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# LOCALISATION OF UNSTABLE $A_p$-ALGEBRAS AND SMITH THEORY*

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## 1. Introduction.

In [6] the Borel-Quillen localisation theorem [4, 10] is reformulated to produce an expression for the mod $p$ cohomology of the fixed point set of a finite $\mathsf{Z}_p$-complex $X$ in terms of the mod $p$ equivariant cohomology of $X$. Here $\mathsf{Z}_p$ denotes the cyclic group of prime order $p$. This result contrasts with previous work where the localised cohomology of the fixed point set is obtained. The key to this extension is to localise in the category of unstable $A_p$-algebras, where $A_p$ is the mod $p$ Steenrod algebra. Many of the classical results concerning the cohomology of fixed point sets then become statements concerning this localisation process.

In this paper we reproduce some of these results giving proofs in the category of unstable $A_p$-algebras. Thus we only require $X$ to have finite mod $p$ cohomology. The finiteness of $X$ as a complex is in fact only used at the final step to identify the result of localisation with the cohomology of the fixed point set. We are motivated by the study of group actions on infinite dimensional $\mathsf{Z}_p$-complexes $X$ with finite mod $p$ cohomology. These arise naturally in the theory of transformation groups [2, 11, 12]. In this case the result of localisation of the equivariant cohomology of $X$ may be thought of as a kind of continuous cohomology for the homotopy fixed point set of the Bousfield Kan $p$-completion of $X$, [5], and results of this paper then give information concerning this continuous cohomology. In [8] we use this to obtain finiteness properties in the form of a "generalised Sullivan conjecture" type result for certain $X$.

## 2. Localisation of Unstable $A_p$-algebras.

Let $H^*$ denote the singular cohomology functor with coefficients in $\mathsf{Z}_p$. Let $\mathscr{K}$ denote the category of unstable $A_p$-algebras and $H\backslash\mathscr{K}$ the category $\mathscr{K}$ under

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$H$, where $H = H^*(BZ_p; Z_p) \in \mathscr{K}$, [9]. Let $S \subset H$ be the multiplicative subset generated by the image of the Bockstein on elements of degree 1. For $M \in H\backslash\mathscr{K}$, the localised algebras $S^{-1}M$ inherits an $A_p$-algebra structure, which has a unique maximal unstable $A_p$-subalgebra denoted $\mathrm{Un}(S^{-1}M)$, [1]. Thus we have a localisation functor $\mathscr{L}: H\backslash\mathscr{K} \to H\backslash\mathscr{K}$ defined by $\mathscr{L}(M) = \mathrm{Un}(S^{-1}M)$. Let $X$ be a finite $Z_p$-complex. The Dwyer-Wilkerson localisation theorem [6] is then

THEOREM 2.1. *There is an $H\backslash\mathscr{K}$-isomorphism*

$$H^*(X^{Z_p}) \cong Z_p \otimes_H \mathscr{L}(H^*_{Z_p}X).$$

Here $X^{Z_p}$ denotes the fixed point set of $X$ and the action of $H$ on $Z_p$ is induced by the augmentation. We shall also require the following theorem of Dwyer-Wilkerson [7]. Let $\bar{H}$ denote the augmentation ideal of $H$.

PROPOSITION 2.2. *For any $M \in H\backslash\mathscr{K}$ there is a natural $H\backslash\mathscr{K}$ isomorphism $\mathscr{L}(M) \cong H \otimes_{Z_p} (\mathscr{L}(M)/\bar{H}\mathscr{L}(M))$.*

We note that the unstable $A_p$-algebra $Z_p \otimes_H \mathscr{L}(M)$ is isomorphic to the unstable $A_p$-algebra Fix $M$ of Lannes [9] and the functor $Z_p \otimes_H \mathscr{L}$ is therefore exact [9].

An augmentation for $M \in H\backslash\mathscr{K}$ is a $H\backslash\mathscr{K}$-map $M \to H$. We shall assume henceforth that $M$ is augmented so that in particular $\mathscr{L}(M)$ is augmented. Denote the augmentation ideal of $M$ by $\bar{M}$.

Let $(\Sigma^r H_k)_+ \in H\backslash\mathscr{K}$ denote the augmented unstable $A_p$-algebra, $p = 2$, with algebra generators $t$ and $x \in \overline{(\Sigma^r H_k)_+}$ in degree 1 and $r + k$ respectively, such that $Sq^i(x) = \binom{k}{i} t^i x$. We write $(\Sigma^r H)_+$ for $(\Sigma^r H_0)_+$.

LEMMA 2.3. *Let $M \in H\backslash\mathscr{K}$. Suppose $\bar{M}$ is freely generated as an $H$-module by a single element. Then $\mathscr{L}(M)$ is $H\backslash\mathscr{K}$ isomorphic to $(\Sigma^r H)_+$ for some $r \geqq 0$.*

PROOF. Suppose $\bar{M}$ is generated by $m \in M^i$. We shall show that $M \cong (\Sigma^r H_k)_+$ for some $r$, $k$ and the result will follow by exactness of $\mathscr{L}$. The proof is by induction on $i$. If $i = 0$ we easily have $\bar{M} = H$. Suppose the lemma is true for all $M$ with generator in dimension $j \leqq i - 1, i \geqq 1$. Let $M \in H\backslash\mathscr{K}$ have a generator $m \in M$ such that $\deg(m) = i$. Suppose $Sq^i m = 0$. Then there exists $M' \in H\backslash\mathscr{K}$ such that $\Sigma \bar{M}' = \bar{M}$. The result then follows by induction. Assume then $Sq^i m = t^i m$. We shall show that if $Sq^i m = t^i m$ and $Sq^j m$ is determined, $0 \leqq j \leqq r$, $r < i - 1$, then $Sq^{r+1} m$ is determined. Since $Sq^0 m = m$ it follows that the action of $A_2$ on $M$ is completely determined. However in $(H_i)_+$ with generator $x \in (\bar{H}_i)^i$ we have $Sq^i(x) = t^i x$ so that we must have $M = (H_i)_+$.

Suppose then $Sq^i m = t^i m$ and $Sq^j m = \delta_j t^j m$ is given, $0 \leqq j \leqq r, r < i - 1$. Then we have

$$Sq^{r+1}Sq^i m = \sum_{0 \leqq c \leqq [\frac{r+1}{2}]} \varepsilon_c Sq^{i+r+1-c} Sq^c m$$
$$= \sum_{0 \leqq c \leqq [\frac{r+1}{2}]} \varepsilon'_c Sq^{i+r+1-c}(t^c m)$$

where the coefficients $\varepsilon_c$, $\varepsilon'_c \in \mathbf{Z}_2$ are given. But $Sq^{i+r+1-c}(t^c m) = 0$ unless $c \geqq \frac{r+1}{2}$, In the case $r$ odd the latter inequality can only be satisfied if $c = \frac{r+1}{2}$ so that

$$Sq^{r+1}Sq^i m = \varepsilon'_{\frac{r+1}{2}} Sq^{i+\frac{r+1}{2}}(t^{\frac{r+1}{2}} m)$$
$$= \varepsilon'_{\frac{r+1}{2}} t^{r+1} Sq^i m = \varepsilon'_{\frac{r+1}{2}} t^{r+i+1} m.$$

In the case that $r$ is even the inequality $\frac{r+1}{2} \leqq c \leqq \left[\frac{r+1}{2}\right]$ has no solution so that $Sq^{r+1}Sq^i m = 0$. In either case $Sq^{r+1}Sq^i m$ is determined. Thus

$$Sq^{r+1}(t^i m) = \sum_{a+b=r+1} \binom{i}{a} x^{i+a} Sq^b m.$$

The left hand side is determined as are all the terms on the right hand side except $t^i Sq^{r+1} m$. Since $M$ is free over $H$ the element $Sq^{r+1} m$ is determined. This concludes the induction step.

## 3. The functor $\mathscr{L}$ and Smith theory.

In this section we present our main results concerning the functor $\mathscr{L}$. We shall again restrict our attention to $p = 2$.

THEOREM 3.1. *Suppose $M \in H\backslash\mathscr{K}$ is finitely generated as an $H$-module. Then for all $s \geqq 0$*

$$\sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))^i \leqq \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H M)^i.$$

PROOF. We may assume without loss of generality that $M$ is free as an $H$-module since $\mathscr{L}$ kills torsion. In particular $M$ is an unstable $A_p$-subalgebra of $\mathscr{L}(M)$. The proof is by induction on the dimension of $M$ as an $H$-module. Suppose $\dim_H(M) = 1$ then the inequality is immediate. Alternatively if $\dim_H(M) = 2$ then, by Lemma 2.3, $M \cong (\Sigma^r H_k)_+$ and $\mathscr{L}(M) \cong (\Sigma^k H)_+$ so that the inequality holds. Now assume the lemma is true for all $M$ with $\dim_H(M) \leqq n-1$, $n \geqq 2$. Let $b \in \overline{\mathscr{L}(M)}$ be non-zero of minimal degree $r$. Then there is an $H\backslash\mathscr{K}$-map $b^*$: $\mathscr{L}(M) \to (\Sigma^r H)_+$ which is onto with $b^*(b)$ non-zero. Let

$a^*: M \to \overline{(\Sigma^r H_k)}_+$ be the onto restriction-projection of $b^*$, and let $M' \in H\backslash\mathscr{K}$ be the kernel of $a^*$. Then $\dim_H(M') = n - 1$ and there is an exact sequence $0 \to \mathscr{L}(M') \to \mathscr{L}(M) \to \Sigma^r H \to 0$. Consequently

$$\sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H M)^i = \begin{cases} \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H M')^i + 1 & s \leqq r + k \\ \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H M')^i & s > r + k \end{cases}$$

$$\geqq \begin{cases} \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M'))^j + 1 & s \leqq r + k \\ \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M'))^j & s > r + k \end{cases}$$

$$\begin{cases} = \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))^i + 1 & s \leqq r \\ \geqq \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))^i + 1 & r < s \leqq r + k. \\ = \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))^i & r + k < s \end{cases}$$

Hence $\sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H M)^i \geqq \sum_{i \geqq s} \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))^i$.

We also have

LEMMA 3.2. *Let $M \in H\backslash\mathscr{K}$. Then $M$ is free as an $H$-module if and only if* $\dim(\mathbf{Z}_2 \otimes_H M) = \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))$

PROOF. From the proof of the above lemma we see that if $M$ is free as an $H$-module then $\dim(\mathbf{Z}_2 \otimes_H M) = \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))$. Suppose then this identity holds. Let $M'$ be the ideal of $M$ consisting of elements of $H$-torsion. Then $M/M' \in H\backslash\mathscr{K}$ is free as an $H$-module so that $\dim(\mathbf{Z}_2 \otimes_H (M/M')) = \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M/M')) = \dim(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))$ where the latter equality follows by exactness of $\mathscr{L}$. Hence $\dim(\mathbf{Z}_2 \otimes_H M) = \dim(\mathbf{Z}_2 \otimes_H (M/M'))$. So $\dim(\mathbf{Z}_2 \otimes_H M') = 0$ and $M$ is free as an $H$-module.

For $M \in \mathscr{K}$ let $\pi_0 M = \mathrm{Hom}_{\mathscr{K}}(M, \mathbf{Z}_2)$. Suppose $\varphi \in \pi_0 M$. Then the component of $M$ containing $\varphi$ is defined to be the unstable $A_p$-algebra $M_\varphi = \mathbf{Z}_2 \otimes_M M$ where $\mathbf{Z}_2$ is an $M$-module by the map $\varphi$. We say $M$ is connected if $\pi_0 M = \mathbf{Z}_2$. Similarly for $M \in H\backslash\mathscr{K}$ let $\pi_0 M = \mathrm{Hom}_{H\backslash\mathscr{K}}(M, H)$. Suppose $\varphi \in \pi_0 M$. Then the component of $M$ containing $\varphi$ is defined to be the unstable $A_p$-algebra under $H$, $M_\varphi = H \otimes_M M$. We say $M$ is connected if $\pi_0 M = \mathbf{Z}_2$. It follows from general properties of $\mathscr{L}$ [9] that

LEMMA 3.3. *Let $M \in H\backslash\mathscr{K}$ and $\varphi \in \pi_0 M$. Let $\bar{\varphi} \in \pi_0(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))$ denote the $\mathscr{K}$ map $\mathbf{Z}_2 \otimes_H \mathscr{L}(\varphi)$. Then $\mathbf{Z}_2 \otimes_H (\mathscr{L}(M_\varphi)) \cong (\mathbf{Z}_2 \otimes_H \mathscr{L}(M))_{\bar{\varphi}}$.*

We shall require

LEMMA 3.4. *Let $M \in H\backslash\mathscr{K}$ be free and finitely generated as an $H$-module. Suppose $M$ is connected. Then for all $m \in \bar{M}$ there is an integer $r$ such that $m^r = 0$.*

PROOF. Let $i: \bar{M} \subset \overline{\mathscr{L}(M)} \cong H \otimes_{\mathbf{Z}_2} \overline{(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))}$. Let $m \in \bar{M}$, necessarily of positive degree, then $i(m^{2^n}) = (\sum_i t^{r_i} \otimes c_i)^{2^n} = \sum_i t^{2^n r_i} \otimes c_i^{2^n}$. Since $Z_2 \otimes_H \mathscr{L}(M)$ is finite and $c_i$ is of positive degree this latter expression is zero for large $n$.

We have

THEOREM 3.5. *Let $M$ be free and finitely generated as an $H$-module. Suppose $Z_2 \otimes_H M$ is generated as an algebra by at most $n$ elements. Then each component of $Z_2 \otimes_H \mathscr{L}(M)$ is generated as an algebra by at most $n$ elements.*

PROOF. By the Lemma 3.3 we may assume $M$ is connected. Let $m_1, \ldots, m_n \in M$ be a minimal generating set for $Z_2 \otimes_H M$ as an algebra. Let $c_1, \ldots, c_r \in \mathscr{L}(M)$ generate $Z_2 \otimes_H \mathscr{L}(M)$ as a graded vector space. Then for some large $k_1$, we have $t^{k_1} m_1 = \sum_i \varepsilon_{i,1} t^{r_i} c_i$ for some $\varepsilon_{i,1} \in Z_2$ and $r_i \geqq 0$, where $t^{r_i} c_i \in M \subset \mathscr{L}(M)$. Now $t^{r_i} c_i = p_i(m_1, \ldots, m_n, t)$, a polynomial in the $m_i$'s with coefficients in $H$. Let $p_i = l_i + q_i$ where $l_i$ is linear in the $m_i$ and $q_i$ is quadratic or higher. Thus $t^{k_1} m_1 = \sum_i \varepsilon_{i,1} l_i$. Similarly we have $t^{k_j} m_j = \sum_i \varepsilon_{i,j} l_i$ for some $\varepsilon_{i,j} \in Z_2, j = 1, \ldots, n$. Hence $\{l_i\}$ spans the homogeneous vector subspace $\langle t^{k_1} m_1, \ldots, t^{k_n} m_n \rangle$ of $M$, and so there is a subcollection, say $\{l_1, \ldots, l_n\}$ after reordering which also spans. Consequently the linear terms in $\{t^{r_1} c_1, \ldots, t^{r_n} c_n\}$ generate $S^{-1}M$ as $S^{-1}H$-algebra since each $m_i$ is nilpotent. It follows that $\{c_1, \ldots, c_n\}$ generates $Z_2 \otimes_H \mathscr{L}(M)$.

The proof of the following theorem is similar to Bredon's proof of the analogous theorem in [3]

THEOREM 3.6. *Let $M \in H \backslash \mathscr{K}$ be free and finitely generated as an $H$-module and suppose each component of $Z_2 \otimes_H M$ satisfies Poincaré duality. Then each component of $Z_2 \otimes_H \mathscr{L}(M)$ satisfies Poincaré duality.*

PROOF. By Lemma 3.3 we may assume $M$ is connected so that $Z_2 \otimes_H \mathscr{L}(M)$ is connected. Let $0 \neq b \in Z_2 \otimes_H \mathscr{L}(M)$ with $\deg(b) > 0$. Following Proposition 2.2 we think of $b$ as an element of $\mathscr{L}(M)$. Then for large $k$ we have $t^k b = \sum_{i \geqq 0} t^i m_i$ for some $m_i \in M$ and $m_0 \notin tM$. If $\deg(m_0) < n$ then there exists $m \in M$ such $m \sum_{i \geqq 0} t^i m_i \notin tM$ since $Z_2 \otimes_H M$ satisfies Poincaré duality. Suppose $m = \sum_{i \geqq 0} t^i a_i$ with $a_i \in \mathscr{L}(M)$. Then $0 \neq m \sum_{i \geqq 0} t^i m = \sum_{i \geqq 0} (t^i a_i)(t^k b) = \sum_{i \geqq 0} t^{i+k} a_i b$. Now suppose $a_i b = 0$ for all $a_i$ of positive degree, then $\sum_{i \geqq 0} t^{i+k} a_i b = t^{\deg(m)+k} b$. Thus $t^{\deg(m)} \sum_{i \geqq 0} t^i m_i = m \sum_{i \geqq 0} t^i m_i \notin tM$, a contradiction. Thus we must have $a_i b \neq 0$ for some $a_i$ of positive degree. Suppose the maximum non-zero dimension of $(Z_2 \otimes_H \mathscr{L}(M))_\varphi$ is $r$. Choosing $\deg(b) = r$ gives a contradiction so we must have in this case $\deg(m_0) = n$.

Now suppose $b$, $b' \in Z_2 \otimes_H \mathscr{L}(M)$ with $\deg(b) = r = \deg(b')$ and both $b, b'$ non-zero. Then as before $t^k b = \sum_{i \geqq 0} t^i m_i$ and $t^k b' = \sum_{i \geqq 0} t^i m_i'$ with $m_0, m_0' \notin tM$ and $\deg(m_0) = n = \deg(m_0')$. Consequently $m_0 - m_0' \in tM$ so that $t^k(b - b') \in tM$ where $\deg(m_i - m_i') < n$, $i \geqq 1$. Cancelling $t$ we obtain $t^{k-1}(b - b') = \sum_{i \geqq 0} t^i m_i''$ with $\deg(b - b') = r$ and $\deg(m_i'') < n$, $i \geqq 0$. This is a contradiction to the maximality of $r$ so we must have $b = b'$ and $\dim(Z_2 \otimes_H \mathscr{L}(M))^r = 1$. Let $b$ be a generator.

Suppose $b' \in Z_2 \otimes_H \mathscr{L}(M)$ with $0 < \deg b' = q < r$. Then $t^q b' = \sum_{i \geqq 0} t^i m_i'$ with $m_0' \notin tM$. If $\deg(m_0') < n$ then the result follows from above so assume $\deg(m_0') = n$. Then $m_0' - m_0 \in tM$ so that $t^k b - t^q b' = \sum_{i \geqq 1} t^i(m_i'')$. Assume $m_j''$ is the first non-zero term then $t^{k-j} b - t^{q-j} b' = j(\sum_{i \geqq j} t^{t-j}(m_i''))$. Since $\deg(m_j'') < n$ there exists $m$ such that $\deg(m) > 0$ and $m \sum_{i \geqq j} t^{i-j}(m_i'')) \notin tM$. Let $m = \sum_{i \geqq 0} t^i a_i$. Then $0 \neq m \sum_{i \geqq j} t^{i-q}(m_i'')) = \sum_{i \geqq j}(t^{k-j+i} a_i b - t^{q-j \to i} a_i b')$ so that $0 \neq \sum_{i \geqq 0} (t^{k+i} a_i b - t^{q+i} a_i b')$. Since $\deg(m) > 0$ and $(Z_2 \otimes_H \mathscr{L}(M))^0$ is connected we may assume that for $\deg(a_i) = 0, a_i = 0$. But $b$ is maximal so we have $0 = \sum_{i \geqq 0} t^{q+i} a_i b'$ so that $a_i b' \neq 0$ for some $a_i$ of positive degree. The result follows.

## 4. Conclusion.

We conclude with a couple of typical applications of the previous results.

THEOREM 4.1. *Let* $M \in H\backslash\mathscr{K}$ *with* $Z_2 \otimes_H M \cong Z_2[x, y]/\{x^2 = y^2 = 0\}$, *with* $\deg(x) = n, \deg(y) = m, n \leqq m$. *Then if* $Z_2 \otimes_H \mathscr{L}(M)$ *is nonzero one of the following must occur*

1. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a, b]/\{a^2 = b^2 = 0\}, \quad 0 < \deg(a) \leqq n, 0 < \deg(b) \leqq m$
2. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a, b]/\{a^3 = b^3 = ab = 0\}, \quad 0 < \deg(a), \deg(b) \leqq n$
3. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a]/\{a^4 = 0\}, \quad 0 < \deg(a) < n$
4. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2 + Z_2[a]/\{a^3 = 0\}, \quad 0 < \deg(a) < n$
5. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a]/\{a^2 = 0\} + Z_2[b]/\{b^2 = 0\}, \quad 0 \leqq \deg(a), \deg(b) \leqq m$
6. $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a]/\{a^2 = 0\}$

Here the isomorphism is of $Z_2$-algebras and the operation $+$ is the sum in this category.

PROOF. By Lemma 3.1 we have $\dim(Z_2 \otimes_H \mathscr{L}(M)) \leqq 4$. But $\chi(Z_2 \otimes_H M) \equiv \chi(Z_2 \otimes_H \mathscr{L}(M)) \bmod 2$. Hence if $Z_2 \otimes_H L(M)$ is nonzero we have either $Z_2 \otimes_H \mathscr{L}(M) \cong Z_2[a]/\{a^2 = 0\}$ or $\dim(Z_2 \otimes_H \mathscr{L}(M)) = 4$. In the latter case, by Lemma 3.2, $M$ is free as an $H$-module. Consequently Theorem 3.6 implies that each component of $Z_2 \otimes_H \mathscr{L}(M)$ satisfies Poincaré duality. The cases 1–5 are then all possible Poincaré duality algebras consistent with the inequalities of Theorem 3.1, except for the inequalites in cases 3, 4, 5 which require further justification. For example, in case 3 Theorem 3.5 implies $\deg(a) \leqq n$. To obtain $\deg(a) < n$ suppose to the contrary $\deg(a) = n$. Let $j: M \subset \mathscr{L}(M)$. Then there is a representative $m \in M$ of $x \in Z_2 \otimes_H M$ such that $j(m) = a$. But then $x^2 \in tM$ so that $a^2 \in t\mathscr{L}(M)$ which contradicts Proposition 2.2. Case 4 is similar so consider case

5. Suppose $\deg(b) > m$. Let $\bar{x}, \bar{y} \in M$ be representatives of $x, y \in \mathbf{Z}_2 \otimes_H M$ respectively, and let $1_a \in \mathbf{Z}_2[a]/\{a^2 = 0\}$, $1_b \in \mathbf{Z}_2[b]/\{b^2 = 0\}$ be units. Then since $\deg(b) > \deg(\bar{x})$ we can express $\bar{x}$ as a polynomial in $1_a$, $1_b$, $a$ with coefficients in $H$. Similarly for $\bar{y}$. But for large $k$, $t^k b$ is a polynomial in $1, \bar{x}, \bar{y}$ with coefficients in $H$ and hence in $1_a$, $1_b$, $a$. This is a contradiction so that $\deg(b) \leqq m$. Similarly $\deg(a) \leqq m$.

THEOREM 4.2. *Let $M \in H\backslash\mathscr{K}$ with $\mathbf{Z}_2 \otimes_H M \cong \mathbf{Z}_2[x]/\{x^{k+1} = 0\}$, $\deg(x) = n$. Then if $\mathbf{Z}_2 \otimes_H \mathscr{L}(M)$ is nonzero each component $(\mathbf{Z}_2 \otimes_H \mathscr{L}(M))_\varphi$ is isomorphic to a $Z_2$-algebra $\mathbf{Z}_2[a]/\{a_\varphi^{h_\varphi+1}\}$ with $\deg(a) \leqq n$. There are at most two components and $\sum_\varphi (h_\varphi + 1) = k + 1$ where the sum is taken across components of $\mathbf{Z}_2 \otimes_H \mathscr{L}(M)$.*

PROOF. This is an easy consequence of Lemmas 3.1, 3.2 and 3.4. To show that there are at most two components of $\mathbf{Z}_2 \otimes_H \mathscr{L}(M)$ note that $\mathrm{Hom}_{\mathscr{K}}(\mathbf{Z}_2 \otimes_H \mathscr{L}(M), \mathbf{Z}_2) \cong \mathrm{Hom}_{H\backslash\mathscr{K}}(M, H)$ by [9]. Thus $\mathrm{Hom}_{H\backslash\mathscr{K}}(M, H)$ is dimension at most 2 since $M$ has 2 generators as an unstable $A_p$-algebra under $H$.

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