MATH. SCAND. 78 (1996), 84–92

ABOUT THE L^p-BOUNDEDNESS OF INTEGRAL OPERATORS WITH KERNELS OF THE FORM $K_1(x-y)K_2(x+y)$

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Abstract.

In this paper we prove the $L^p(R^n)$-boundedness, $1 < p < \infty$, of integral operators with kernels of the form $K_1(x-y)K_2(x+y)$ for a wide class of functions K_1, K_2 satisfying certain homogeneity condition.

1. Introduction.

Let q be a real number, $1 < q < \infty$. We set q' the conjugate exponent of q given by $q^{-1} + q'^{-1} = 1$. Also for $g: R^n \to C$ we define $g^{(j,q)}(x) = 2^{jn/q}g(2^j x)$, and $g_{(j)}(x) = g(2^j x)$.

Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ be two families of measurable functions on R^n with support contained in $\{t: 2^{-1} \leqq |t| \leqq 2\}$ such that

$$\|\varphi_j\|_{q_0} \leqq c_1, \ \|\psi_j\|_{q_1} \leqq c_2 \tag{1.1}$$

for some $q_0 > q$, $q_1 > q'$, $c_1 > 0$, $c_2 > 0$, and for all $k \in Z$.

In this paper we prove the L^p boundedness, $1 < p < \infty$, of the integral operator defined by

$$Tf(\xi) = \int_{R^n} K_1(\xi - y)K_2(\xi + y)f(y)\,dy$$

where $K_1(x) = \sum_{j\in Z} \varphi_j^{(j,q)}(x)$, and $K_2(x) = \sum_{j\in Z} \psi_j^{(j,q')}(x)$.

The operator $f \to K_1 * f$ is well known since K_1 is in weak-L^q.

The particular case $K_1(x) = \Omega_1(x)|x|^{-\alpha}$ and $K_2(x) = \Omega_2(x)|x|^{-n+\alpha}$ with Ω_1 and Ω_2 in $L^\infty(R^n)$, $0 < \alpha < n$, is treated in [G-U] and it is a generalization of the one dimensional case developed in [R-Sj].

[1] Partially supported by CONICOR, CONICET and SECYT.
Received August 24, 1994.

The singular operator obtained taking $q = 1, q' = \infty$ is studied in [G-S-U], for suitable hypothesis about the families $\{\varphi_j\}_{j\in Z}$ and $\{\psi_j\}_{j\in Z}$.

2. Preliminaries.

We begin with the following observation.

LEMMA 2.1. *Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ be two families of measurable and non negative functions on R^n with support contained in $\{t\colon 2^{-1} \leqq |t| \leqslant 2\}$ satisfying* (1.1). *Then there exists $c = c(c_1, c_2, n, q, q_0, q_1) > 0$ such that, for $l \in Z$ we have*

$$\sum_{j>-l} \int_{R^n} \varphi_j^{(j,q)}(x)\psi_{-l}^{(-l,q')}(2\xi - x)\,dx \leqq c$$

PROOF. We take δ satisfying $0 < \delta < q - 1$ and $(q-\delta)' < q_1$. Then we have

$$\sum_{j>-l} \int_{R^n} \varphi_j^{(j,q)}(x)\psi_{-l}^{(-l,q')}(2\xi - x)\,dx$$

$$= \sum_{j>-l} 2^{jn(q^{-1}-(q-\delta)^{-1})} \int_{R^n} \varphi_j^{(j,q-\delta)}(x)\psi_{-l}^{(-l,q')}(2\xi - x)\,dx$$

$$\leqq \sum_{j>-l} 2^{jn(q^{-1}-(q-\delta)^{-1})} \|\varphi_j^{(j,q-\delta)}\|_{q-\delta} \|\psi_{-l}^{(-l,q')}\|_{(q-\delta)'}$$

$$= \sum_{j>-l} 2^{jn(q^{-1}-(q-\delta)^{-1})} \|\varphi_j\|_{q-\delta} 2^{-ln(q'^{-1}-(q-\delta)'^{-1})} \|\psi_{-l}\|_{(q-\delta)'}$$

$$\leqq c 2^{ln((q-\delta)^{-1}-q^{-1})} 2^{ln(q^{-1}-(q-\delta)^{-1})} = c$$

where c depends only on c_1, c_2, n, q, and δ. This proves the lemma.

LEMMA 2.2. *Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ be two families of measurable and non negative functions on R^n with support contained in $\{t\colon 2^{-1} \leqq |t| \leqq 2\}$ satisfying* (1.1). *Let G_j be the linear operator defined by*

$$G_j f(\xi) = \int_{R^n} \varphi_j(x)\psi_j(2\xi - x) f(\xi - x)\,dx$$

Then there exists a positive constant $c = c(c_1, c_2, n, p)$ such that

$$\|G_j f\|_1 \leqq c\,\|f\|_p$$

for $j \in Z$, $1 \leqq p \leqq \infty$ and $f \in L^p(R^n)$.

PROOF. We first estimate $G_j|f|$. By the Jensen inequality

$$\|G_j|f|\|_1^p$$
$$\leqq \left(\int_{R^n\times R} \varphi_j(x)\psi_j(2\xi - x)\,dx\,d\xi\right)^{p-1} \int_{R^n\times R} \varphi_j(x)\psi_j(2\xi - x)\,|f(\xi - x)|^p\,dx\,d\xi$$
$$= \|(\varphi_j * \psi_j)_{(1)}\|_1^{p-1} \int_{R^n\times R} \varphi_j(x)\psi_j(2\xi - x)\,|f(\xi - x)|^p\,dx\,d\xi$$

we choose $\delta > 0$ such that $1 < (q - \delta)^{-1} + q'^{-1} < 2$ and we define r by $1 + r^{-1} = (q - \delta)^{-1} + q'^{-1}$. Since $\operatorname{supp}(\varphi_j * \psi_k) \subseteq \{x: |x| \leqq 4\}$ we can write

$$\|(\varphi_j * \psi_j)_{(1)}\|_1 \leqq c' \,\|(\varphi_j * \psi_j)_{(1)}\|_r \leqq c' \,\|\varphi_j\|_{q-\delta}\,\|\psi_j\|_{q'} \leqq c''$$

with c'' independent of j. Then if we denote by $f^{\vee}(t) = f(-t)$ we obtain

$$\|G_j|f|\|_1^p \leqq c'' \int_{R^n} \varphi_j(x)((\psi_j)_{(1)} * |f^{\vee}|^p)(x/2)\,dx \leqq C\,\|\varphi_j\|_q\,\|\psi_j\|_{q'}\,\||f^{\vee}|^p\|_1 \leqq c\,\|f\|_p^p.$$

To study $G_j f$ we write $f = f^+ - f^-$ as usual, and the lemma follows.

LEMMA 2.3. *Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ be two families of functions satisfying the hypothesis of the lemma 2.2. Then for $1 < p < \infty$, there exists $\gamma > 0$ and $c_{p,\gamma} > 0$ such that*

$$\|G_j f\|_{p+\gamma} \leqq c_{p,\gamma}\,\|f\|_p \text{ for all } j \in Z.$$

PROOF. Let q_0, q_1 as in (1.1). We prove first the case $p'^{-1} > q_0^{-1} + q_1^{-1}$.

We define α, β by $q_0 = q + \alpha$, $q_1 = q + \beta$, then there exists $t \in (0, 1)$ such that $p'^{-1} = (q + t\alpha)^{-1} + (q + t\beta)^{-1}$. Since $\operatorname{supp}(\varphi_j)$ and $\operatorname{supp}(\psi_j)$ are contained in $\{x: 2^{-1} \leqq |x| \leqq 2\}$ we have $\|\varphi_j\|_{q+t\alpha} \leqq c'$ and $\|\psi_j\|_{q'+t\beta} \leqq c''$, with c', c'' depending only of c_1, c_2, q_0, q_1, and n. Then we can write

$$|G_j f(\xi)| \leqq \left(\int_{R^n} \varphi_j(x)^{p'}\psi_j(2\xi - x)^{p'}\,dx\right)^{1/p'} \left(\int_{R^n} |f(\xi - x)|^p\,dx\right)^{1/p}$$
$$\leqq (\|\varphi_j^{p'}\|_{(q+t\alpha)/p'}\,\|\psi_j^{p'}\|_{(q'+t\beta)/p'})^{1/p'}\,\|f\|_p \leqq c_3\,\|f\|_p.$$

with c_3 independent of ξ and j, then

$$\|G_j f\|_\infty \leqq c_3\,\|f\|_p. \tag{2.4}$$

We define p_0 by $1 - p_0^{-1} = q_0^{-1} + q_1^{-1}$. Then (2.4), lemma 2.2 and the Marcinkiewicz interpolation theorem ([S-W]) give us

$$\|G_j\|_{p,s} \leqq c_{p,s} \quad \text{for all} \quad j \in Z,\ p > p_0,\ 1 \leqq s \leqq \infty. \tag{2.5}$$

With $\|G_j\|_{p,s}$ we denote the operator norm of G_j from $L^p(R^n)$ into $L^s(R^n)$.

Next we take the adjoint operator G_j^* given by

$$G_j^* g(\xi) = \int_{R^n} \varphi_j^\vee(x)\psi_j(2\xi - x)g(\xi - x)\,dx$$

Applying (2.5) to G_j^* we obtain

(2.6) $\quad \|G_j\|_{p,s} = \|G_j^*\|_{s',p'} \leqq c_{s',p'}$ for all $j \in Z$ and $p < s < p_0'$

The case $p_0' \leqq p \leqq p_0$ follows from (2.5), (2.6) and the Marcinkiewicz interpolation theorem. ([S-W])

3. The main result.

Theorem 3.1. *Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ be two families of measurable functions on R^n with support contained in $\{t: 2^{-1} \leqq |t| \leqq 2\}$ satisfying* (1.1). *Then*

$$Tf(\xi) = \int_{R^n} K_1(\xi - y)K_2(\xi + y)f(y)\,dy$$

where $K_1(x) = \sum_{j\in Z} \varphi_j^{(j,q)}(x)$, and $K_2(x) = \sum_{j\in Z} \psi_j^{(j,q')}(x)$, is a well defined and bounded operator on $L^p(R^n)$, $1 < p < \infty$.

The proof of this theorem will be a direct consequence of the following results.

For $l \in Z$ we set $A_l = \{t: 2^l \leqq |x| \leqq 2^{l+1}\}$. A direct application of lemma 2.1 gives us

LEMMA 3.2. *Let $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ as in theorem* 3.1. *Then the linear operator defined by*

$$T_1 f(\xi) = \sum_{l\in Z} \chi_{A_l}(\xi) \int_{|x| \leqq 2^{l-1}} \sum_{j,k\in Z} |\varphi_j^{(j,q)}(x)|\,|\psi_k^{(k,q')}(2\xi - x)|\,f(\xi - x)\,dx$$

is well defined a.e. $\xi \in R^n$ *and bounded on $L^p(R^n)$, $1 \leqq p \leqq \infty$.*

PROOF. It is enough to assume that the functions $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ and f are non negative. We first prove the boundedness of T_1 on $L^\infty(R^n)$. We take $x \in R^n$ such that $|x| \leqq 2^{l-1}$. Then $\varphi_j^{(j,q)}(x) = 0$ for $j < -l$, *and* $\psi_k^{(k,q)}(2\xi - x) = 0$ for $\xi \in A_l$ and $k \notin [-l-4, -l+1]$. So

$$T_1f(\xi) = \sum_{l\in Z} \chi_{A_l}(\xi) \int_{|x|\leqq 2^{l-1}} \sum_{\substack{j\geqq -l\\ -4\leqq i\leqq 1}} \varphi_j^{(j,q)}(x)\psi_{-l+i}^{(-l+i,q')}(2\xi - x)f(\xi - x)\,dx$$

$$\leqq \|f\|_\infty \sum_{l\in Z} \chi_{A_l}(\xi) \int_{R^n} \sum_{\substack{j\geqq -l\\ -4\leqq i\leqq 1}} \varphi_j^{(j,q)}(x)\psi_{-l+i}^{(-l+i,q')}(2\xi - x)\,dx$$

And so lemma 2.1 implies that T_1 is bounded on $L^\infty(R^n)$.

To see the boundedness of T_1 on $L^1(R^n)$ we write

$$\int_{R^n} T_1f(\xi)\,d\xi = \sum_{l\in Z} \sum_{\substack{j>-l\\ -4\leqq i\leqq 1}} \int_{A_l} \int_{|x|\leqq 2^{l-1}} \varphi_j^{(j,q)}(x)\psi_{-l+i}^{(-l+i,q')}(2\xi - x)f(\xi - x)\,dx\,d\xi$$

We set $A_{r,s} = \{t\colon 2^r \leqq |t| \leqq 2^s\}$. Then for $\xi \in A_l$ and $|x| \leqq 2^{l-1}$ we have $2^{l-1} \leqq |\xi - x| \leqq 2^{l+2}$. So we can replace in the above integral $f(\xi - x)$ by $\chi_{A_{l-1,l+2}}(\xi - x)f(\xi - x)$. Now we change the integration order to obtain

$$\|T_1f\|_1 \leqq \sum_{l\in Z} \sum_{\substack{j>-l\\ -4\leqq i\leqq 1}} \int_{|x|\leqq 2^{l-1}} \varphi_j^{(j,q)}(x) \int_{R^n} \psi_{-l+i}^{(-l+i,q')}(2\xi - x)f\chi_{A_{l-1,l+2}}(\xi - x)\,d\xi\,dx$$

$$= \sum_{l\in Z} \sum_{\substack{j>-l\\ -4\leqq i\leqq 1}} 2^{-n/q'} \int_{|x|\leqq 2^{l-1}} \varphi_j^{(j,q)}(x)(\psi_{-l+i}^{(-l+i+1,q')} * (f\chi_{A_{l-1,l+2}})^\vee)(x/2)\,dx$$

Now we choose δ as in lemma 2.1. Then the last sum is bounded by

$$c\sum_{l\in Z} \sum_{\substack{j>-l\\ -4\leqq i\leqq 1}} \|\varphi_j^{(j,q)}\|_{q-\delta} \|\psi_{-l+i}^{(-l+i+1,q'} * (f\chi_{A_{l-1,l+2}})^\vee\|_{(q-\delta)}$$

$$\leqq c\sum_{l\in Z} \sum_{\substack{j>-l\\ -4\leqq i\leqq 1}} \|\varphi_j^{(j,q)}\|_{q-\delta} \|\psi_{-l+i}^{(l+i+1,q')}\|_{(q-\delta)'} \|f\chi_{A_{l-1,l+2}}\|_1$$

So we apply lemma 2.1 in order to obtain

$$\|T_1f\|_1 \leqq c' \sum_{l\in Z} \|f\chi_{A_{l-1.l+2}}\|_1 \leqq c''\,\|f\|_1$$

where in the last inequality we use the finite overlapping property of the family $\{f\chi_{A_{l-1,l+2}}\}_{l\in Z}$

The boundedness of T_1 on $L^p(R^n)$, $1 < p < \infty$, follows now from the Marcinkiewicz interpolation theorem ([S-W]).

LEMMA 3.3. *Let* $\{\varphi_j\}_{j\in Z}$,$\{\psi_j\}_{j\in Z}$ *as in theorem* 3.1. *Then the linear operator defined by*

$$T_2f(\xi) = \sum_{l\in Z} \chi_{A_l}(\xi) \int_{2^{l-1}\leqq|x|\leqq 2^{l+2}} \sum_{j,k\in Z} |\varphi_j^{(j,q)}(x)|\,|\psi_k^{(k,q')}(2\xi - x)|\,f(\xi - x)\,dx$$

is bounded in $L^p(R^n)$ for $1 < p < \infty$.

PROOF. As in lemma 3.2, it is enough to assume that the functions $\{\varphi_j\}_{j\in Z}$ and f are non negative. Along this proof c will denote a positive constant, not necessarily the same, even in a same chain of inequalities

We set

$$U_1 f(\xi) = \sum_{l\in Z} \chi_{A_l}(\xi) \int\limits_{\substack{2^{l-1}\leqq|x|\leqq 2^{l+2}\\ |2\xi - x|\leqq|\xi/2|}} \sum_{j,k\in Z} \varphi_j^{(j,q)}(x)\psi_k^{(k,q')}(2\xi - x) f(\xi - x)\,dx$$

we also put

$$U_2 f(\xi) = \sum_{l\in Z} \chi_{A_l}(\xi) \int\limits_{\substack{2^{l-1}\leqq|x|\leqq 2^{l+2}\\ |2\xi - x|\geqq|\xi/2|}} \sum_{j,k\in Z} \varphi_j^{(j,q)}(x)\psi_k^{(k,q')}(2\xi - x) f(\xi - x)\,dx$$

We have $T_2 f = U_1 f + U_2 f$. We first prove that U_1 is bounded on $L^p(R^n)$, $1 < p < \infty$. We note that for $|2\xi - x| \leqq |\xi/2|$ and $2^{l-1} \leqq |x| \leqq 2^{l+2}$ we have $2^{l-1} \leqq |\xi - x| \leqq 2^{l+3}$ then wee can replace in the above integral $f(\xi - x)$ by $f\chi_{A_{l-1,l+3}}(\xi - x)$. Moreover the sum is extended only over $-l-2 \leqq j \leqq -l+1$ and $k \geqq -l$. Then we have

$$\|U_1 f\|_1 \leqq \sum_{l\in Z} \int\limits_{R^n} \Bigg(\int\limits_{R^n} \sum_{\substack{-1\leqq i\leqq 1\\ k\geqq -l}} \varphi_{-l+i}^{(-l+i,q)}(x)\psi_k^{(k,q')}(2\xi - x) f\chi_{A_{l-1,l+3}}(\xi - x)\,dx \Bigg) d\xi$$

We apply the same argument as in the proof of the lemma 3.2 to obtain $\|U_1 f\|_1 \leqq c\,\|f\|_1$

On the other hand, for $\xi \in A_l$,

$$U_1 f(\xi) \leqq \sum_{l\in Z} \int\limits_{R^n} \sum_{\substack{-1\leqq i\leqq 1\\ k\geqq -l}} \varphi_j^{(j,q)}(x)\psi_k^{(k,q')}(2\xi - x) f(\xi - x)\,dx$$

Now we can apply the lemma 2.1 in order to obtain $\|U_1 f\|_\infty \leqq c\,\|f\|_\infty$ for some positive constant $c > 0$. The Marcinkiewicz interpolation theorem ([S-W]) give us the L^p-boundedness of U_1, $1 < p < \infty$.

We consider now $U_2 f$. By the Marcinkiewicz interpolation theorem ([S-W]), it is enough to check that U_2 is of weak type $p - p$ for $1 < p < \infty$. We take $f \in L^p(R^n)$ such that $\|f\|_p = 1$. We have

$$U_2 f(\xi) = \sum_{l\in Z} \sum_{\substack{-2\leqq i\leqq 1\\ -3\leqq m\leqq 1}} \chi_{A_l}(\xi) \int\limits_{\substack{2^{l-1}\leqq|x|\leqq 2^{l+2}\\ |2\xi - x|\geqq|\xi/2|}} \varphi_{-l+i}^{(-l+i,q)}(x)\psi_{-l+m}^{(-l+m,q')}(2\xi - x) f(\xi - x)\,dx$$

We set

$$T_{r;i,m}f(\xi) = \int_{R^n} (\varphi_{r+i})_{(i)}(x)(\psi_{r+m})_{(m)}(2\xi - x)f(\xi - x)\,dx.$$

A change of variable shows that

$$U_2 f(\xi) \leqq 2^n \sum_{l \in Z} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} \chi_{A_l}(\xi) T_{-l;i,m}(f_{(l)})(2^{-l}\xi)$$

Given $\lambda > 0$ we set

$$J_1 f(\xi) = \sum_{l < l(\lambda)} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} \chi_{A_l}(\xi) T_{-l;i,m}(f_{(l)})(2^{-l}\xi)$$

$$J_2 f(\xi) = \sum_{l \geqq l(\lambda)} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} \chi_{A_l}(\xi) T_{-l;i,m}(f_{(l)})(2^{-l}\xi)$$

where $l(\lambda)$ is an integer such that $2^{l(\lambda)-1} < \lambda^{-p/n} \leqq 2^{l(\lambda)}$

We assert that J_1, J_2 satisfy the weak type $p - p$ inequality. The lemma 2.2 implies that there exists some positive constant c such that

$$\|J_1 f\|_1 \leqq \sum_{l < l(\lambda)} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} 2^{ln} \|T_{-l;i,m}(f_{(l)})\|_1 \leqq c \sum_{l < l(\lambda)} 2^{ln(1-1/p)} \|f\|_p$$

with c independent of λ. Now we can write

$$|\{\xi: J_1 f(\xi) > \lambda/2\}| \leqq c\lambda^{-1} \|J_1 f\|_1 \leqq c\lambda^{-1} 2^{l(\lambda)n(1-1/p)} \|f\|_p \leqq c\lambda^{-p}$$

with c independent of f and λ.

On the other hand, we choose $\gamma > 0$ as in the lemma 2.3. Then

$$\|J_2 f\|_{p+\gamma} \leqq \sum_{l \geqq l(\lambda)} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} \|(T_{-l;i,m}(f_{(l)}))_{(-l)} \chi_{A_l}\|_{p+\gamma}$$

$$\leqq \sum_{l \geqq l(\lambda)} \sum_{\substack{-2 \leqq i \leqq 1 \\ -3 \leqq m \leqq 1}} 2^{ln/(p+\gamma)} \|T_{-l;i,m}(f_{(l)})\|_{p+\gamma}$$

$$\leqq c \sum_{l \geqq l(\lambda)} 2^{ln/(p+\gamma)} \|f_{(l)}\|_p \leqq c \sum_{l \geqq l(\lambda)} 2^{ln((p+\gamma)^{-1} - p^{-1})} \|f\|_p \leqq c 2^{l(\lambda)n((p+\gamma)^{-1} - p^{-1})}.$$

Then

$$|\{\xi: J_2 f(\xi) > \lambda/2\}| \leqq c\lambda^{-(p+\gamma)} \|J_2 f\|_{p+\gamma}^{p+\gamma} \leqq c\lambda^{-(p+\gamma)} 2^{l(\lambda)n(1-(p+\gamma)/p)} \leqq c\lambda^{-p}.$$

LEMMA 3.4. *Let* $\{\varphi_j\}_{j \in Z}$, $\{\psi_j\}_{j \in Z}$ *as in theorem* 3.1. *Then the linear operator defined by*

$$T_3 f(\xi) = \sum_{l \in Z} \chi_{A_l}(\xi) \int_{2^{l+2} \leqq |x|} \sum_{j,k \in Z} |\varphi_j^{(j,q)}(x)|\,|\psi_k^{(k,q')}(2\xi - x)|f(\xi - x)\,dx$$

is well defined a.e. $\xi \in R^n$ and bounded on $L^p(R^n)$, $1 < p < \infty$.

PROOF. As above we can assume that the functions $\{\varphi_j\}_{j\in Z}$, $\{\psi_j\}_{j\in Z}$ and f are non negative. Then

$$T_3 f(\xi) \leqq \int_{2|\xi| \leqq |x|} \sum_{j,k\in Z} \varphi_j^{(j,q)}(x)\psi_k^{(k,q')}(2\xi - x) f(\xi - x)\, dx$$

So it is enough to estimate the boundedness on $L^p(R^n)$, $1 < p < \infty$, of the operator

$$Kf(\xi) = \int_{2|\xi| \leqq |x|} \sum_{j,k\in Z} \varphi_j^{(j,q)}(x)\psi_k^{(k,q')}(2\xi - x) f(\xi - x)\, dx$$

We take $g \in L^{p'}(R^n)$, $g \geqq 0$, then Tonelli's theorem implies

$$\int_{R^n} Kf(\xi)g(\xi)\, d\xi = \int_{R^n} f(u)K^{\#}g(u)\, du$$

where

$$K^{\#}g(u) = \int_{2|u-x|<|x|} \sum_{j,k\in Z} \varphi_j^{\vee\,(j,q)}(x)\psi_k^{(k,q')}(2u - x) g(u - x)\, dx$$

Since for $u \in A_l$, $\{x\colon 2|u - x| < |x|\} \subseteq \{x\colon |x| < 2|u|\} \subseteq \{x\colon |x| \leqq 2^{l+2}\}$, then lemmas 3.2 and 3.3 imply the boundedness of $K^{\#}$ on $L^{p'}$. Then

$$\|Kf\|_p = \sup_{\|g\|_{p'}=1,\, g\geqq 0} \int_{R^n} Kf(\xi)g(\xi)\, d\xi \leqq c\,\|f\|_p$$

for some positive constant c.

PROOF OF THEOREM 3.1. A change of variables and dominated convergence, show that it is enough to prove the boundedness, on $L^p(R^n)$, of the operator given by

$$\tilde{T}f(\xi) = \int_{R^n} \sum_{j,k\in Z} |\varphi_j^{(j,q)}(x)|\, |\psi_k^{(k,q')}(2\xi - x)|\, f(\xi - x)\, dx$$

Now, $\tilde{T}f = T_1 f + T_2 f + T_3 f$ and the theorem follows from the lemmas 3.2, 3.3 and 3.4.

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