{D}^{\mathrm{op}} \to \mathbf{Top}$ and $\mathscr{E} \to \mathbf{Top}$.

$$\tilde{X} = \overline{X \circ T} = (\mathscr{E} \times_{\mathbf{Top}} \mathscr{D}^{\mathrm{op}})^{\mathrm{op}},$$

is a $\mathscr{D}$-topos, and its sections form the topos $\Gamma(\tilde{X})$. Explicitly, the objects of $\Gamma(\tilde{X})$ are the families $\mathscr{F} = (\mathscr{F}(u))_{u \in \mathrm{Ob}(\mathscr{D})}$, where for each $u\colon G/H \to G/K$, $\mathscr{F}(u)$ is a sheaf on X^K, such that each morphism (α, β) of $\mathscr{D}$, as in 1.1, induces a morphism

$$\mathscr{F}(\alpha, \beta)\colon \mathscr{F}(u) \to X(\alpha)_*\mathscr{F}(u');$$

these morphisms are functorial in the following sense: Firstly, $\mathscr{F}(\mathrm{id}_u)$ is the identity of $\mathscr{F}(u)$ for every $u \in \mathrm{Ob}(\mathscr{D})$, and for $(\alpha, \beta)\colon u \to u'$, $(\alpha', \beta')\colon u' \to u''$, the morphism $\mathscr{F}((\alpha', \beta') \circ (\alpha, \beta)) = \mathscr{F}(\alpha' \circ \alpha, \beta \circ \beta')$ is the composite

$$\mathscr{F}(u) \to X(\alpha)_*\mathscr{F}(u') \to X(\alpha)_*(X(\alpha')_*\mathscr{F}(u'')) = X(\alpha' \circ \alpha)_*\mathscr{F}(u'')$$

of $\mathscr{F}(\alpha, \beta)$ followed by $X(\alpha)_*\mathscr{F}(\alpha', \beta')$. Those objects $\mathscr{F}$ for which $\mathscr{F}(u)$ is a sheaf of abelian groups for all u and $\mathscr{F}(\alpha, \beta)$ is a homomorphism for all (α, β), form the category $\mathrm{Mod}(\Gamma(\tilde{X}))$.

An abelian group valued functor $F\colon \mathscr{D} \to \mathbf{Ab}$ determines an object F/X of $\mathrm{Mod}(\Gamma(\tilde{X}))$ in the following way: For any $u\colon G/H \to G/K$, $(F/X)(u)$ is the constant sheaf $F(u)/X^K$ with stalks $F(u)$ on X^K; the morphism $(F/X)(\alpha, \beta)$ induced by (α, β) of 1.1 is the composite

$$F(u)/X^K \xrightarrow{F(\alpha, \beta)} F(u')/X^K \to X(\alpha)_*(F(u')/X^{K'}),$$

where the second arrow is induced by the identity between the corresponding constant presheaves.

In particular, if $m\colon \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ is a contravariant coefficient system, we apply the preceding construction to the functor $m \circ S$, where $S\colon \mathscr{D} \to \mathrm{Or}(G)^{\mathrm{op}}$ is the "source" functor $u \mapsto G/H$. The result is an object m/X of $\mathrm{Mod}(\Gamma(\tilde{X}))$ such that $(m/X)(u)$ is the constant sheaf $m(G/H)/X^K$ on X^K.

Let $P\colon \mathscr{D}^{\mathrm{op}} \to \mathbf{Top}$, $u \mapsto$ point, be the constant functor. The evident natural transformation $X \circ T \to P$ determines a morphism

$$c = (c_*, c^*) \colon \tilde{X} = \overline{X \circ T} \to \mathscr{D} \times \mathbf{Set}$$

of $\mathscr{D}$-topoi. We note that $\mathrm{Mod}(\Gamma(\bar{P})) = \mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$ is the category of functors $\mathscr{D} \to \mathbf{Ab}$. We consider the global section functor $\Gamma \colon \mathrm{Mod}(\Gamma(\tilde{X})) \to \mathbf{Ab}$, which is the composite

$$\Gamma \colon \mathrm{Mod}(\Gamma(\tilde{X})) \xrightarrow{\Gamma(c_*)} \mathrm{Mod}(\Gamma(\bar{P})) = \mathbf{Hom}(\mathscr{D}, \mathbf{Ab}) \xrightarrow{\varepsilon_*} \mathbf{Ab}; \tag{1.2}$$

here

$$\Gamma(c_*) \colon \mathscr{F} \mapsto [u \mapsto \Gamma(X^K, \mathscr{F}(u))]$$

$$\varepsilon_* \colon F \mapsto \varprojlim_{\mathscr{D}} F.$$

If $\mathscr{F}$ is an object of $\mathrm{Mod}(\Gamma(\tilde{X}))$, then the nth cohomology group of the topos $\Gamma(\tilde{X})$ with coefficients $\mathscr{F}$ is by definition

$$H^n(\Gamma(\tilde{X}); \mathscr{F}) = R^n\Gamma(\mathscr{F}),$$

where $R^n\Gamma$ is the nth derived functor of the functor Γ of 1.2.

We can now state our main result, whose proof will occupy sections 2–6 below:

THEOREM 1.3. *Let X be a paracompact G-space and $m \colon \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ a contravariant coefficient system. For any $n \in \mathbb{N}$, there is a natural isomorphism*

$$H^n(\Gamma(\tilde{X}); m/X) \cong \bar{H}^n_G(X; m),$$

where $\bar{H}^n_G(X; m)$ is the nth equivariant Alexander-Spanier cohomology group of X with coefficients m, see [9].

2. The Alexander-Spanier resolution.

For a topological space Y, an abelian group M and $n \in \mathbb{N}$, let $\mathscr{C}^n(Y; M)$ be the sheaf on Y associated to the presheaf

$$V \mapsto C^n(V; M), \quad V \subset Y \text{ open},$$

where $C^n(V; M)$ is the abelian group of all functions $V^{n+1} \to M$. Then the sequence of sheaves on Y,

$$0 \to M \to \mathscr{C}^0(Y; M) \xrightarrow{d^0} \mathscr{C}^1(Y; M) \xrightarrow{d^1} \mathscr{C}^2(Y; M) \to \cdots \tag{2.1}$$

is exact, the sheaves $\mathscr{C}^n(Y; M)$ are fine and, for paracompact (and Hausdorff) Y,

$$\Gamma(Y, \mathscr{C}^n(Y; M)) \cong C^n(Y; M)/C^n_0(Y; M) = \bar{C}^n(Y; M), \tag{2.2}$$

where the second identity is the definition of the nth Alexander-Spanier cochain group of Y with coefficients M (cf. [15, p. 307]). These facts are well-known, see for example [2, I,7]. We recall one consequence: if Y is paracompact, then the sheaf cohomology $H^n(Y; M)$ is isomorphic to the Alexander-Spanier cohomology $\bar{H}^n(Y; M)$.

We remark that the sheaves $\mathscr{C}^n(Y; M)$ are functorial in both M and Y. Namely, a homomorphism $\varphi: M_1 \to M_2$ obviously induces a morphism

$$\varphi_*: \mathscr{C}^n(Y; M_1) \to \mathscr{C}^n(Y; M_2),$$

while a continuous map $f: Y_1 \to Y_2$ induces homomorphisms

$$C^n(V; M) \to C^n(f^{-1}V; M), \quad V \subset Y_2 \text{ open},$$

which in turn determine a morphism of sheaves

$$f^*: \mathscr{C}^n(Y_2; M) \to f_*\mathscr{C}^n(Y_1; M).$$

Let now X be a G-space and $m: \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ a contravariant coefficient system. For each $n \in \mathbf{N}$ we define an object $\mathscr{C}^n(m/X)$ of $\mathrm{Mod}(\Gamma(\tilde{X}))$ as follows: If $u: G/H \to G/K$, we set

$$\mathscr{C}^n(m/X)(u) = \mathscr{C}^n(X^K; m(G/H)),$$

and for the morphism (α, β) of 1.1 we define $\mathscr{C}^n(m/X)(\alpha, \beta)$ to be the composite

$$\mathscr{C}^n(X^K; m(G/H)) \xrightarrow{m(\beta)_*} \mathscr{C}^n(X^K; m(G/H')) \xrightarrow{X(\alpha)^*} X(\alpha)_*\mathscr{C}^n(X^{K'}; m(G/H')).$$

The exact sequences 2.1 for $M = m(G/H)$, $Y = X^K$ combine to give the exact sequence

$$(2.3) \qquad 0 \to m/X \to \mathscr{C}^0(m/X) \xrightarrow{d^0} \mathscr{C}^1(m/X) \xrightarrow{d^1} \mathscr{C}^2(m/X) \to \cdots$$

in $\mathrm{Mod}(\Gamma(\tilde{X}))$. We call $\mathscr{C}^{\cdot}(m/X)$ the *Alexander-Spanier resolution* of m/X.

Now we begin the computation of $R\Gamma(m/X)$. First of all, by 2.3,

$$(2.4) \qquad R\Gamma(m/X) \xrightarrow{\simeq} R\Gamma(\mathscr{C}^{\cdot}(m/X)),$$

an isomorphism in the derived category $D^+(\mathbf{Ab})$. Secondly, because $\Gamma = \varepsilon_* \circ \Gamma(c_*)$, where ε_* and $\Gamma(c_*)$ are induced by morphisms of $\mathbf{Z}$-topoi, we have

$$(2.5) \qquad R\Gamma \cong R\varepsilon_* \circ R\Gamma(c_*),$$

see [16, 5.4]. Furthermore, if the G-space X is paracompact, so are the closed subspaces X^K $(K \leq G)$; because the sheaves $\mathscr{C}^n(X^K; m(G/H))$ on X^K are fine, we see that in this case the objects $\mathscr{C}^n(m/X)$ of $\mathrm{Mod}(\Gamma(\tilde{X}))$ are $\Gamma(c_*)$-acyclic. Thus

$$(2.6) \qquad R\Gamma(c_*)(\mathscr{C}^{\cdot}(m/X)) \cong \Gamma(c_*)(\mathscr{C}^{\cdot}(m/X)), \quad X \text{ paracompact}.$$

Let us consider the cochain complex $\bar{A}^{\cdot} = \bar{A}^{\cdot}(X; m)$ in $\mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$ defined by

$$\begin{aligned}(2.7)\qquad &\bar{A}^n(u) = \bar{C}^n(X^K; m(G/H)),\\ &\bar{A}^n(\alpha,\beta)\colon \bar{C}^n(X^K; m(G/H)) \xrightarrow{m(\beta)_*} \bar{C}^n(X^K; m(G/H'))\\ &\qquad\qquad \xrightarrow{X(\alpha)^*} \bar{C}^n(X^{K'}; m(G/H'))\end{aligned}$$

for $u\colon G/H \to G/K$ and (α, β) as in 1.1. By 1.2, 2.2 and the definition of $\mathscr{C}^{\cdot}(m/X)$ we have

$$(2.8)\qquad \Gamma(c_*)(\mathscr{C}^{\cdot}(m/X)) \cong \bar{A}^{\cdot}(X; m), \quad X \text{ paracompact.}$$

The formulae 2.4, 2.5, 2.6 and 2.8 together show that for a paracompact G-space X,

$$R\Gamma(m/X) \cong R\varepsilon_*(\bar{A}^{\cdot}(X; m)).$$

Hence, to prove theorem 1.3, it is enough to prove the following two results:

PROPOSITION 2.9. *For any G-space X we have a natural isomorphism*

$$\varepsilon_*(\bar{A}^{\cdot}(X; m)) \cong \bar{C}_G^{\cdot}(X; m),$$

where the right hand side is the equivariant Alexander-Spanier cochain complex of X with coefficients m, see [9].

PROPOSITION 2.10. *If X is a paracompact G-space, then the canonical morphism*

$$\varepsilon_*(\bar{A}^{\cdot}(X; m)) \to R\varepsilon_*(\bar{A}^{\cdot}(X; m))$$

is an isomorphism in $D^+(\mathbf{Ab})$.

Proposition 2.9 will be proved in the next section, while the more difficult proof of proposition 2.10 is given sections 4–6.

We note that in the terminology of [1], $R^n\varepsilon_*(\bar{A}^{\cdot}(X; m))$ is the nth cohomology group of the category Or(G) with coefficients in the complex $\bar{A}^{\cdot}(X; m)$ of natural systems.

3. The projective limit.

Let $F\colon \mathscr{D} \to \mathbf{Ab}$ be a functor. Then $\varepsilon_*(F)$, the projective limit of F, consists of all families

$$x = (x(u)) \in \prod_{u \in \mathrm{Ob}(\mathscr{D})} F(u),$$

such that, for any morphism $(\alpha, \beta)\colon u \to u'$ of $\mathscr{D}$, the identity

$$(3.1)\qquad F(\alpha, \beta)(x(u)) = x(u')$$

holds. In fact the elements $x_H = x(\mathrm{id}_{G/H})$, $H \leqq G$, determine $x(u)$ for arbitrary $u\colon G/H \to G/K$. Namely, the commutative squares

$$\begin{array}{ccc} G/H & \xleftarrow{\text{id}} & G/H \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle u} \\ G/H & \xrightarrow{u} & G/K \end{array} \qquad\qquad \begin{array}{ccc} G/K & \xleftarrow{u} & G/H \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle u} \\ G/K & \xrightarrow{\text{id}} & G/K \end{array}$$

give the morphisms $(u, \text{id})\colon \text{id}_{G/H} \to u$ and $(\text{id}, u)\colon \text{id}_{G/K} \to u$ of $\mathscr{D}$, and by 3.1, $x(u) = F(u, \text{id})(x_H) = F(\text{id}, u)(x_K)$. It follows easily that we can identify

$$(3.2) \qquad \varepsilon_*(F) = \left\{ (x_H) \in \prod_{H \leqq G} F(\text{id}_{G/H}) \,\middle|\, \begin{array}{l} F(u, \text{id})(x_H) = F(\text{id}, u)(x_K) \\ \text{for } u\colon G/H \to G/K \end{array} \right\}.$$

Let us now return to the functors $\bar{A}^n = \bar{A}^n(X; m)\colon \mathscr{D} \to \mathbf{Ab}$ of 2.7 ($n \in \mathsf{N}$). We also consider the analogous functors $A^n = A^n(X; m)\colon \mathscr{D} \to \mathbf{Ab}$ with

$$A^n(u) = C^n(X^K; m(G/H))$$

for $u\colon G/H \to G/K$; here the locally zero cochains have not been factored out. The canonical surjections $C^n(X^K; m(G/H)) \to \bar{C}^n(X^K; m(G/H))$ induce natural morphisms $A^n \to \bar{A}^n$. By 3.2, $\varepsilon_*(A^n)$ and $\varepsilon_*(\bar{A}^n)$ consist of all families $c = (c_H)_{H \leqq G}$ and $\gamma = (\gamma_H)_{H \leqq G}$ such that $c_H \in C^n(X^H; m(G/H))$, $\gamma_H \in \bar{C}^n(X^H; m(G/H))$ and

$$X(u)^*(c_H) = m(u)_*(c_K), \quad X(u)^*(\gamma_H) = m(u)_*(\gamma_K)$$

for $u\colon G/H \to G/K$. Comparing this with the definition of the equivariant cochain group $C^n_G(X; m)$ in [9, section 1], we see that we can identify

$$(3.3) \qquad C^n_G(X; m) = \varepsilon_*(A^n).$$

Our objective is to prove proposition 2.9, that is

$$\bar{C}^n_G(X; m) \cong \varepsilon_*(\bar{A}^n).$$

Here $\bar{C}^n_G(X; m) = C^n_G(X; m)/C^n_{G,0}(X; m)$, i.e., we factor out those cochains which are locally zero with respect to an open G-covering of X in the sense of [9]. Hence, to prove 2.9, it is enough to verify

LEMMA 3.4. *The canonical morphism $\varepsilon_*(A^n) \to \varepsilon_*(\bar{A}^n)$ is surjective with kernel $C^n_{G,0}(X; m)$.*

For the proof of 3.4 we need another lemma concerning the coverings of X and $X^H, H \leqq G$:

LEMMA 3.5. *Suppose for every $H \leqq G$ we are given a finite number of open coverings of X^H. Then there exists an open G-covering $\mathscr{U}$ of X such that the covering $\mathscr{U} \cap X^H = \{U \cap X^H \mid U \in \mathscr{U}\}$ of X^H is a refinement of each of the given coverings of X^H, for all $H \leqq G$.*

PROOF. Let $x \in X$. We can find an open G_x-invariant neighbourhood U_x of x in X with the following properties:

1) If $gx \notin X^H$ for some $H \leqq G$, $g \in G$, then $gU_x \subset X \backslash X^H$.
2) If $gx \in X^H$ for some $H \leqq G$, $g \in G$, then $gU_x \cap X^H$ is contained in a set from each of the given coverings of X^H.

These requirements impose only a finite number of conditions on U_x; in 1) we also need the fact that the fixed point sets are closed in X. Then if $x' = gx\ (g \in G)$ is in the orbit of x, let $U_{x'} = gU_x$. Now $\mathcal{U} = \{U_x \mid x \in X\}$ is the required G-covering of X.

PROOF OF 3.4. We show first that $C^n_{G,0}(X;m) = \ker[\varepsilon_*(A^n) \to \varepsilon_*(\bar{A}^n)]$. The inclusion $\subset$ is clear. For the converse, let $c = (c_H) \in \varepsilon_*(A^n)$ map to 0 in $\varepsilon_*(\bar{A}^n)$. This means that $c_H \in C^n(X^H; m(G/H))$ is locally zero with respect to an open covering $\mathcal{V}_H$ of X^H, for every $H \leqq G$. Choose, by 3.5, an open G-covering $\mathcal{U}$ of X such that $\mathcal{U} \cap X^H$ is a refinement of $\mathcal{V}_H$ for each $H \leqq G$. Then c is locally zero with respect to $\mathcal{U}$.

To prove the surjectivity of $\varepsilon_*(A^n) \to \varepsilon_*(\bar{A}^n)$, take $\gamma = (\gamma_H) \in \varepsilon_*(\bar{A}^n)$, and a representative $c_H \in C^n(X^H; m(G/H))$ of $\gamma_H \in \bar{C}^n(X^H; m(G/H))$ for each $H \leqq G$. Then the cochain

$$X(u)^*(c_H) - m(u)_*(c_K) \in C^n(X^K; m(G/H))$$

is locally zero with respect to an open covering $\mathcal{V}_K(u)$ of X^K, for every $u\colon G/H \to G/K$. By 3.5 we find an open G-covering $\mathcal{U}$ of X such that for $K \leqq G$, $\mathcal{U} \cap X^K$ is a refinement of each of the coverings $\mathcal{V}_K(u)$, $u\colon G/H \to G/K$. We remark that because $\mathcal{U}$ is a G-covering, $X(u)^{-1}(U \cap X^H) \in \mathcal{U} \cap X^K$ for $U \in \mathcal{U}$, $u\colon G/H \to G/K$.

Put $Z = \bigcup_{U \in \mathcal{U}} U^{n+1} \subset X^{n+1}$ and define $c'_H \in C^n(X^H; m(G/H))$ by

$$c'_H(x_0, \ldots, x_n) = \begin{cases} c_H(x_0, \ldots, x_n), & \text{if } (x_0, \ldots, x_n) \in Z \cap (X^H)^{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $c' = (c'_H) \in \varepsilon_*(A^n)$ and $c' \mapsto \gamma \in \varepsilon_*(\bar{A}^n)$.

To end this section, we note that there is a relative version of proposition 2.9: If X' is a G-subspace of X, then

$$\varepsilon_*(\bar{A}^{\cdot}(X, X'; m)) \cong \bar{C}^{\cdot}_G(X, X'; m), \tag{3.6}$$

where $\bar{A}^n(X, X'; m)(u) = \bar{C}^n(X^K, X'^K; m(G/H))$ for $u\colon G/H \to G/K$. This follows from the definition of the relative cochain groups, 2.9 applied to X and X', and the left exactness of ε_*.

4. The derived functor of the projective limit, first reductions.

In this section we begin the proof of proposition 2.10, i.e., that the natural morphism $\varepsilon_*(\bar{A}^\cdot(X;m)) \to R\varepsilon_*(\bar{A}^\cdot(X;m))$ is an isomorphism in $D^+(\mathbf{Ab})$, if the G-space X is paracompact, which we henceforth assume. By section 3, we already know that $\varepsilon_*(\bar{A}^\cdot(X;m)) \cong \bar{C}^\cdot_G(X;m)$.

Let $(H_1), (H_2), \ldots, (H_r)$ be the distinct conjugacy classes of subgroups of G, ordered in such a way that

$$(H_i) < (H_j) \Rightarrow i > j;$$

then $H_1 = G$ and $H_r = \{e\}$, the trivial subgroup. For $i \in \{1, 2, \ldots, r\}$ we set

$$X^{(H_i)} = GX^{H_i} = \{x \in X \mid (H_i) \leqq (G_x)\}$$

$$X_i = X^{(H_1)} \cup X^{(H_2)} \cup \ldots \cup X^{(H_i)}.$$

Now

$$\varnothing = X_0 \subset X_1 \subset \ldots \subset X_{r-1} \subset X_r = X$$

are closed G-subsets of X, and $x \in X_i \backslash X_{i-1}$ implies $(G_x) = (H_i)$, i.e., (H_i) is the only isotropy type in $X_i \backslash X_{i-1}$. Because we have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \bar{C}^\cdot_G(X_i, X_{i-1}; m) & \to & \bar{C}^\cdot_G(X_i; m) & \to & \bar{C}^\cdot_G(X_{i-1}; m) & \to & 0 \\ 0 & \to & \bar{A}^\cdot(X_i, X_{i-1}; m) & \to & \bar{A}^\cdot(X_i; m) & \to & \bar{A}^\cdot(X_{i-1}; m) & \to & 0, \end{array}$$

$i \in \{1, 2, \ldots, r\}$, a five-lemma argument shows that for 2.10 it is enough to prove that $\varepsilon_*(\bar{A}^\cdot(X_i, X_{i-1}; m)) \to R\varepsilon_*(\bar{A}^\cdot(X_i, X_{i-1}; m))$ is an isomorphism in $D^+(\mathbf{Ab})$ for $i \in \{1, 2, \ldots, r\}$. Thus we are reduced to proving

PROPOSITION 4.1. *Let Y be a closed G-subspace of the paracompact G-space X and $K \leqq G$ a subgroup. If every orbit in $X \backslash Y$ has type (K), then the natural morphism*

$$\varepsilon_*(\bar{A}^\cdot(X, Y; m)) \to R\varepsilon_*(\bar{A}^\cdot, (X, Y; m))$$

is an isomorphism in $D^+(\mathbf{Ab})$.

In the rest of this section, as well as in sections 5 and 6 below, K, X and Y are as in 4.1. Let W be the group $\mathrm{Map}_G(G/K, G/K)$. Recall that W is isomorphic to NK/K in such a way that an element $a \in NK$ corresponds to the G-map $gK \mapsto ga^{-1}K$. Consider the twisted product

$$G/K \times_W X^K = (G/K \times X^K)/W,$$

where W acts on $G/K \times X^K$ via $(\alpha, (gK, z)) \mapsto (\alpha(gK), X(\alpha^{-1})(z))$ $(\alpha \in W, g \in G,$

$z \in X^K$). The group G acts on $G/K \times_W X^K$ by left translations on the factor G/K, and we have a G-map

$$f: G/K \times_W (X^K, Y^K) \to (X, Y), \quad [gK, z] \mapsto gz.$$

Clearly f is closed, and because all the orbits of $X \backslash Y$ are of type (K), f gives a bijection $G/K \times_W (X^K \backslash Y^K) \xrightarrow{\sim} X \backslash Y$, see [3, II 5.11].

LEMMA 4.2. *The G-map f induces quasi-isomorphisms*

$$\begin{array}{ccc} \bar{A}^{\cdot}(X, Y; m) & \to & \bar{A}^{\cdot}(G/K \times_W (X^K, Y^K); m) \\ \varepsilon_*(\bar{A}^{\cdot}(X, Y; m)) & \to & \varepsilon_*(\bar{A}^{\cdot}(G/K \times_W (X^K, Y^K); m)). \end{array}$$

PROOF. The first assertion follows immediately from the strong excision property of Alexander-Spanier cohomology, [15, 6.6.5]. The second assertion is a consequence of 3.6 and the equivariant analogue of [15, 6.6.5], which can easily be proved with the aid of the equivariant tautness property, [9, 5.1].

Hence we may assume in 4.1 that $(X, Y) = G/K \times_W (X^K, Y^K)$.

Let $\mathscr{I}_K$ be the full subcategory of $\mathscr{D}$ whose only object is $\mathrm{id}_{G/K}$. We have an evident restriction functor $\mathbf{Hom}(\mathscr{D}, \mathbf{Ab}) \to \mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab})$, $F \mapsto F | \mathscr{I}_K$. The formula $\alpha \mapsto (\alpha, \alpha^{-1})$ defines an isomorphism $W \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\mathrm{id}_{G/K}, \mathrm{id}_{G/K})$, and therefore we may identify

$$\mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab}) = \mathbf{Z}W\text{-Mod}, \tag{4.3}$$

the category of left $\mathbf{Z}W$-modules. In this identification the projective limit functor $\varepsilon_*^K: \mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab}) \to \mathbf{Ab}$ becomes simply $(\cdot) \mapsto (\cdot)^K$. For any F in $\mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$, there is an obvious natural transformation $\varepsilon_*(F) \to \varepsilon_*^K(F | \mathscr{I}_K)$.

In the situation of 4.1 we obtain the commutative square

$$\begin{array}{ccc} \varepsilon_*(\bar{A}^{\cdot}) & \xrightarrow{2} & \varepsilon_*^K(\bar{A}^{\cdot} | \mathscr{I}_K) \\ \downarrow & & \downarrow 1 \\ R\varepsilon_*(\bar{A}^{\cdot}) & \xrightarrow{3} & R\varepsilon_*^K(\bar{A}^{\cdot} | \mathscr{I}_K) \end{array} \tag{4.4}$$

in $D^+(\mathbf{Ab})$, where $\bar{A}^{\cdot} = \bar{A}^{\cdot} = \bar{A}^{\cdot}(X, Y; m)$. Thus, to prove 4.1, it suffices to show the arrows 1, 2 and 3 in 4.4 are isomorphisms. We treat the arrow 1 in this section, leaving 2 and 3 to sections 5 and 6 below, respectively.

It follows from 4.3 that $\bar{A}^{\cdot} | \mathscr{I}_K$ is identified with the complex $\bar{C}^{\cdot}(X^K, Y^K; m(G/K))$ of $\mathbf{Z}W$-modules, where $\alpha \in W$ acts on (X^K, Y^K) via the homeomorphism $X(\alpha): X^K \xrightarrow{\sim} X^K$, and on $m(G/K)$ via $m(\alpha^{-1})$. Also, $\varepsilon_*^K(\bar{A}^{\cdot} | \mathscr{I}_K) = \bar{C}^{\cdot}(X^K, Y^K; m(G/K))^W$ and

$$R^n\varepsilon_*^K(\bar{A}^{\cdot} | \mathscr{I}_K) = H^n(W; \bar{C}^{\cdot}(X^K, Y^K; m(G/K))),$$

the group cohomology of W with coefficients in the complex $\bar{C}^{\cdot}(X^K, Y^K; m(G/K))$. The fact that 1 in 4.4 is an isomorphism therefore follows from

LEMMA 4.5. *$H^i(W; \bar{C}^n(X^K, Y^K; m(G/K))) = 0$ for $n \in \mathbf{N}$, $i > 0$.*

PROOF. In the notation of [15, p. 311], we have

$$\bar{C}^n(X^K, Y^K; m(G/K)) \cong \varinjlim \operatorname{Hom}(C_n(\mathcal{U})/C_n'(\mathcal{U}'), m(G/K)), \tag{4.6}$$

where the limit can be taken over open W-coverings $\mathcal{U}$ of X^K. Because Y^K is closed in X^K, we may also assumethat $U \in \mathcal{U} \setminus \mathcal{U}'$ implies $U \subset X^K \setminus Y^K$. To prove that $\bar{C}^n(X^K, Y^K; m(G/K))$ is W-acyclic, it suffices, by [4, VII (4.6)], to show that the Hom-modules in 4.6 are W-acyclic. But since W acts freely on $X^K \setminus Y^K$, $C_n(\mathcal{U})/C_n'(\mathcal{U}')$ is a free $\mathbf{Z}W$-module. Therefore $\operatorname{Hom}(C_n(\mathcal{U})/C_n'(\mathcal{U}'), m(G/K))$ is isomorphic to a product of coinduced $\mathbf{Z}W$-modules $\operatorname{Hom}(\mathbf{Z}W, m(G/K))$ (see [4, III (5.7), (5.9)]), and hence is W-acyclic.

5. Some properties of equivariant Alexander-Spanier cohomology.

In this section we prove that the map 2 in 4.4 is an isomorphism. By 3.6 and 4.3 it is enough to show that

$$\bar{C}_G^{\cdot}(X, Y; m) \xrightarrow{\sim} \bar{C}^{\cdot}(X^K, Y^K; m(G/K))^W \tag{5.1}$$

is an isomorphism, where $(X, Y) = G/K \times_W (X^K, Y^K) \cong G \times_{NK} (X^K, Y^K)$ and $W = \operatorname{Map}_G(G/K, G/K) \cong NK/K$. For the proof of 5.1 we present three elementary properties of equivariant Alexander-Spanier cohomology, which were not covered in [9].

Let now $H \leqq G$ be a subgroup and Z an H-space. To the G-coefficient system $m\colon \operatorname{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ we associate the H-coefficient system $m_H\colon \operatorname{Or}(H)^{\mathrm{op}} \to \mathbf{Ab}$, $m_H(H/H') = m(G/H')$; note that $G/H' \cong G \times_H H/H'$ for $H' \leqq H$.

PROPOSITION 5.2. *There is a natural isomorphism*

$$\bar{C}_G^{\cdot}(G \times_H Z; m) \xrightarrow{\sim} \bar{C}_H^{\cdot}(Z; m_H).$$

PROOF. We have

$$G \times_H Z \cong \coprod_{gH \in G/H} gH \times_H Z,$$

and the sets $gH \times_H Z$, $gH \in G/H$, form an open G-covering $\mathcal{U}_0$ of $G \times_H Z$. Every open H-covering of $Z \cong eH \times_H Z \subset G \times_H Z$ extends in an evident way to an open G-covering of $G \times_H Z$, and every open G-covering of $G \times_H Z$, which is a refinement of $\mathcal{U}_0$, is obtained in this way from an open H-covering of Z.

Now, it is enough to consider cochains of $G \times_H Z$ subordinate to $\mathcal{U}_0$ (see p. 183 in [9]). Such a G-equivariant cochain $\gamma \in \bar{C}_G^n(G \times_H Z; m)$ determines by restriction an H-equivariant cochain in $\bar{C}^n_H(Z; m_H)$, and by G-equivariance, γ is determined by its restriction. This proves the assertion.

Applying the relative version of 5.2, we obtain

$$\bar{C}^{\cdot}_G(X, Y; m) \cong \bar{C}^{\cdot}_G(G \times_K (X^K, Y^K); m) \xrightarrow{\sim} \bar{C}^{\cdot}_{NK}(X^K; m_{NK}). \tag{5.3}$$

Here the normal subgroup K of NK acts trivially on X^K.

Proposition 5.4. *Suppose Z is a G-space, where a normal subgroup N of G acts trivially. Then there is a natural isomorphism*

$$\bar{C}^{\cdot}_G(Z; m) \xrightarrow{\sim} \bar{C}^{\cdot}_{G/N}(Z; m_{G/N})$$

with $m_{G/N}\colon \mathrm{Or}(G/N)^{\mathrm{op}} \to \mathbf{Ab}$ *defined by* $m_{G/N}\colon (G/N)/(H/N) \mapsto m(G/H)$ *for* $N \leqq H \leqq G$.

Proof. Clearly, an open G-covering of Z is the same thing as an open G/N-covering of Z. On the other hand, a cochain $c = (c_H)_{H \leqq G} \in C^n_G(Z; m)$ is determined by those $c_H\colon (Z^{H'})^{n+1} \to m(G/H')$, where $N \leqq H' \leqq G$; namely, if $H \leqq G$ is arbitrary and we let $H' = HN \leqq G$, then the diagram

$$\begin{array}{ccc}
(Z^H)^{n+1} & \xrightarrow{\quad c_H \quad} & m(G/H) \\
\| & & \uparrow \\
(Z^{H'})^{n+1} & \xrightarrow{\quad c_{H'} \quad} & m(G/H') \\
\| & & \| \\
(Z^{H'/N})^{n+1} & & m_{G/N}((G/N)/(H'/N))
\end{array}$$

commutes. Hence the natural map $C^n_G(Z; m) \to C^n_{G/N}(Z; m_{G/N})$ is an injection. For surjectivity, let the functions $c_{H'}\colon (Z^{H'})^{n+1} \to m(G/H')$ $(N \leqq H' \leqq G)$ represent an element of $C^n_{G/N}(Z; m_{G/N})$. Then we can define $c_H\colon (Z^H)^{n+1} \to m(G/H)$ for all $H \leqq G$ by requiring the above diagram to commute, and in fact $(c_H)_{H \leqq G} \in C^n_G(Z; m)$ is a G-equivariant cochain. Here we need to observe that every G-map $G/H \to G/L$, given by $gH \mapsto gaL$ with $a^{-1}Ha \leqq L$, induces a G-map $G/H' \to G/L'$, where $H' = HN$, $L' = LN$, by $gH' \mapsto gaL'$; namely, the normality of N implies that $a^{-1}H'a = a^{-1}Ha \cdot a^{-1}Na \leqq L'$.

The relative version of this result applied to the right hand side of 5.3 gives

$$\bar{C}^{\cdot}_{NK}(X^K, Y^K; m_{NK}) \xrightarrow{\sim} \bar{C}^{\cdot}_W(X^K, Y^K; m_W). \tag{5.5}$$

Here the group $W \cong NK/K$ acts freely on $X^K \backslash Y^K$.

Proposition 5.6. *Suppose Z is a G-space, $Z' \subset Z$ is a G-subspace and G acts freely on $Z \backslash Z'$. Then there is an isomorphism*

$$\bar{C}_G(Z, Z'; m) \xrightarrow{\sim} \bar{C}^{\cdot}(Z, Z'; m(G))^G.$$

PROOF. By assumption, $Z^H = Z'^H$ for $\{e\} < H \leqq G$. By 3.6 we have $\bar{C}_G(Z, Z'; m) \cong \varepsilon_*(\bar{A}^{\cdot}(Z, Z'; m))$, and now

$$\bar{A}^{\cdot}(Z, Z'; m)\colon u \mapsto \begin{cases} \bar{C}^{\cdot}(Z, Z'; m(G)) & \text{for } u\colon G \xrightarrow{\sim} G \\ 0 & \text{for other } u. \end{cases}$$

Thus it is clear that $\varepsilon_*(\bar{A}^{\cdot}(Z, Z'; m)) \xrightarrow{\sim} [\bar{A}^{\cdot}(Z, Z'; m)(\mathrm{id}_G)]^{\mathrm{Aut}_{\mathscr{D}}(\mathrm{id}_G)}$.

This result applied to the right hand side of 5.5 gives

$$(5.7) \qquad \bar{C}_W^{\cdot}(X^K, Y^K; m_W) \xrightarrow{\sim} \bar{C}^{\cdot}(X^K, Y^K; m(G/K))^W.$$

5.1 now follows from 5.3, 5.5 and 5.7.

6. The derived functor of the projective limit, conclusion.

In this section we prove that the arrow 3 in 4.4, that is

$$R\varepsilon_*(\bar{A}^{\cdot}(X, Y; m)) \to R\varepsilon_*^K(\bar{A}^{\cdot}(X, Y; m) \mid \mathscr{I}_K),$$

is an isomorphism in $D^+(\mathbf{Ab})$, for $(X, Y) = G/K \times_W (X^K, Y^K)$.

At this point we must recall, how the derived functors of projective limit functors ε_* can be computed. Let $\mathscr{C}$ be a (small) category and consider $\varepsilon_*\colon \mathbf{Hom}(\mathscr{C}, \mathbf{Ab}) \to \mathbf{Ab}$. For $F^{\cdot} \in D^+(\mathbf{Hom}(\mathscr{C}, \mathbf{Ab}))$ we have

$$R\varepsilon_*(F^{\cdot}) = \varepsilon_*(E^{\cdot}),$$

where $F^{\cdot} \to E^{\cdot}$ is a resolution of $F^{\cdot}$ (i.e., a quasi-isomorphism) such that each E^n is a product of elementary objects of $\mathbf{Hom}(\mathscr{C}, \mathbf{Ab})$; the *elementary object* $A_x \in \mathbf{Hom}(\mathscr{C}, \mathbf{Ab})$ determined by an abelian group A and an object $x \in \mathrm{Ob}(\mathscr{C})$ is defined by

$$A_x\colon y \mapsto A^{\mathrm{Hom}_{\mathscr{C}}(y, x)}, \quad y \in \mathrm{Ob}(\mathscr{C}).$$

Then the functor $(\cdot)_x\colon \mathbf{Ab} \to \mathbf{Hom}(\mathscr{C}, \mathbf{Ab})$ defined by $A \mapsto A_x$ is right adjoint to the evaluation functor $e_x\colon \mathbf{Hom}(\mathscr{C}, \mathbf{Ab}) \to \mathbf{Ab}$, $e_x(F) = F(x)$. This method of computing $R\varepsilon_*$, based on the ε_*-acyclicity of elementary objects, is well-known; see for example [6], where $\mathscr{C}$ is assumed to be the category associated to an ordered set.

Let $\mathscr{D}_K$ be the full subcategory of $\mathscr{D}$ with $\mathrm{Ob}(\mathscr{D}_K) = \{u\colon G/H \to G/K \mid H \leqq G\}$, and $\iota\colon \mathscr{D}_K \hookrightarrow \mathscr{D}$ the inclusion functor. By [14, 1.2.10], the restriction functor ι^*: $\mathbf{Hom}(\mathscr{D}, \mathbf{Ab}) \to \mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab})$, $\iota^* F = F \mid \mathscr{D}_K$, has a right adjoint $\iota_*\colon \mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab}) \to \mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$, whose explicit construction is the following: If $F \in \mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab})$, and $u\colon G/H \to G/L$ is an object of $\mathscr{D}$, then

(6.1) $$\iota_*(F)(u) = \varprojlim_{u\backslash\mathscr{D}_K} F,$$

the projective limit taken over the category $u\backslash\mathscr{D}_K$ with objects $(v,(\alpha,\beta))$, where v is an object of $\mathscr{D}_K$ and $(\alpha,\beta)\colon u\to v$ is a morphism of $\mathscr{D}$; a morphism $(v,(\alpha,\beta))\to(v',(\alpha',\beta'))$ in $u\backslash\mathscr{D}_K$ is a morphism $(\varphi,\psi)\colon v\to v'$ of $\mathscr{D}_K$ such that $(\alpha',\beta')=(\varphi,\psi)\circ(\alpha,\beta)$.

Let $\mathscr{A}$ be the full subcategory of $u\backslash\mathscr{D}_K$ with objects $(\alpha\circ u,(\alpha,\mathrm{id}_{G/H}))$, $\alpha\colon G/L\to G/K$. Every morphism of $\mathscr{A}$ has the form

$$(\gamma,\mathrm{id}_{G/H})\colon(\alpha\circ u,(\alpha,\mathrm{id}_{G/H}))\xrightarrow{\sim}(\gamma\circ\alpha\circ u,(\gamma\circ\alpha,\mathrm{id}_{G/H})),$$

where $\gamma\in W=\mathrm{Map}_G(G/K,G/K)$. Suppose $(v,(\alpha,\beta))$ is an object of $u\backslash\mathscr{D}_K$. Then $(\mathrm{id}_{G/K},\beta)$ is a morphism $(\alpha\circ u,(\alpha,\mathrm{id}_{G/K}))\to(v,(\alpha,\beta))$. On the other hand, every morphism from an object of $\mathscr{A}$ to $(v,(\alpha,\beta))$ can be written as $(\gamma,\beta)\colon(\alpha'\circ u,(\alpha',\mathrm{id}_{G/H}))\to(v,(\alpha,\beta))$ for some $\gamma\in W$, so $(\gamma,\beta)=(\mathrm{id}_{G/K},\beta)\circ(\gamma,\mathrm{id}_{G/H})$. Therefore it is enough to take the projective limit in 6.1 over the subcategory $\mathscr{A}$, and we can identify

(6.2) $$(\iota_*F)(u)\cong\Bigg[\prod_{\alpha\colon G/L\to G/K}F(\alpha\circ u)\Bigg]^W,$$

the fixed point set for the right action of the group $W=\mathrm{Map}_G(G/K,G/K)$ on the product $\prod_\alpha F(\alpha\circ u)$ such that, if $\gamma\in\mathrm{W}$ and $a=(a_\alpha)\in\prod_\alpha F(\alpha\circ u)$, then

$$a\cdot\gamma=(b_\alpha),\quad b_\alpha=F(\gamma^{-1},\mathrm{id})(a_{\gamma\circ\alpha}).$$

LEMMA 6.3. *The functor ι_* is exact, preserves products and maps the elementary object of* **Hom**$(\mathscr{D}_K,$ **Ab**$)$ *determined by an abelian group A and object v of $\mathscr{D}_K$ to the elementary object of* **Hom**$(\mathscr{D},$ **Ab**$)$ *determined by the same A and v.*

PROOF. The exactness of ι_* is clear by 6.2, because W acts freely on $\mathrm{Map}_G(G/L,G/K)$. Being a right adjoint functor, ι_* trivially preserves products. Finally, the last assertion follows from the fact that, given an object v of $\mathscr{D}_K$, the composite of the functors $(\cdot)_v\colon\mathbf{Ab}\to\mathbf{Hom}(\mathscr{D}_K,\mathbf{Ab})$ and $\iota_*\colon\mathbf{Hom}(\mathscr{D}_K,\mathbf{Ab})\to\mathbf{Hom}(\mathscr{D},\mathbf{Ab})$ is right adjoint to the composite

$$\mathbf{Hom}(\mathscr{D},\mathbf{Ab})\xrightarrow{\iota^*}\mathbf{Hom}(\mathscr{D}_K,\mathbf{Ab})\xrightarrow{e_v}\mathbf{Ab},$$

which equls $e_v\colon\mathbf{Hom}(\mathscr{D},\mathbf{Ab})\to\mathbf{Ab}$.

The significance of the functor ι_* for the computation of $R\varepsilon_*(\bar{A}^{\cdot}(X,Y;m))$ is shown by

LEMMA 6.4. *$\bar{A}^{\cdot}(X,Y;m)$ is quasi-isomorphic to $\iota_*(\bar{A}^{\cdot}(X,Y;m)\,|\,\mathscr{D}_K)$.*

PROOF. Recall that we may take $(X, Y) = G/K \times_W (X^K, Y^K)$. We consider the values of the functors on an object $u: G/H \to G/L$ of $\mathscr{D}$. We have an obvious closed map

$$(G/K)^L \times_W (X^K, Y^K) \to [G/K \times_W (X^K, Y^K)]^L. \tag{6.5}$$

Because W acts freely on $X^K \backslash Y^K$, we see that $(G\backslash K)^L \times_W (X^K \backslash Y^K)$ is mapped bijectively to $[G\backslash K \times_W (X^K \backslash Y^K)]^L$. Note further that $(G/K)^L \cong \mathrm{Map}_G(G/L, G/K)$. By lemma 6.6 below, there is a natural isomorphism

$$\bar{C}^{\cdot}((G/K)^L \times_W (X^K, Y^K); m(G/H)) \cong \left[\prod_{(G/K)^L} \bar{C}^{\cdot}(X^K, Y^K; m(G/H))\right]^W$$

Thus the strong excision property of Alexander-Spanier cohomology, which was already used in the proof of 4.2, implies that the maps 6.5 induce the required quasi-isomorphism.

LEMMA 6.6. *Suppose S is a free G-set, Z a G-space and M an abelian group. Then there is a natural isomorphism*

$$\bar{C}^{\cdot}(S \times_G Z; M) \xrightarrow{\sim} \left[\prod_S \bar{C}^{\cdot}(Z; M)\right]^G$$

PROOF. A suitable natural homomorphism is defined by the composite

$$\bar{C}^{\cdot}(S \times_G Z; M) \to \bar{C}^{\cdot}(S \times Z; M) \xrightarrow{\sim} \prod_{s \in S} \bar{C}^{\cdot}(\{s\} \times Z; M) \cong \prod_S \bar{C}^{\cdot}(Z; M),$$

where the first map is induced by the canonical surjection $S \times Z \to S \times_G Z$ and the second map is the isomorphism of [15, 6.4.8]. Choose a set $S_0 \subset S$ of representatives for the G-orbits of S. Because

$$S \times_G Z \cong \coprod_{s \in S_0} \{s\} \times Z,$$

both sides of the claim of the lemma become isomorphic to $\prod_{S_0} \bar{C}^{\cdot}(Z; M)$, and in this identification the above natural homomorphism becomes the identity.

Take now a resolution $\bar{A}^{\cdot}(X, Y; m) \mid \mathscr{D}_K \to E^{\cdot}$ of $\bar{A}^{\cdot}(X, Y; m) \mid \mathscr{D}_K$ such that each E^n is a product of elementary objects of $\mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab})$. By 6.2 and 6.4,

$$\bar{A}^{\cdot}(X, Y; m) \to \iota_*(\bar{A}^{\cdot}(X, Y; m) \mid \mathscr{D}_K) \to \iota_*(E^{\cdot})$$

is a resolution of $\bar{A}^{\cdot}(X, Y; m)$ and each $\iota_*(E^n)$ is a product of elementary objects of $\mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$. Thus we can compute.

$$\begin{aligned} R\varepsilon_*(\bar{A}^{\cdot}(X, Y; m)) &\cong \varepsilon_*(\iota_*(E^{\cdot})) \\ &= \varepsilon_*(E^{\cdot}) \cong R\varepsilon_*(\bar{A}^{\cdot}(X, Y; m) \mid \mathscr{D}_K). \end{aligned} \tag{6.7}$$

On the second line here, ε_* means the projective limit functor $\mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab}) \to \mathbf{Ab}$; the equality $\varepsilon_*(\iota_*(E^{\cdot})) = \varepsilon_*(E^{\cdot})$ follows from the fact that for an elementary object A_v, there is the identity $\varepsilon_*(A_v) = A$.

Next we consider the restriction functor $\mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab}) \to \mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab})$. Clearly this functor is exact, preserves products and carries an elementary object of $\mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab})$ to a product of elementary objects of $\mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab})$. This last claim is due to the fact that the group $\mathrm{Hom}_{\mathscr{D}}(\mathrm{id}_{G/K}, \mathrm{id}_{G/K}) \cong \mathrm{W}$ acts freely on $\mathrm{Hom}_{\mathscr{D}}(\mathrm{id}_{G/K}, v)$ for any object v of $\mathscr{D}_K$, and thus, for an abelian group A, $A_v \,|\, \mathscr{I}_K = A^{\mathrm{Hom}(\mathrm{id}, v)}$ splits as a product of factors isomorphic to $A^{\mathrm{Hom}(\mathrm{id}, \mathrm{id})}$. Furthermore we have

LEMMA 6.8. *For any* $F \in \mathbf{Hom}(\mathscr{D}_K, \mathbf{Ab})$, *the natural map* $\varepsilon_*(F) \to \varepsilon_*^K(F \,|\, I_K)$ *is an isomorphism.*

PROOF. If $u\colon G/H \to G/K$ is an object of $\mathscr{D}_K$, then $(\mathrm{id}_{G/K}, u)$ is a morphism $\mathrm{id}_{G/K} \to u$; if $(\alpha, \beta)\colon \mathrm{id}_{G/K} \to u$ is an arbitrary morphism, i.e., the square

$$\begin{array}{ccc} G/K & \xleftarrow{\ \beta\ } & G/H \\ \Big\| & & \Big\downarrow{\scriptstyle u} \\ G/K & \xrightarrow{\ \alpha\ } & G/K \end{array}$$

commutes, then $(\alpha, \beta) = (\mathrm{id}_{G/K}, u) \circ (\alpha, \alpha^{-1})$, where (α, α^{-1}) lies in the group $\mathrm{Hom}_{\mathscr{D}}(\mathrm{id}_{G/K}, \mathrm{id}_{G/K}) \cong W$. The assertion follows from this observation.

Let $\bar{A}^{\cdot}(X, Y; m) \,|\, \mathscr{D}_K \to E^{\cdot}$ be the above resolution. Then $\bar{A}^{\cdot}(X, Y; m) \,|\, \mathscr{I}_K \to E^{\cdot} \,|\, \mathscr{I}_K$ is a resolution of $\bar{A}^{\cdot}(X, Y; m) \,|\, \mathscr{I}_K$ and each $E^n \,|\, \mathscr{I}_K$ is a product of elementary objects of $\mathbf{Hom}(\mathscr{I}_K, \mathbf{Ab})$. Thus

$$\begin{aligned} (6.9) \qquad R\varepsilon_*(\bar{A}^{\cdot}(X, Y; m) \,|\, \mathscr{D}_K) &\cong \varepsilon_*(E^{\cdot}) \\ &= \varepsilon_*^K(E^{\cdot} \,|\, \mathscr{I}_K) \cong R\varepsilon_*^K(\bar{A}^{\cdot}(X, Y; m) \,|\, \mathscr{I}_K). \end{aligned}$$

The identities 6.7 and 6.9 together show that the arrow 3 in 4.4 is an isomorphism.

7. The spectral sequence of an equivariant map.

Let $f\colon X \to Y$ be a G-map between paracompact G-spaces. The map f induces a natural transformation $X \circ T \to Y \circ T$ (notation as in section 1); for an object $u\colon G/H \to G/K$ of $\mathscr{D}$, the map $(X \circ T)(u) \to (Y \circ T)(u)$ is simply $f^K\colon X^K \to Y^K$. This natural transformation determines a morphism

$$f = (f_*, f^*)\colon \Gamma(\tilde{X}) \to \Gamma(\tilde{Y})$$

of topoi. Explicitly, if $\mathscr{F} = (\mathscr{F}(u))_{u \in \mathrm{Ob}(\mathscr{D})}$ is an object of $\Gamma(\tilde{X})$, then $f_*(\mathscr{F}) = (\mathscr{G}(u))_{u \in \mathrm{Ob}(\mathscr{D})}$, where

$$\mathscr{G}(u) = (f^K)_*(\mathscr{F}(u)), \quad u\colon G/H \to G/K;$$

the functor f^* has a similar description.

Let $\mathscr{F}$ be an object of $\mathrm{Mod}(\Gamma(\tilde{X}))$. Associated to the morphism $f\colon \Gamma(\tilde{X}) \to \Gamma(\tilde{Y})$ there is, by [16, 5.3], a spectral sequence, called the Cartan-Leray spectral sequence, with

$$E_2^{pq} = H^p(\Gamma(\tilde{Y}); R^q f_*(\mathscr{F})), \tag{7.1}$$

converging to $H^{p+q}(\Gamma(\tilde{X}); \mathscr{F})$. This is in fact a special case of the Grothendieck spectral sequence of composite functors, [8, 2.5.4]. In 7.1 the object $R^q f_*(\mathscr{F}) \in \mathrm{Mod}(\Gamma(\tilde{Y}))$ can be described as follows: if $u\colon G/H \to G/K$ is an object of $\mathscr{D}$, then

$$R^q f_*(\mathscr{F})(u) = R^q(f^K)_*(\mathscr{F}(u)),$$

and $R^q(f^K)_*(\mathscr{F}(u))$ is the sheaf associated to the presheaf

$$V \mapsto H^q((f^K)^{-1}(V); \mathscr{F}(u)), \qquad V \subset Y^K \text{ open}, \tag{7.2}$$

on Y^K.

Taking $\mathscr{F} = m/X$ and combining 1.3 with 7.1, we obtain

PROPOSITION 7.3. *For a G-map $f\colon X \to Y$ between paracompact G-spaces and any coefficient system $m\colon \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ there is a spectral sequence with*

$$E_2^{pq} = H^p(\Gamma(\tilde{Y}); R^q f_*(m/X)),$$

converging to $\bar{H}^{p+q}(X; m)$.

Now we attempt to give a more concrete interpretation of the E_2-term of the spectral sequence of 7.3 in some simple cases. First of all, let Y be a point. Then $\Gamma(\tilde{Y}) = \mathbf{Hom}(\mathscr{D}, \mathbf{Ab})$, $H^p(\Gamma(\tilde{Y}); \cdot) = R^p\varepsilon_*$ and $R^q f_*(m/X)$ is

$$[u\colon G/H \to G/K] \mapsto \bar{H}^q(X^K; m(G/H)).$$

If we use the more suggestive notation $R^p\varepsilon_* = \varprojlim_{\mathscr{D}}{}^p$, we have

COROLLARY 7.4. *For a paracompact G-space X and any coefficient system $m\colon \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$ there is a spectral sequence with*

$$E_2^{pq} = \varprojlim_{\mathscr{D}}{}^p \bar{H}^q(X^K; m(G/H)),$$

converging to $\bar{H}_G^{p+q}(X; m)$.

To get a special case of this special case, assume now further that G acts freely on the paracompact space X. Then $X^K = \varnothing$, unless $K = \{e\}$, so the functor $R^q f_*(m/X)$ vanishes on the object $u: G/H \to G/K$ of $\mathscr{D}$, unless u is a G-map $G \xrightarrow{\simeq} G$. Now, if $F: \mathscr{D} \to \mathbf{Ab}$ is any functor such that $F(u) = 0$ unless $u: G \xrightarrow{\simeq} G$, it can be seen as in 6.8 and 6.9 that

$$\varprojlim_{\mathscr{D}} F \cong F(\mathrm{id}_G)^G, \varprojlim_{\mathscr{D}}{}^p F \cong H^p(G; F(\mathrm{id}_G)).$$

In particular,

$$\varprojlim_{\mathscr{D}}{}^p \bar{H}^q(X^K; m(G/H)) \cong H^p(G; \bar{H}^q(X; m(G)));$$

in the $\mathsf{Z}G$-module $\bar{H}^q(X; m(G))$ the action of $g \in G$ is induced by the action of g^{-1} on X and $m(r_g)$ on $m(G)$. Furthermore, by [9, 6.4], the freeness of the G-action on X implies that $\bar{H}_G^{p+q}(X; m) \cong H^{p+q}(X/G; \mathscr{M})$, where $\mathscr{M}$ is the locally constant sheaf on X/G with stalks $m(G)$ described in [9, 6.5]. In fact $\mathscr{M}$ only depends on the $\mathsf{Z}G$-module $M = m(G)$. Altogether we have obtained the classical Cartan-Leray spectral sequence of the covering space $X \to X/G$ (see [5, p. 355]):

COROLLARY 7.5. *If the group G acts freely on the paracompact space X and M is a $\mathsf{Z}G$-module, then 7.4 gives a spectral sequence with*

$$E_2^{pq} \cong H^p(G; H^q(X; M)),$$

converging to $H^{p+q}(X/G; \mathscr{M})$.

For another application of 7.3, let now $f: X \to Y$ be a G-fibration. Then for each $K \leqq G$, the map $f^K: X^K \to Y^K$ is an ordinary fibration. Assume further that the spaces Y^K are localy contractible; this is the case if, for example, Y is a $G-$CW-complex. If V is a neighbourhood of $y \in Y^K$ such that $V \hookrightarrow Y^K$ is homotopic to the constant map $V \to \{y\}$, then over V the fibration f^K is fibre homotopy equivalent to the trivial fibration $V \times F^K \to V$, where $F = f^{-1}(y)$. Therefore it follows from 7.2 and the homotopy invariance of sheaf cohomology with constant coefficients that the sheaf $R^q f_*^K(m(G/H))$ is constant with stalks $H^q(F^K; m(G/H))$ on V for $H \leqq G$. Thus we have proved

PROPOSITION 7.6. *Suppose that in 7.3 f is a G-fibration and the fixed point spaces Y^K are locally contractible. Then $R^q f_*(m/X)$ is a family of locally constant sheaves.*

If, in the situation of 7.6, we regard $H^p(\Gamma(\tilde{Y}); R^q f_*(m/X))$ as equivariant Alexander-Spanier cohomology of Y with local coefficients $R^q f_*(m/X)$, 7.3 and 7.6 give an equivariant version of the Serre spectral sequence for the G-fibration f. We note that if all fixed point sets of Y are non-empty and simply connected, then the above coefficient system is essentially a functor $\mathscr{D} \to \mathbf{Ab}$. However, it does not factor through an ordinary contravariant coefficient system $\mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Ab}$.

REFERENCES

1. H.-J. Baues and G. Wirsching, *Cohomology of small categories*, J. Pure Appl. Algebra 38 (1985), 187–211.
2. G. E. Bredon, *Sheaf theory*, McGraw-Hill, New York-Toronto-London, 1967.
3. G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York-London, 1972.
4. K. S. Brown, *Cohomology of Groups*, Springer-Verlag, New York-Heidelberg-Berlin, 1982.
5. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.
6. R. Deheuvels, *Homologie des ensembles ordonnés et des espaces topologiques*, Bull. Soc. Math. France 90 (1962), 261–321.
7. D. Grayson, *Higher algebraic K-theory* II [after D. Quillen], Lecture Notes in Math. 551 (1976), 217–240.
8. A. Grothendieck, *Sur quelques points d'algebre homologique*, Tohoku Math. J. 9 (1957), 119–221.
9. H. Honkasalo, *Equivariant Alexander-Spanier cohomology*, Math. Scand. 63 (1988), 179–195.
10. H. Honkasalo, *Equivariant Alexander-Spanier cohomology for actions of compact Lie groups*, Math. Scand. 67 (1990), 23–34.
11. H. Honkasalo and E. Laitinen, *Equivariant Lefschetz classes in Alexander-Spanier cohomology*, Osaka J. Math. (to appear).
12. S. Illman, *Equivariant singular homology and cohomology*, Mem. Amer. Math. Soc. 156 (1975).
13. I. Moerdijk and J.-A. Svensson, *The equivariant Serre spectral sequence*, Proc. Amer. Math. Soc. 118 (1993), 263–278.
14. B. Saint-Donat, *Techniques de descente cohomologique*, Lecture Notes in Math. 270 (SGA 4, Tome 2) (1972), 83–162.
15. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York-Toronto-London, 1966.
16. J.-L. Verdier, *Cohomology dans les topos*, Lecture Notes in Math. 270 (SGA 4, Tome 2) (1972), 1–82.

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